Angles and Lines

Click this to start.

9.1. Angles and Lines 1

When solving problems in geometry, we often work with angles and lines. If knowing how some particular angles get made and work, we can see their concept, and learn better important angles that can help us solve many problems.
It is often the case indeed, some angles can help us get the solutions easy and fast. They are the same angles.

Fig. 1

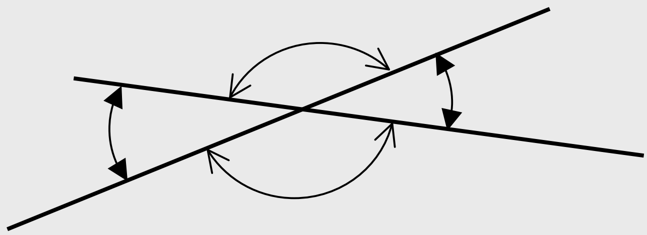
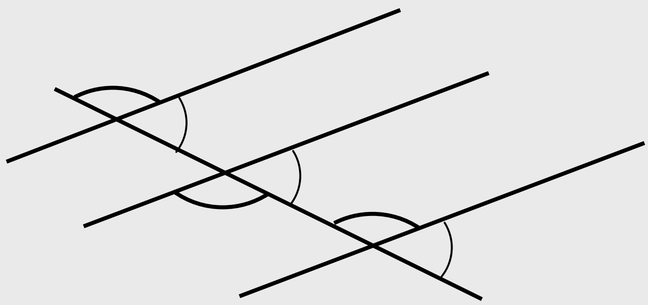


Fig. 2



If knowing thus, those same angles in a concrete manner, that is, if understanding how those same angles get made and work, we can get their concept, and can solve many problems fast enough and with a lot of ease.

So let’s see now, what the angles are, & how they are equal.

Those angles are made by lines, rays, or line segments crossing each other, and some of them are as follows.

Corresponding angles and ***Alternate*** angles

Fig. 3

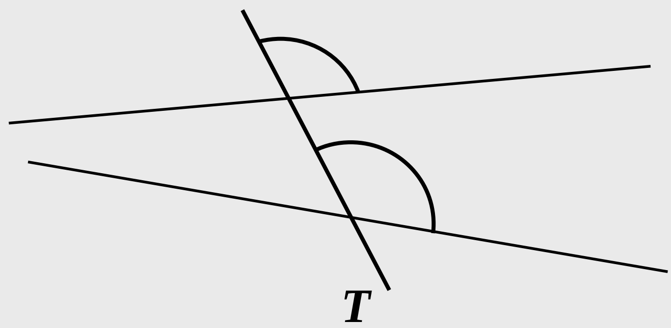
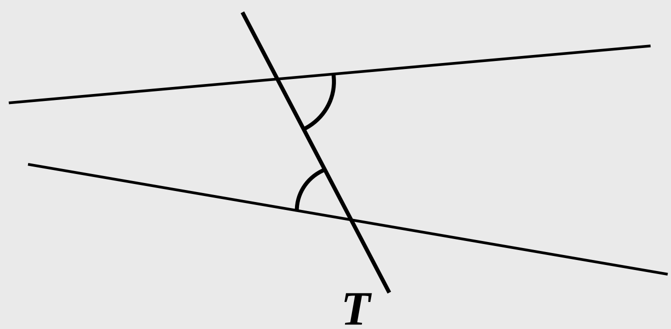


Fig. 4



They are the same if lines are ***parallel***. And the lines are crossed by another line called the ***transversal*** as the line ***T*** shown above. So a transversal is a line intersecting lines.

And the angles are in a pair. They work in a pair. So working with or finding the angles, we consider two angles at a time.

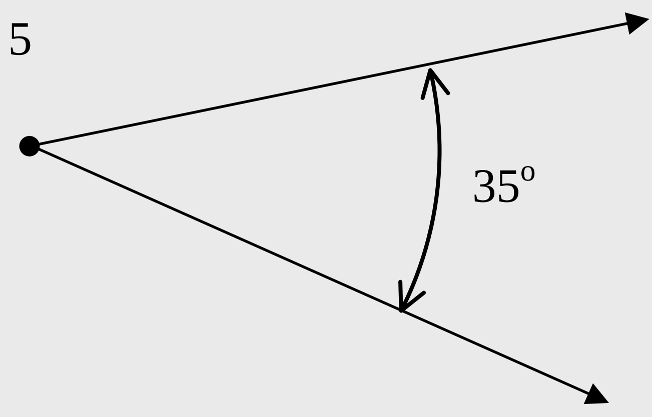
As stated earlier, getting their concept, we can learn better other important angles, so the more familiar, the better.

To be familiar, we want to know first, how they are made or identified, along with the basics on angles, and do examples.

So, together with some basics, let's now begin with how those angles get made or paired, and see and do examples.

First off, what do we mean by an angle?

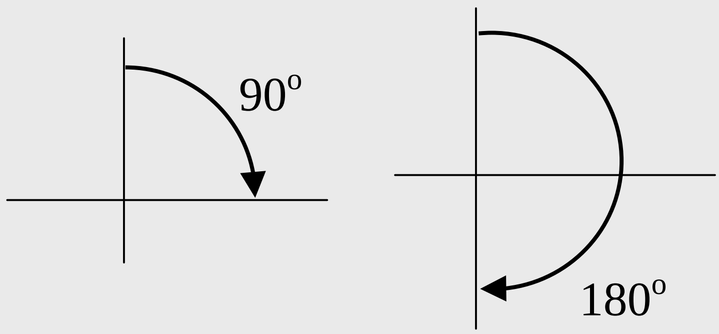
Fig. 5



An angle can mean an amount of ***turning*** or an amount of ***difference in direction***.

Opening a door, for instance, usually, we turn the knob 90^o or more clockwise. So making a quarter turn or more, we can open a door.   What angle then, is a half turn?

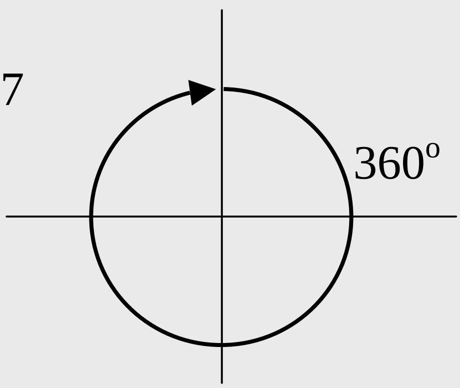
Fig. 6



It's 180^o. So making a half turn, the knob turns 180^o.

And making a complete or full turn, it turns 360^o.

Fig. 7

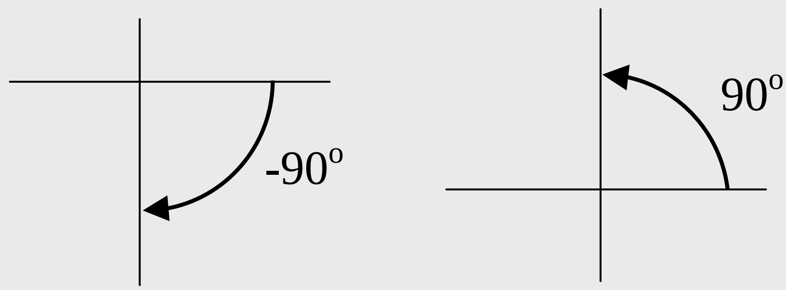


Also, as numbers, angles can be negative and 0, as well as positive.

If turning is made ***clockwise***, we can take the angle as a ***negative*** angle. And expressing a negative angle, we use a minus sign, the way we do to a number.

So for instance, indicating an angle of a negative ninety degrees, we can put it this way: -90^o, read as negative ninety degrees or minus ninety degrees.

Fig. 8



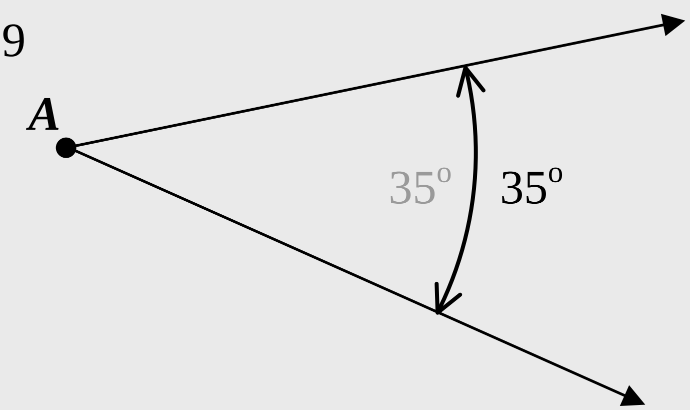
If turning is made ***counterclockwise***, the angle is ***positive***.

And writing an angle with no minus sign, we mean it's positive, so we don't use a plus sign. For instance, indicating an angle of a positive ninety degrees, we can just put it this way: 90^o. And often times, we call the amount of an angle its value.

Next, naming an angle, we usually use a label for it, and can use a letter as a label, together with a math sign like this: $\angle$, and we can call it an angle sign. So specifying an angle, for instance, we can do so this way: $\angle \boldsymbol{A} = 35^{o}$, which means therefore, the angle $\boldsymbol{A}$ is 35^{o}.

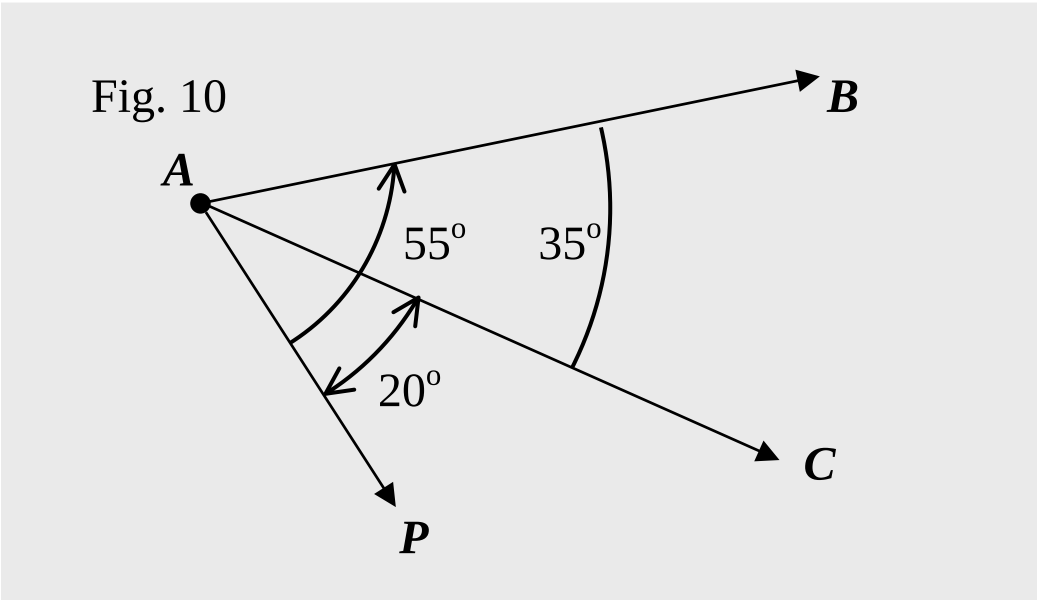
And indicating an angle visually, we often use a wedge, along with an arc with or without an arrowhead. And then, normally, we put a label, the name, at the vertex of the wedge, and specify the amount of the angle, next to the arc, either inside or outside the arc.

Fig. 9



What if however, two or more angles share the same vertex?

If more than an angle share a vertex, we can use three letters for the name of each angle for clarity purposes. And as long as the name of the vertex is in the middle, the order of the letters doesn't matter.



So we can specify each angle above the way as follows.

$\angle BAC = 35^{\circ}$ or $\angle CAB = 35^{\circ}$.

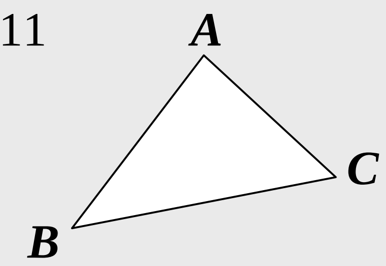
$\angle CAP = \angle PAC = 20^{\circ}$.

$\angle BAP = \angle PAB = 55^{\circ}$.

By the way, naming a polygon as a triangle, we can use names of vertices the way as follows.

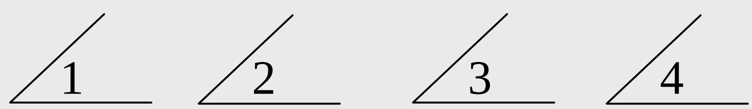
The triangle in the figure below can be called a ***triangle ABC.*** And indicating a triangle, we often use a math symbol, and use it this way: ΔABC, read as a triangle ABC, so we use a small triangle for the symbol, and put it in front of the name of the triangle.

Fig. 11



We can, of course, use others for labels, too, to name angles. It seems some people like to use numbers for those labels. They name angles the way as follows.

Fig 12



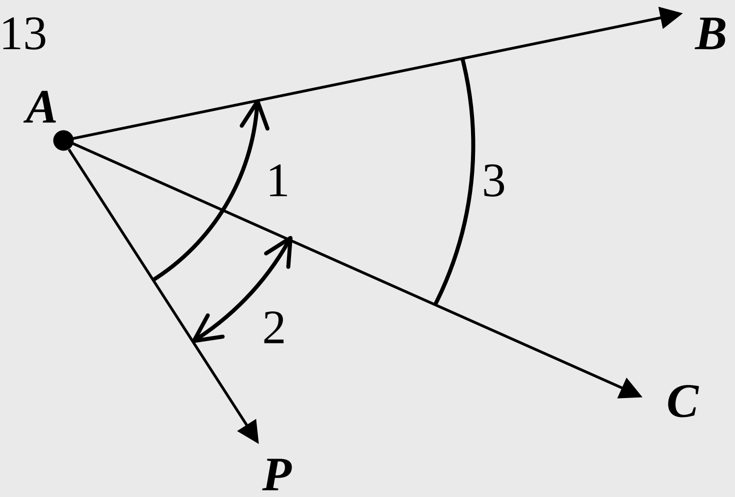
Then, we can specify angles this way:

$\angle 1 = 35^o$, $\angle 2 = 35^o$, $\angle 3 = 25^o$, $\angle 4 = 65^o$

and we can have these: $\angle 1 = \angle 2$, and $\angle 1 \neq \angle 3$.

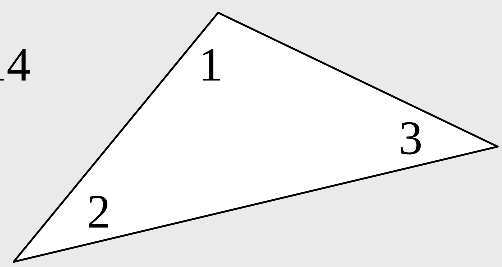
Numbers can be convenient to name angles if more than an angle share a vertex. So as shown below, we can use a number to name each angle with no ambiguity.

Fig. 13



And we can name angles in a triangle this way, too:

Fig. 14

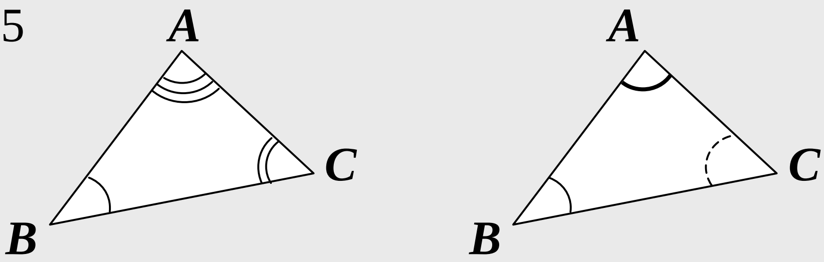


So we can specify each angle above the way as follows.

For example, $\angle 1 = 100^{o}$, $\angle 2 = 45^{o}$, and $\angle 3 = 35^{o}$.

Next, if indicating different angles, we can use a different number of arcs or a different arc for each angle.

Fig. 15

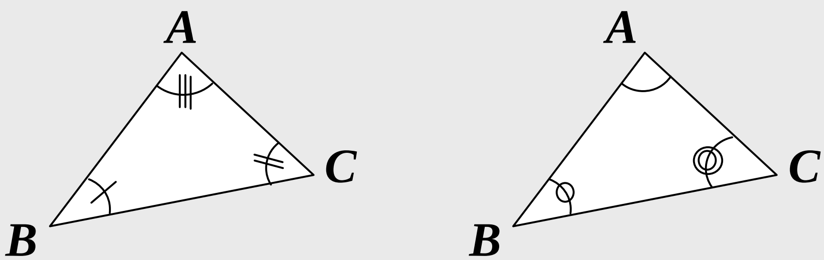


So in ΔABC in the figure above, the three angles are different, that is, we have this: $\angle A \neq \angle B \neq \angle C$.

And of course, we can indicate different angles other ways, too, as the ways shown below.

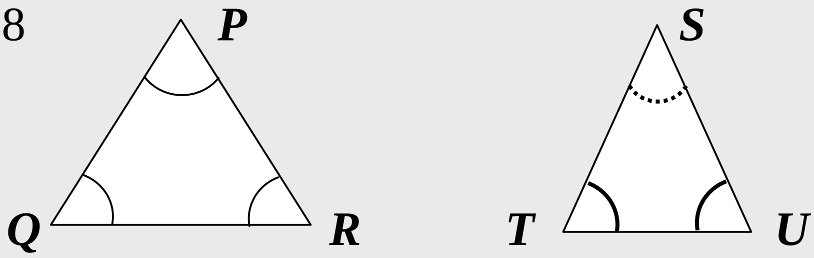
Fig 16

Fig. 17



And if all the three angles are the same, we can use the same arc for each angle.

Fig. 18



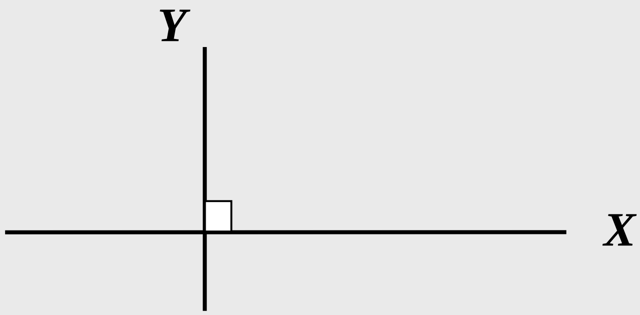
So in $\Delta \boldsymbol{PQR}$, all the three angles are the same, so we have this: $\angle \boldsymbol{P} = \angle \boldsymbol{Q} = \angle \boldsymbol{R}$, which means, $\angle \boldsymbol{P} = \angle \boldsymbol{Q} = \angle \boldsymbol{R} = 60^{o}$, because the sum of all the three angles in a triangle is 360^{o}. And if the three angles are equal, it's a regular triangle, which is equilateral, so the three sides are equal, too.

And in $\Delta \boldsymbol{STU}$, the two angles, $\angle \boldsymbol{T}$ and $\angle \boldsymbol{U}$ are the same, but $\angle \boldsymbol{S}$ is different, so we have this: $\angle \boldsymbol{S} \neq \angle \boldsymbol{T} = \angle \boldsymbol{U}$.

Showing thus, some angles are equal or different, we often use arcs the ways as above.

By the way, showing 90^o, we often use a small square or rectangle the way below.

Fig. 19



So the angle between the two lines $\boldsymbol{X}$ and $\boldsymbol{Y}$ is 90^o, that is, the two lines are perpendicular to each other. And using a math symbol, ⊥, we can quickly put the idea above this way: $\boldsymbol{X} \perp \boldsymbol{Y}$, which is saying thus, $\boldsymbol{X}$ is perpendicular to $\boldsymbol{Y}$.

Next, working with angles, we often do arithmetic with angles, and can do it mostly the way we do it with numbers.

So for instance, assuming $\angle \boldsymbol{A} = 25^o$ and $\angle \boldsymbol{B} = 45^o$, and $\angle \boldsymbol{C}$ is the sum of the two angles, we can get $\angle \boldsymbol{C}$ this way:

$\angle \boldsymbol{C} = \angle \boldsymbol{A} + \angle \boldsymbol{B} = 25^o + 45^o = 70^o$.

So we get this: $\angle \boldsymbol{C} = 70^o$.

And assuming again, $\angle \boldsymbol{A} = 25^{o}$ and $\angle \boldsymbol{B} = 45^{o}$, we can subtract $\angle \boldsymbol{A}$ from $\angle \boldsymbol{B}$ this way: $\angle \boldsymbol{B} - \angle \boldsymbol{A} = 45^{o} - 25^{o} = 20^{o}$.

Next, assuming an angle $\boldsymbol{D}$ is the sum of twice the angle $\boldsymbol{A}$ above and three times the angle $\boldsymbol{B}$, we can get the angle $\boldsymbol{D}$ doing the arithmetic the way as follows.

$\angle \boldsymbol{D} = 2\angle \boldsymbol{A} + 3\angle \boldsymbol{B} = 2 \text{ x } 25^{o} + 3 \text{ x } 75^{o} = 50^{o} + 225^{o} = 275^{o}$,

so we get this: $\angle \boldsymbol{D} = 275^{o}$.

And next, assuming an angle $\boldsymbol{E}$ is half the angle $\boldsymbol{D}$, we can get the angle $\boldsymbol{E}$ this way: $\angle \boldsymbol{E} = \dfrac{1}{2}\angle \boldsymbol{D} = \dfrac{275^{o}}{2} = 137.5^{o}$.

So like numbers, we can multiply or divide an angle by a number if the number is not a zero, of course, in the case of divisions. Unlike numbers, though, we don't normally do multiplications or divisions by an angle.

Dividing an angle by another angle, though, we can mean something.   What then is it?

It’s a ratio, the ratio between the two angles.

For instance, dividing 90^o by 45^o, we get 2, which is a ratio of 90^o to 45^o, and it’s saying that 90^o is twice 45^o.

And dividing 30^o by 90^o, we get a third, $\frac{1}{3}$, which is a ratio of 30^o to 90^o, and it’s saying that 30^o is a third of 90^o.

Now, assuming next, an angle $\boldsymbol{X}$ is bigger than an angle $\boldsymbol{Y}$, and the angle $\boldsymbol{Y}$ is bigger than or equal to 75^o, we can put the idea this way: $\angle \boldsymbol{X} > \angle \boldsymbol{Y} \geq 75^o$.

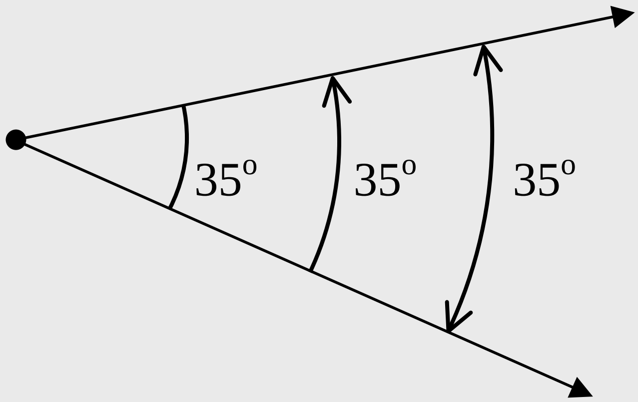
And we call ‘>’ and ‘≥’ relational operators.

The first one is used to show that one is bigger or smaller than the other.

And the second one is used to show that one is greater than or equal to the other, or that one is less than or equal to the other.

Next, when indicating the amount of difference in direction, we can use an angle, too. We can say, for instance, the difference between the directions of the two flights is 35^o.

Fig. 20



So indicating an amount of turning or an amount of difference in direction, we can use an angle.

And indicating it visually, we often use an arc with or without arrows, so we can use an arc one of the three ways shown above. Note that the length of the arc doesn't matter.

The arc length depends on the distance from the vertex, that is, the radius of the arc. The longer the radius, the longer the arc. It doesn't change the angle, the amount of turning or the amount of difference in direction.

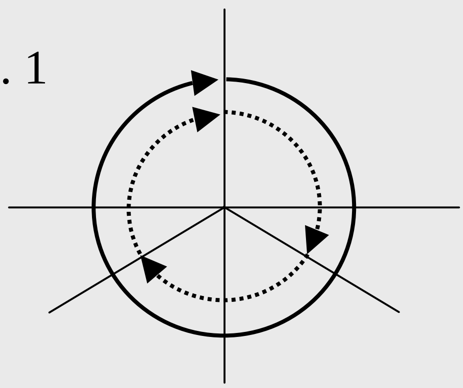
We'll continue with some examples in the next lessons.

9.1.1. Examples 1 in Angles and Lines 1

What do we mean by an angle?

An angle can mean an amount of turning. If making a complete or full turn, it turns 360^{o}. And just saying a turn when doing math, we mean a complete turn or a full turn.

Fig. 1



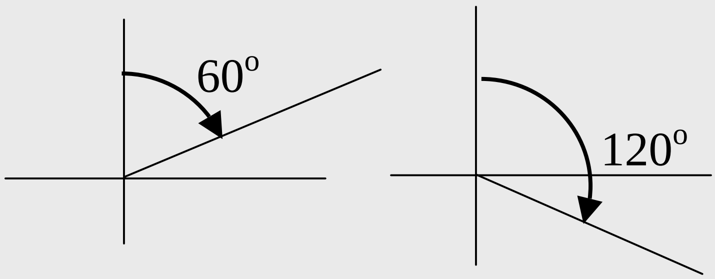
Suppose now, for instance, opening a special door, we turn the knob 240^{o} or more clockwise.

And 120 is a third of 360, and 240 is two thirds of 360, so if we want to open the special door, we need to turn the knob two thirds of a full turn or more.
So making two thirds of a turn or more, we can open the door.   What then is a sixth turn, a sixth of a complete turn?

Making a complete or full turn, it turns 360^o.

So making one sixth of a turn, it turns one sixth of 360^o.

Fig. 2



And one sixth of 360 is 60, so the angle one sixth of a turn makes is 60^o.

That is to say that $\frac{1}{6}$ of 360 is 60, so $\frac{1}{6}$ of 360^o is 60^o.

Making thus, a sixth of a turn, the knob turns 60^o.

What angles then are those turns as follows?

1. one and a half turn

2. two and a quarter turn

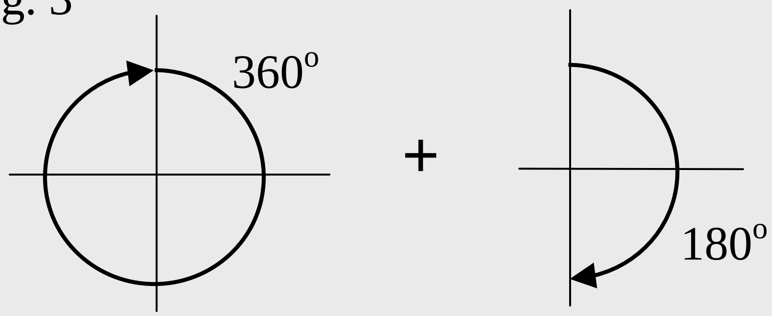
3. five twelfths of a turn

1. one and a half turn

It's more than a full turn, and is the sum of a full turn and half a full turn.

In short, it's a turn and a half.

Fig. 3



A full turn is 360^o, and half a full turn is 180^o.

So one and a half turn is this: $360^o + 180^o = 540^o$.

And we have this: $1+\frac{1}{2}=\frac{3}{2}=3 \text{ x } \frac{1}{2}$.

So we can get the sum of the angles the way below, too:

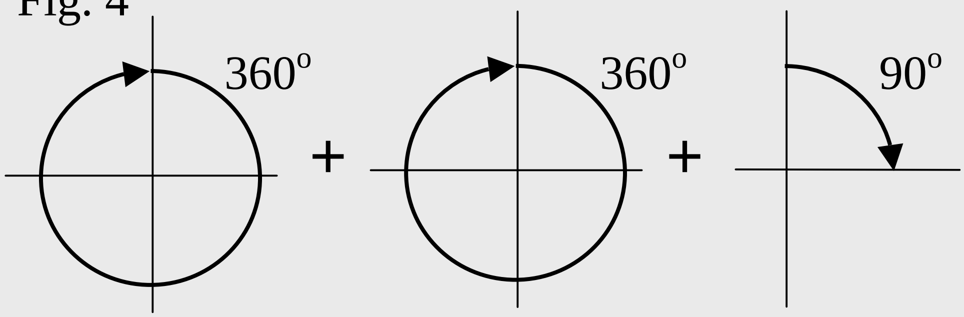
$$\frac{3}{2} \text{ x } 360 = 3 \text{ x } \frac{1}{2} \text{ x } 360 = 3 \text{ x } \frac{360}{2} = 3 \text{ x } 180 = 540$$

2. two and a quarter turn

It's more than two full turns, and is as follows.

the sum of two full turns and a quarter of a full turn

Fig. 4



And a full turn is 360^o and a quarter of a full turn is 90^o.

So two and a quarter turn is this:

$2 \text{ x } 360^o + 90^o = 720^o + 90^o = 810^o$.

And we have this: $2+\frac{1}{4}=\frac{9}{4}=9 \text{ x } \frac{1}{4}$.

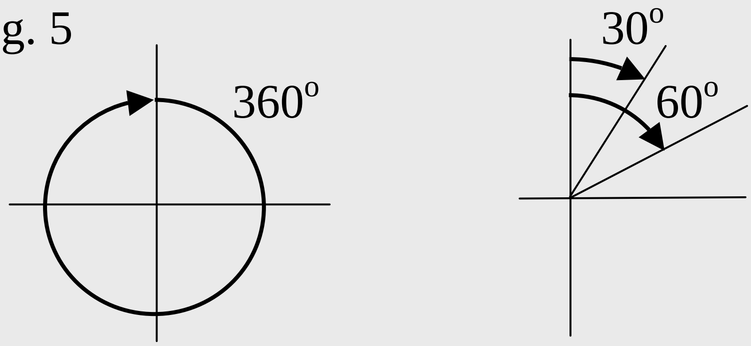
So we can get the sum of the angles the way below, too:

$$\frac{9}{4} \text{ x } 360 = 9 \text{ x } \frac{1}{4} \text{ x } 360 = 9 \text{ x } \frac{360}{4} = 9 \text{ x } 90 = 810$$

3. five twelfths of a turn

It's less than a full turn, and is five twelfths of a full turn.

Fig. 5



And a full turn is 360^{o} and one twelfth of a full turn is 30^{o}, because one twelfth of 360 is 30.

In other words, $\frac{1}{12} \text{ x } 360 = \frac{360}{12} = 30$.

So five twelfths of a turn is this: $5 \text{ x } 30^{o}$ = 150^{o}.

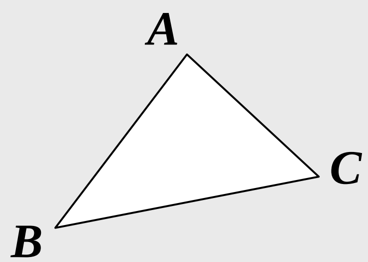
And we can get the angle the way below, too:

$$\frac{5}{12} \text{ x } 360 = 5 \text{ x } \frac{1}{12} \text{ x } 360 = 5 \text{ x } \frac{360}{12} = 5 \text{ x } 30 = 150$$

9.1.2. Examples 2 in Angles and Lines 1

The triangle in the figure below can be called a triangle ***ABC***. And indicating a triangle, we often use a math symbol, and use it this way: **Δ*ABC***, read as a triangle ***ABC***, so we put a small triangle in front of the name of the triangle.

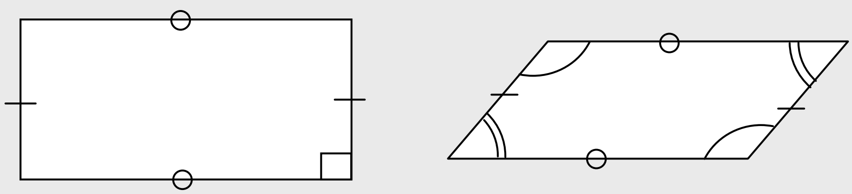
Fig. 1



We can, of course, use a single letter; for instance, we can just name it this way, too: **Δ*Q***, which is read as a triangle ***Q***, and can be thus, another name for **Δ*ABC***.

How then can you name the polygons as follows?

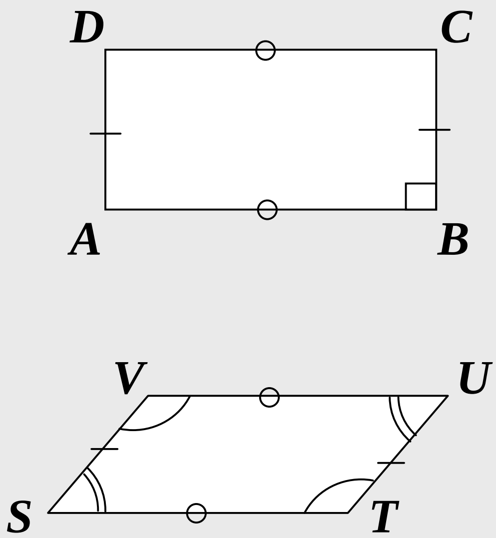
Fig. 2



Garb a piece of paper now, try naming them yourself, and then, compare yours with the suggestions in the next page.

One is a rectangular parallelogram, often just called a rectangle, and the other is just a parallelogram. And using letters, we can label each polygon the way as follows.

Fig. 3



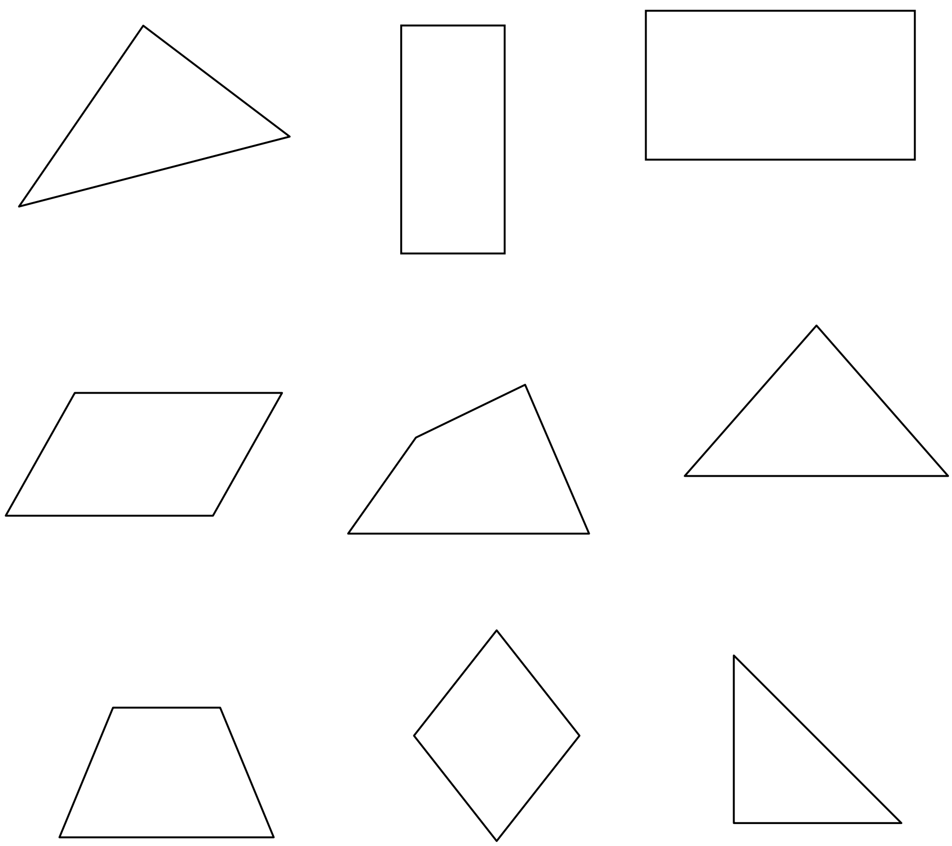
Then, we can call the rectangle a rectangle ***ABCD***, and can call the parallelogram a parallelogram ***STUV***. You don't have to italicize the letters; it's only a personal preference.

Using a small rectangle as a math symbol, we can name the rectangle this way: ▭***ABCD***, or this way: □ ***ABCD***, and thus, either way, it's read as a rectangle ***ABCD***.

And using a small parallelogram, we can name the parallelogram this way: ▱***STUV***.

Now, it's your turn again. Some examples are as follows. So, try now, naming those polygons as triangles, tetragons as squares, rectangles, trapezoids, etc.

You can, of course, try yours, too. Learning things in math, you don't just copy things in your mind. But you do the copy in your writing, too.
Put it in your writing, try it yourself, and see what happens; you never know what to ask, the real question, until you try it.

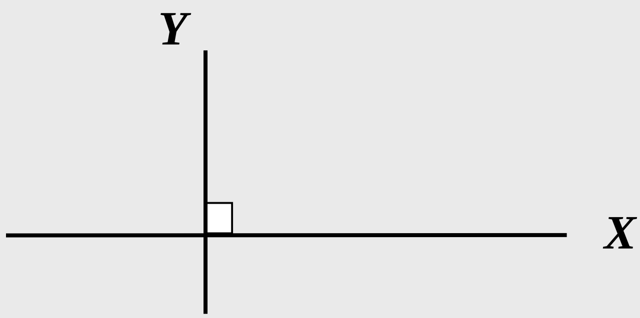


Next, among angles we often use, the one most often used is a right angle, which is 90^o.

What math symbol then, do we use showing that the angle between lines or line segments is a right angle?

Showing that lines make 90^{o} in a figure or graph, we often use a small square or rectangle as a math symbol, and use it the way as follows.

Fig. 4



So the angle between the two lines $\boldsymbol{X}$ and $\boldsymbol{Y}$ is 90^{o}, that is, the two lines are perpendicular to each other.

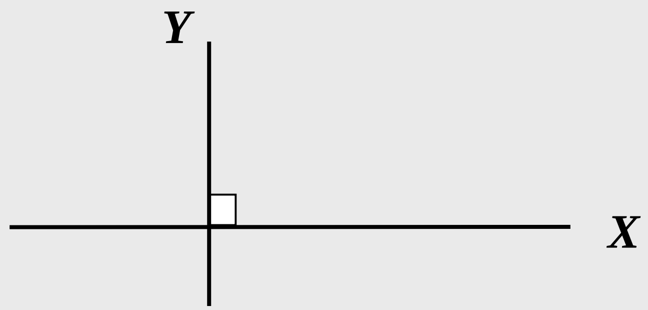
Thus, perpendicular means making the angle of 90^{o}.

And making a math expression, we often use a math sign that indicates, for instance, two lines are perpendicular to each other. So the sign shows the two make a right angle.

What then is it, the math sign?

It can be called a perpendicular sign.
Assuming the angle between the two lines ***X*** and ***Y*** is 90°, that is, the two are perpendicular to each other, we can indicate the fact using a math sign, and the sign is this: ⊥.

Fig. 4



So using it, we can quickly put the idea in the figure above this way: ***X*** ⊥ ***Y***, which is thus, saying that the line ***X*** is perpendicular to the line ***Y***. In short, ***X*** is perpendicular to ***Y***.

So now, make math expressions showing the ideas below.

1. ***A*** and ***B*** are perpendicular to each other.
2. ***C*** is perpendicular to ***D***.
3. The angle between ***U*** and ***V*** is 90°.

Also, put in your figure all the ideas in 1, 2, and 3 above.
And show that a tetragon in your figure is a rectangle. Then, compare yours with the suggestions in the next page.

If A and B are perpendicular, we can put it this way: $A \perp B$.

Fig. 5

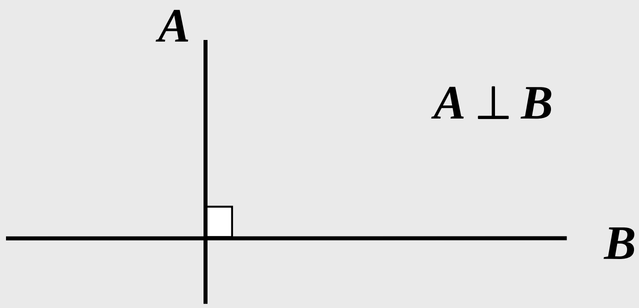


Fig. 6

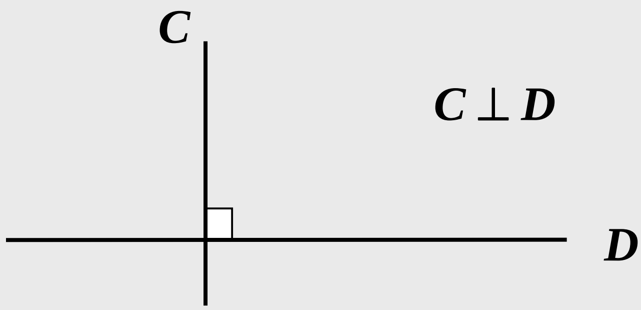
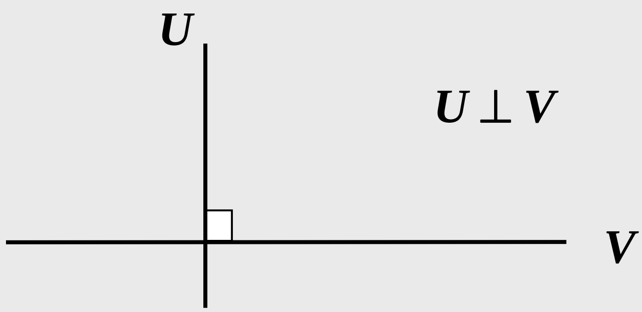


Fig. 7



So we can use a small square or rectangle to show that two lines or two line segments make a right angle, 90^o, that is, the two are perpendicular to each other.

Also, when making a math expression showing that two lines or two segments are perpendicular to each other, we can use a perpendicular sign, which is this: ⊥.

And showing that a tetragon is a rectangle, we can put it in a figure the way as follows.

Fig. 8

A rectangle is a parallelogram where the four angles are equal, that is, the four angles are right angles. Why?

If the four angles are equal, each of the four is a right angle, because the sum of the four is 360^o, a quarter of which is 90^o, which is a right angle.
And for the same reason, It is enough to show that three angles are right angles.

Also, showing that a tetragon is a rectangle, we can put it in a figure any of the ways as follows.

Fig. 9

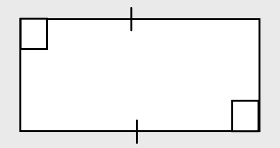


Fig. 10

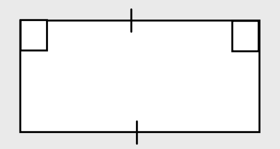
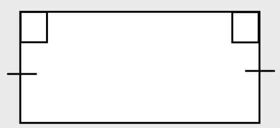


Fig. 11

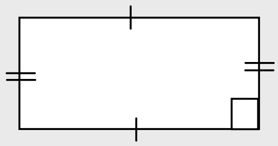
Fig. 12



If in a tetragon, a pair of opposite sides are equal and two angles are right angles, the tetragon is a rectangle.

That’s not it. Showing that a tetragon is a rectangle, we can put it in a figure the way as follows, too.

Fig. 13



So if in a tetragon, each pair of opposite sides are equal and an angle is a right angle, the tetragon is a rectangle.

And we can put the same this way, too: If in a parallelogram, an angle is a right angle, the parallelogram is a rectangle.

It’s because in a parallelogram, each pair of opposite sides are equal, which means, also, each pair of opposite sides are parallel.

What then is a parallelogram?

A parallelogram is a tetragon where each pair of opposite sides are parallel.

In a tetragon, each pair of opposite sides are parallel if and only if each pair of opposite sides are equal.

Oftentimes though, when we quickly show a rectangle in a figure, we just put it this way:

Fig. 14

It's not necessarily a rectangle, of course. It may not be a parallelogram, either. It can be even a trapezium, which is a tetragon with no parallel sides or a tetragon with four different sides.

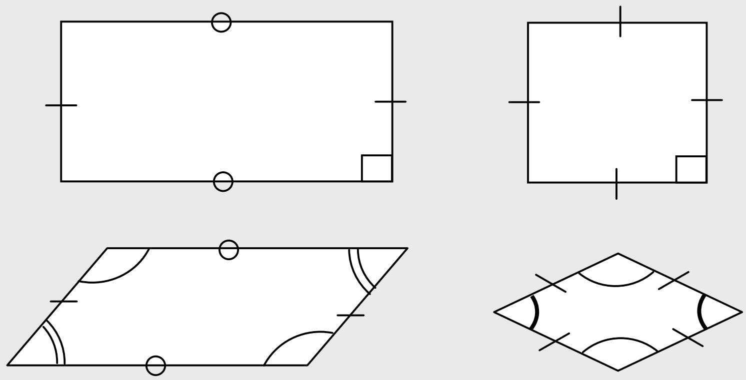
Though it's not correct, we just assume that it's a rectangle.

By the way, a tetragon is often called a quadrangle or a quadrilateral, too. Among tetragons, we have trapeziums and trapezoids, and trapezoids include parallelograms.

And parallelograms include rectangles, squares, and rhombuses. Squares are special, because rectangles include squares, and rhombuses include squares, too. So a square can be called a rectangular rhombus, and can be called an equilateral and equiangular tetragon, too.

In a square, therefore, all the components are equal; all the four sides are equal, and all the four angles are equal, too.

Fig. 9



9.1.3. Examples 3 in Angles and Lines 1

Working with angles, we often do arithmetic with angles, and can do it mostly the way we do it with numbers.

So for instance, assuming $\angle \boldsymbol{A} = 15^{o}$ and $\angle \boldsymbol{B} = 75^{o}$, and $\angle \boldsymbol{C}$ is the sum of the two angles, we can get $\angle \boldsymbol{C}$ this way:

$\angle \boldsymbol{C} = \angle \boldsymbol{A} + \angle \boldsymbol{B} = 15^{o} + 75^{o} = 90^{o}$.

So we get this: $\angle \boldsymbol{C} = 90^{o}$.

What then, is $\angle \boldsymbol{D}$ if $\angle \boldsymbol{D}$ is the sum of two angles, and the two are these: $\angle \boldsymbol{X} = 65^{o}$ and $\angle \boldsymbol{Y} = 85^{o}$?

Assuming $\angle \boldsymbol{X} = 65^{o}$ and $\angle \boldsymbol{Y} = 85^{o}$, and $\angle \boldsymbol{D}$ is the sum of the two angles, we can get $\angle \boldsymbol{D}$ this way:

$\angle \boldsymbol{D} = \angle \boldsymbol{X} + \angle \boldsymbol{Y} = 65^{o} + 85^{o} = 150^{o}$.

In short, $\angle \boldsymbol{D} = 65^{o} + 85^{o} = 150^{o}$.

And assuming again, $\angle \boldsymbol{X} = 65^{o}$ and $\angle \boldsymbol{Y} = 85^{o}$, we can subtract $\angle \boldsymbol{X}$ from $\angle \boldsymbol{Y}$ this way: $\angle \boldsymbol{Y} - \angle \boldsymbol{X} = 85^{o} - 65^{o} = 20^{o}$.

What then, is $\angle \boldsymbol{E}$ if $\angle \boldsymbol{E} = \angle \boldsymbol{P} - \angle \boldsymbol{Q}$, where $\angle \boldsymbol{P} = 95^{o}$ and $\angle \boldsymbol{Q} = 35^{o}$?

And assuming $\angle \boldsymbol{P} = 95^o$ and $\angle \boldsymbol{Q} = 35^o$, we can subtract $\angle \boldsymbol{Q}$ from $\angle \boldsymbol{P}$ this way: $\angle \boldsymbol{P} - \angle \boldsymbol{Q} = 95^o - 35^o = 60^o$.

So we get this: $\angle \boldsymbol{E} = \angle \boldsymbol{P} - \angle \boldsymbol{Q} = 95^o - 35^o = 60^o$.

In short, $\angle \boldsymbol{E} = 95^o - 35^o = 60^o$.

Next, assuming an angle $\boldsymbol{R}$ is the sum of twice the angle $\boldsymbol{P}$ above and three times the angle $\boldsymbol{Q}$, we can get the angle $\boldsymbol{R}$ doing the arithmetic the way as follows.

$\angle \boldsymbol{R} = 2\angle \boldsymbol{P} + 3\angle \boldsymbol{Q} = 2 \text{ x } 95^o + 3 \text{ x } 35^o = 190^o + 105^o =$ 295^o, so we get this: $\angle \boldsymbol{R} = 295^o$.

Now, it's your turn. So, find an angle $\boldsymbol{G}$ which is the sum of three times the angle $\boldsymbol{P}$ above and five times the angle $\boldsymbol{Q}$.

We have these: $\angle P = 95^o$ and $\angle Q = 35^o$.

So, assuming the angle R is the sum of three times the angle P above and five times the angle Q, we can get the angle R doing the arithmetic the way as follows.

$\angle R = 3\angle P + 5\angle Q = 3 \text{ x } 95^o + 5 \text{ x } 35^o = 285^o + 175^o =$

460^o, so we get this: $\angle R = 460^o$.

In short, $\angle R = 3 \text{ x } 95^o + 5 \text{ x } 35^o = 285^o + 175^o = 460^o$.

And next, assuming an angle H is half the angle R, we can get the angle H this way: $\angle H = \frac{1}{2}\angle R = \frac{460^o}{2} = 230^o$.

What then is an angle T if the angle T is a third of the angle R above?

Since $\angle \boldsymbol{R}$ = 460°, and $\angle \boldsymbol{T}$ is a third of $\angle \boldsymbol{R}$, we can get $\angle \boldsymbol{T}$ this way: $\angle \boldsymbol{T} = \frac{1}{3}\angle \boldsymbol{R} = \frac{460^{\circ}}{3}$.

So $\angle \boldsymbol{T} = \frac{460^{\circ}}{3}$.

Thus, like numbers, we can multiply or divide an angle by a number if the number is not a zero, of course, in the case of divisions. Unlike numbers, though, we don't multiply or divide by an angle.

Dividing an angle by another angle, though, we can mean something. It means a ratio, the ratio between the two angles. So it says how big or small one angle is compared to the other.

For instance, dividing 90° by 45°, we get 2, which means, 90° is twice 45°.

And dividing 30° by 90°, we get a third, which is this: $\frac{1}{3}$, which means, 30° is a third of 90°.

Next, moving on to the uses of relational operators, we can make an example as follows.

Assuming an angle ***X*** is bigger than an angle ***Y***, and the angle ***Y*** is bigger than or equal to 75^{o}, we can put the idea this way: $\angle \boldsymbol{X} > \angle \boldsymbol{Y} \geq 75^{o}$.

And we call '>' and '≥' relational operators.

Suppose now, an angle ***V*** is less than an angle ***W***, and the angle ***W*** is less than or equal to 95^{o}.

How then can we put the relations among the three angles, the angle ***V***, the angle ***W***, and 95^{o} using the relational operators used in the example above?

Assuming an angle V is less than an angle W, and the angle W is less than or equal to 95^o, we can put the idea this way: $\angle V < \angle W \leq 95^o$. And that's about it in the basics on arithmetic with angles, which can look bit different from plane numbers.

So you may want to do bit more examples, which are as follows. Try mental math. If it's too hard, use pen and paper. This practice is only for your familiarity with angle arithmetic.

1. $15^o + 24^o =$

2. $23^o + 35^o =$

3. $35^o + 26^o =$

4. $27^o + 45^o =$

5. $79^o - 23^o =$

6. $88^o - 35^o =$

7. $928^o - 216^o =$

Do not use a calculator. No need to hurry. Use paper. Doing calculations on paper, use a pen, not a pencil. If a mistake is made, just cross that out, put the right one next to or below the mistake crossed out, and continue the calculation.

Do not interrupt your calculation erasing mistakes. Just strike through a mistake with one single line like this: ~~394~~

1. $15^o + 24^o = 39^o$
2. $23^o + 35^o = 58^o$
3. $35^o + 26^o = 61^o$
4. $27^o + 45^o = 72^o$
5. $79^o – 23^o = 56^o$
6. $88^o – 35^o = 53^o$
7. $928^o – 216^o = 712^o$

No need to do all those examples followed by these. More are provided, simply because some people want to do more, since it takes more for them to get used to. Some people do, and some people don’t. By the way, fast calculation doesn’t necessarily mean math skill of high caliber.

1. $25^o - 13^o =$

2. $35^o - 25^o =$

3. $35^o - 26^o =$

4. $2 \text{ x } 15^o + 3 \text{ x } 24^o =$

5. $2 \text{ x } 23^o + 3 \text{ x } 35^o =$

6. $3 \text{ x } 35^o + 2 \text{ x } 26^o =$

7. $3 \text{ x } 25^o - 2 \text{ x } 14^o =$

8. $3 \text{ x } 35^o - 2 \text{ x } 35^o =$

9. $3 \text{ x } 35^o - 2 \text{ x } 26^o =$

10. $\frac{70^o}{2} =$

11. $\frac{720^o}{3} =$

1. $25^o - 13^o = 12^o$

2. $35^o - 25^o = 10^o$

3. $35^o - 26^o = 9^o$

4. $2 \times 15^o + 3 \times 24^o = 30^o + 72^o = 102^o$

5. $2 \times 23^o + 3 \times 35^o = 46^o + 105^o = 151^o$

6. $3 \times 35^o + 2 \times 26^o = 105^o + 52^o = 157^o$

7. $3 \times 25^o - 2 \times 14^o = 75^o - 28^o = 47^o$

8. $3 \times 35^o - 2 \times 35^o = 105^o - 70^o = 35^o$

9. $3 \times 35^o - 2 \times 26^o = 105^o - 52^o = 53^o$

10. $\frac{70^o}{2} = 35^o$

11. $\frac{720^o}{3} = 240^o$

1. $47^o - 25^o =$

2. $73^o + 29^o =$

3. $75^o + 88^o =$

4. $2 \times 37^o + 4 \times 25^o =$

5. $7 \times 33^o + 2 \times 19^o =$

6. $7 \times 45^o + 8 \times 78^o =$

7. $4 \times 25^o - 2 \times 37^o =$

8. $7 \times 33^o - 2 \times 19^o =$

9. $8 \times 77^o - 7 \times 45^o =$

10. $\frac{90^o}{2} =$

11. $\frac{170^o}{5} =$

1. $47^o - 25^o = 22^o$

2. $73^o + 29^o = 102^o$

3. $75^o + 88^o = 163^o$

4. $2 \times 37^o + 4 \times 25^o = 74^o + 100^o = 174^o$

5. $7 \times 33^o + 2 \times 19^o = 231^o + 38^o = 269^o$

6. $7 \times 45^o + 8 \times 78^o = 315^o + 624^o = 939^o$

7. $4 \times 25^o - 2 \times 37^o = 100^o - 74^o = 26^o$

8. $7 \times 33^o - 2 \times 19^o = 231^o - 38^o = 193^o$

9. $8 \times 77^o - 7 \times 45^o = 616^o - 315^o = 301^o$

10. $\frac{90^o}{2} = 45^o$

11. $\frac{170^o}{5} = 34^o$

1. $728^o + 997^o =$

2. $328^o – 213^o =$

3. $457^o – 249^o =$

4. $9 \text{ x } 98^o – 7 \text{ x } 27^o =$

5. $4 \text{ x } 35^o – 3 \text{ x } 24^o =$

6. $3 \text{ x } 73^o – 2 \text{ x } 95^o =$

7. $7 \text{ x } 28^o + 9 \text{ x } 97^o =$

8. $3 \text{ x } 25^o + 4 \text{ x } 34^o =$

9. $2 \text{ x } 93^o + 3 \text{ x } 75^o =$

10. $\frac{71^o}{2} =$

11. $\frac{276^o}{3} =$

1. $728^o + 997^o = 1725^o$

2. $328^o - 213^o = 115^o$

3. $457^o - 249^o = 208^o$

4. $9 \times 98^o - 7 \times 27^o = 882^o - 189^o = 693^o$

5. $4 \times 35^o - 3 \times 24^o = 140^o - 72^o = 68^o$

6. $3 \times 73^o - 2 \times 95^o = 219^o - 190^o = 29^o$

7. $7 \times 28^o + 9 \times 97^o = 196^o + 873^o = 1069^o$

8. $3 \times 25^o + 4 \times 34^o = 75^o + 136^o = 211^o$

9. $2 \times 93^o + 3 \times 75^o = 186^o + 365^o = 551^o$

10. $\frac{71^o}{2} = 35.5^o$

11. $\frac{276^o}{3} = 92^o$

1. $584^o - 439^o =$

2. $728^o - 249^o =$

3. $368^o - 289^o =$

4. $3 \text{ x } 96^o - 2 \text{ x } 96^o =$

5. $4 \text{ x } 55^o - 2 \text{ x } 77^o =$

6. $7 \text{ x } 93^o - 3 \text{ x } 99^o =$

7. $3 \text{ x } 79^o + 2 \text{ x } 96^o =$

8. $2 \text{ x } 77^o + 4 \text{ x } 55^o =$

9. $7 \text{ x } 93^o + 3 \text{ x } 99^o =$

10. $\frac{703^o}{2} =$

11. $\frac{70^o}{2} + \frac{30^o}{2} =$

1. $584^o - 439^o = 45^o$

2. $728^o - 249^o = 479^o$

3. $368^o - 289^o = 79^o$

4. $3 \times 96^o - 2 \times 96^o = 1 \times 96 = 96^o$

5. $4 \times 55^o - 2 \times 77^o = 20 \times 11^o - 14 \times 11^o = 66^o$

6. $7 \times 93^o - 3 \times 99^o = 651^o - 297^o = 354^o$

7. $3 \times 79^o + 2 \times 96^o = 237^o + 192^o = 429^o$

8. $2 \times 77^o + 4 \times 55^o = 154^o + 220^o = 374^o$

9. $7 \times 93^o + 3 \times 99^o = 651^o + 297^o = 948^o$

10. $\frac{703^o}{2} = 351.5^o$

11. $\frac{70^o}{2} + \frac{30^o}{2} = 35^o + 15^o = 50^o$

1. $\frac{720^{o}}{2}+\frac{25^{o}}{2}=$

2. $\frac{90^{o}}{2}+\frac{96^{o}}{3}=$

3. $\frac{70^{o}}{2}-\frac{30^{o}}{4}=$

4. $\frac{720^{o}}{3}+\frac{25^{o}}{2}=$

5. $\frac{90^{o}}{2}-\frac{45^{o}}{4}=$

6. $\frac{70^{o}}{2}+\frac{37^{o}}{4}=$

1. $\frac{720^o}{2}+\frac{25^o}{2}$ = $360^o + 12.5^o = 372.5^o$

$= \frac{720^o + 25^o}{2} = \frac{745^o}{2}$

2. $\frac{90^o}{2}+\frac{96^o}{3}$ = $45^o + 32^o = 77^o$

3. $\frac{70^o}{2}-\frac{30^o}{4}$ = $35^o - 7.5^o = 27.5^o$

$= \frac{140^o - 30^o}{4} = \frac{110^o}{4}$

4. $\frac{720^o}{3}+\frac{25^o}{2}$ = $240^o + 12.5^o = 252.5^o$

$= \frac{1440^o + 75^o}{6} = \frac{1515^o}{6}$

5. $\frac{90^o}{2}-\frac{45^o}{4}$ = $45^o - 11.25^o = 33.75^o$

$= \frac{180^o - 45^o}{4} = \frac{135^o}{4}$

6. $\frac{70^o}{2}+\frac{37^o}{4}$ = $35^o + 9.25^o = 44.25^o$

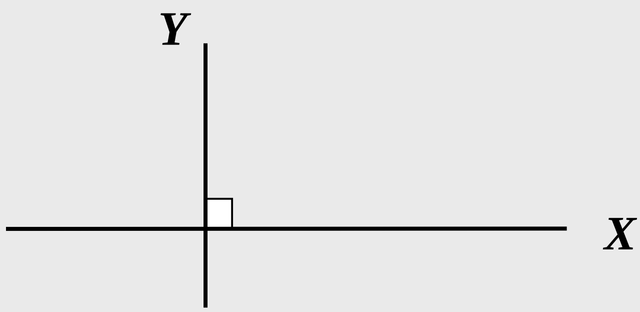
$= \frac{140^o + 37^o}{4} = \frac{177^o}{4}$

9.1.4. Examples 4 in Angles and Lines 1

So, what do we mean by an angle?

An angle can mean an amount of turning. Also, it can mean an amount difference in direction.

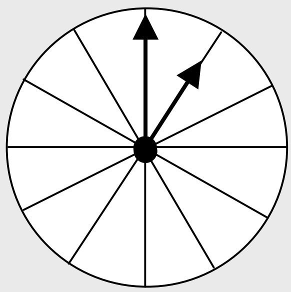
Fig. 1



The difference in direction between the two lines X and Y in the figure above is 90^{o}. The directional difference is 90^{o}.

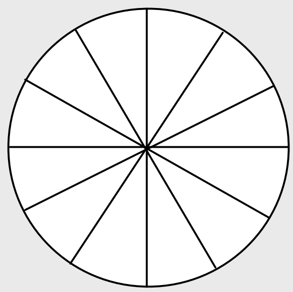
So, for another instance, if a clock has twelve marks along its edge, and every mark indicates every hour on the hour, we can say that the directional difference between the two hands of the clock is 30^{o} when it's one o'clock on the hour.

Fig. 2



It's because the angle between the two line segments connecting the center of the clock and the two consecutive marks respectively is 30^{o}, since there are 12 marks equally spaced along the clock's edge, and a full turn is 360^{o}.

Fig. 2



So the directional difference between the two hands is 60^{o} at two o'clock on the hour. And if it's three o'clock on the hour, the difference is 90^{o}.

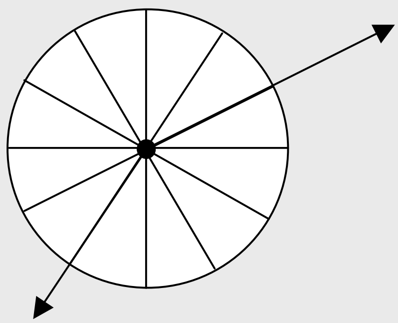
Assuming thus, an airplane is flying due north, and another is flying due west, we can say that the directional difference between the two flights is 90^{o}.

Suppose now, one airplane is flying in the direction of two o'clock and another is flying in the direction of seven o'clock.

What then is the directional difference between the two flights?

It is 150^o, because the two flight paths are away from each other in the amount of 5 marks indicating hours in a clock.

Fig. 3

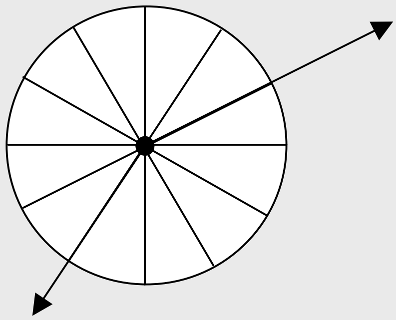


In a clock, the two hands pointing at two consecutive marks are 30^o away from each other, so the two flight paths are 150^o away from each other in terms of direction.

What then, do we mean by such a difference in terms of direction, that is, a directional difference?

It’s an angle. We just call it an angle. So in the case of the two flight paths, the angle between the two is 150^o.

Fig. 3

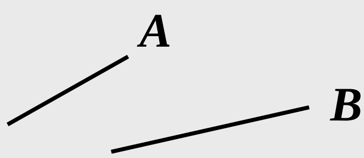


It’s because, as stated earlier, the angle between the two line segments connecting the center of a clock and the two consecutive marks respectively is 30^o, since there are 12 marks equally spaced along the clock’s edge, and a full turn is 360^o.

So the directional difference between two lines is called the angle between the two lines. In short, an angle is a directional difference, which is a difference in direction.

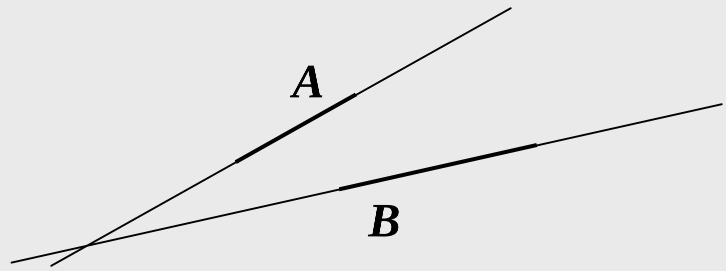
What then is the angle between the two shown below?

Fig. 4



Taking the angle between the two segments as A and B not shown to be meeting each other, we can put two lines on top of the two or extend the two segments the way below.

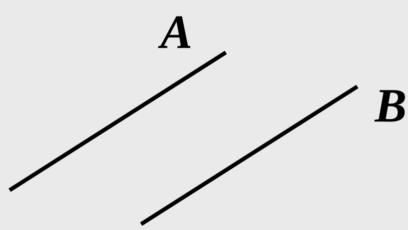
Fig. 5



And then, we can take the angle between the two lines or the two extensions.

Suppose now, the two lines ***A*** and ***B*** in the figure below are parallel to each other.

Fig. 6

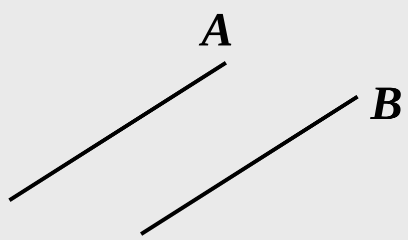


Then, we say that there is no difference in direction between the two lines. That is, the two lines have the same directions.

What then, is the angle between the two lines?

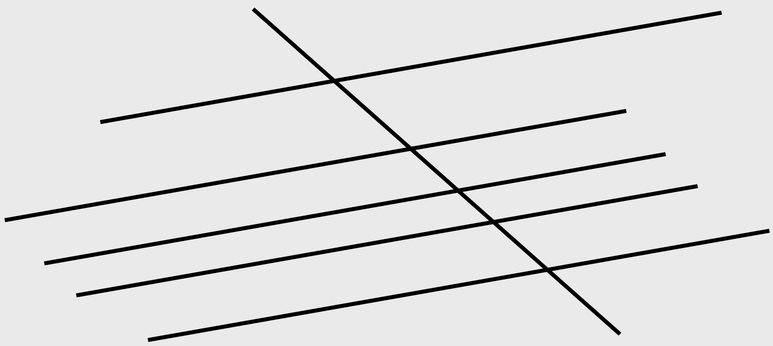
The angle is 0^o, because between the two lines, there is no difference in direction.

Fig. 6



So if some lines are parallel to each other, the angle between any two of all the lines is 0^o, because there is no difference in direction among all the lines parallel.

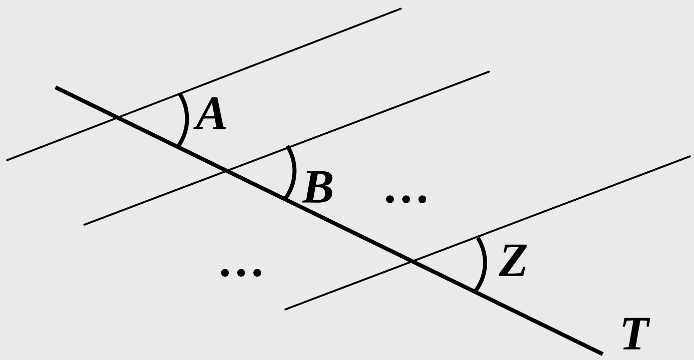
Fig. 7



A line therefore, crossing parallel lines makes the ***same angle*** with each and every one of all those parallel lines.

We often use the fact solving problems.

Fig. 8



Assuming all the lines crossed by ***T*** are parallel, we get this:

$\angle \boldsymbol{A} = \angle \boldsymbol{B} = \ldots = \angle \boldsymbol{Z}$. The angles are all the same.

Why the same, though?

Again, it's because the angle between any two parallel lines is 0^{o}, so there is ***the same difference in direction*** between the line ***T*** and each of all the parallel lines. Thus, ***the angle*** between the line ***T*** and each parallel line is ***the same***.
And we'll cover the fact above in another lesson again.

It's very important to keep the fact above, because it's often used, very often; we often use it when learning things in math as well as solving problems.

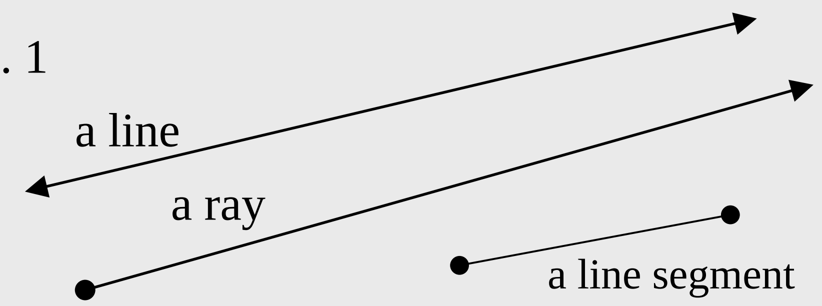
9.2. Angles and Lines 2

Next, doing geometry, we often use a math object called a line, and often use it with angles, together with its parts. And, of course, we often use lines doing other math, too. So as well as angles, you may want to know about them very well.

Not just knowing them only, but getting the concept, too, we can do many problems. So you want to learn them so that you get the concept of a line, along with its parts, together with angles that lines and their parts can make.

And among the parts, we have two kinds: rays and line segments. Sometimes, we call a ray a half line, and oftentimes, just saying a segment, we mean a line segment.

Fig. 1



So doing the math called geometry, we frequently use lines, rays, and segments. And you are going to learn them now so that you can see what they are and how they work so that you can do many problems, and do them fast enough.

Why learning lines, though? Lines are simple, aren't they? We can simply draw some lines like these:

Fig. 2

So a line is simple and easy, isn't it? Thus, the math with it should be simple and easy, too, shouldn't it? So why bother learning it?

It has, of course, nothing complicated, and is one of the simplest in math. Depending on how we understand it, though, math can be easier or can get harder. Understanding matters.

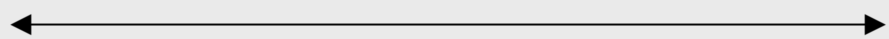
It's not just a line. It's a line in math.
So what do we mean by a line in math?

In math, we don't take as lines the ones in Fig. 2 below; the ones in the figure are not actual lines in math.

Fig. 2

Saying lines doing math, we don't just mean things like strings or ropes. We mean mathematical lines.

Fig. 3



What line then, is a mathematical line?

Saying a line in math, we mean a line as a concept. And it's not a material object. It's an idea, a math concept.

So an actual line in math? We can see it only if we understand it. Understanding it, we can see it, and can actually see it in our thoughts only. Understanding matters.

And that’s what you gonna get here: Understanding.

So a line in math is not quite the same as a line in daily life as a fishing line or a sideline in a basketball court.

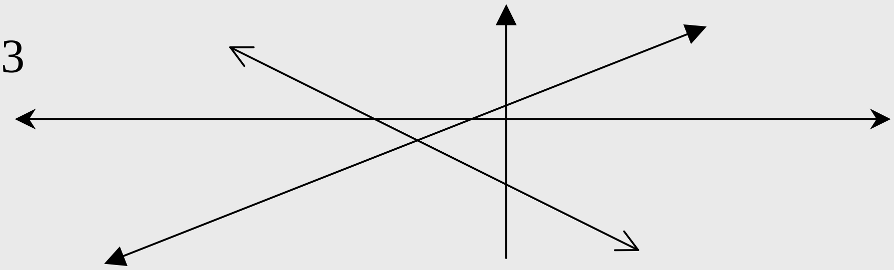


Fig. 3

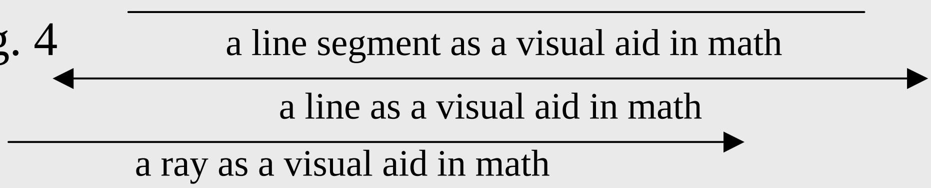
lines as visual aids in math

We use lines in many kinds in real world. In real world, we can see, for instance, lines that make lanes on pavements, lines that make tennis courts, and lines that make ruled sheets in notebooks. They are not lines in math, though.

Saying a line in real world, we usually mean a long mark or stroke which is long in proportion to its thickness or width like a streak or string. And depending on its nature or the situation where it’s used, it can have many other meanings, too, like a line where we wait for our turns or a timeline where things happen in a chronological order.

Saying a line in daily life in real world, we mean something long and thin like a tight guitar string or a long thin streak of ink that looks like one of those shown below.

Fig. 4



In daily life, strings are called lines, but not actual lines in math. Though looking quite straight and even, they can't be perfect. In math, all lines are ***perfectly straight and even***.

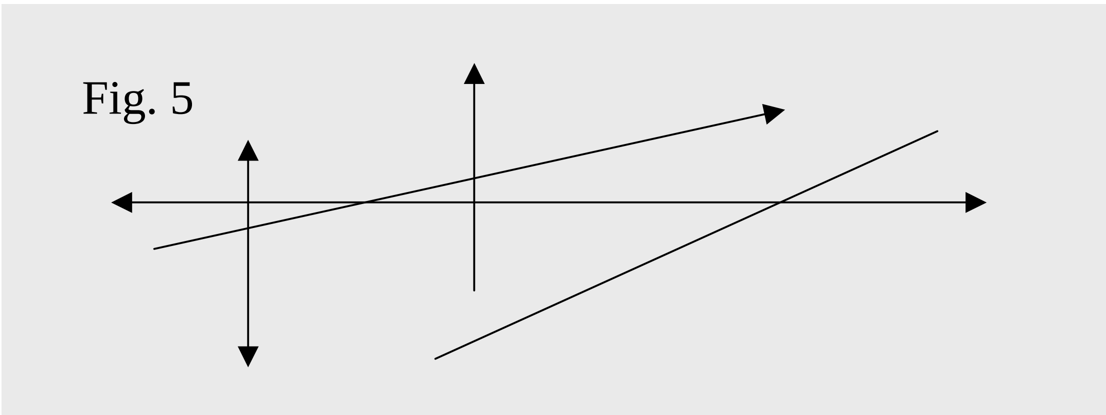
Unlike a line in real world as a sideline in a basketball court or an extension cable, a ***line in math*** is not a material object. And it's not an image on a monitor screen, either. We can't actually see it in our eyes. It's not visible.

We can only think of it, can think about it, and can use it if we can do this: Understanding it getting the concept.

So if we understand it getting the concept, we can use it solving many problems.

Thus, if we want to see it, we need to understand it getting the concept, which is the only way we can see it right. And the same is true of every thing else in math.

So it’s not a material object. Though sounds weird, it has no thickness. Showing a line in math, though, we usually use a long thin streak of ink as one of those shown below.



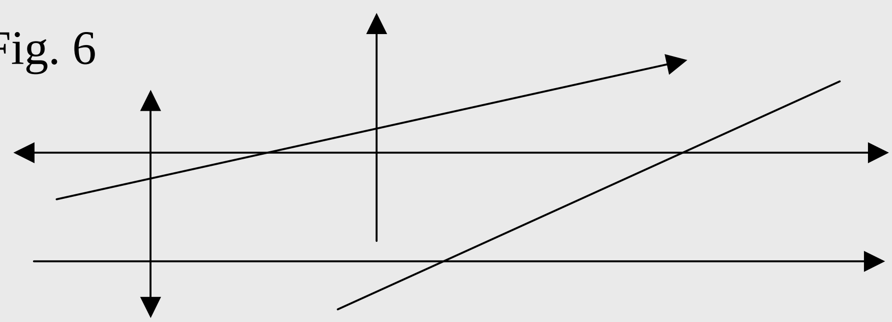
All those in the figure above are no more than ***visual aids*** that can help imagine a line in math. Thus, they are only analogous to it, a mathematical line, so they have some characteristics of it, and are not actual lines in math.

Since a line in math has no thickness, of all lines in math, none is thicker or thinner than any other. In math, therefore, all lines themselves are the same, and can ***actually*** exist in our thoughts only. Getting the concept, we understand it.

How then, can we get the understanding?
How can we get its concept?

If understanding its nature, characteristics or properties, and how it gets made, we understand it getting the concept.
We get conceptual understanding.

Fig. 6



And getting conceptual understanding, we can see what it's about and how it works, so that we can make use of it solving problems. Conceptual understanding matters.

And we have another reason that *a line in math* is other than *a line in daily life*.

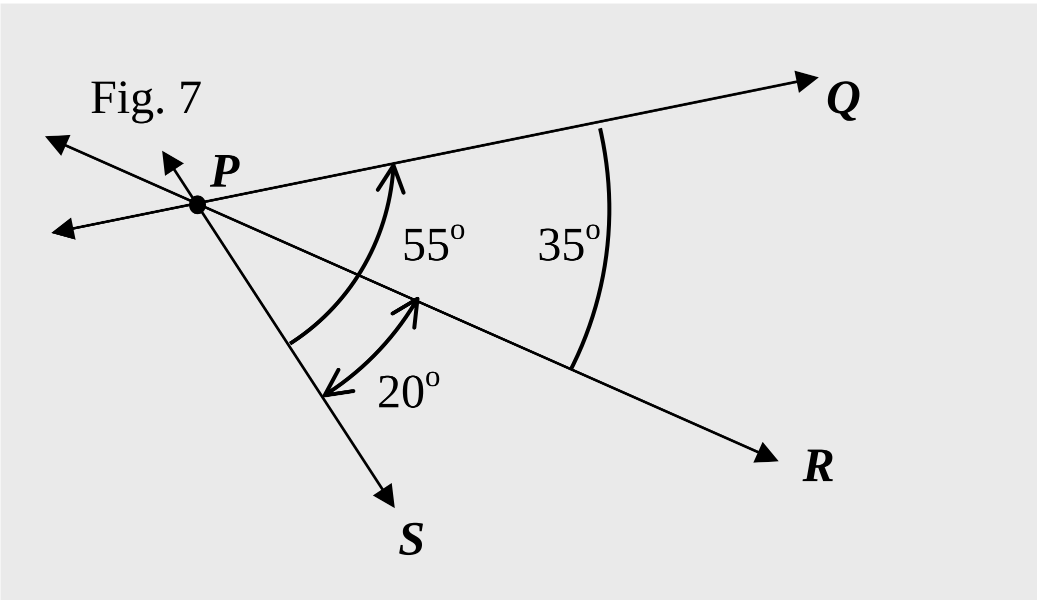
It's the length. No matter how long it may be, if it's a line in material world, its length is finite. If it's a line in math, though, its length is infinite.

So in sum, there are at least two differences between a line in real world and a line in math.

One is this: In real world, no line can be infinite in length. In the world of math, every line is of length infinite.

And the other difference is this: A line in real world has a thickness. In math, though, every line has no thickness. Why then no thickness?

Suppose, for instance, we have two angles as follows.

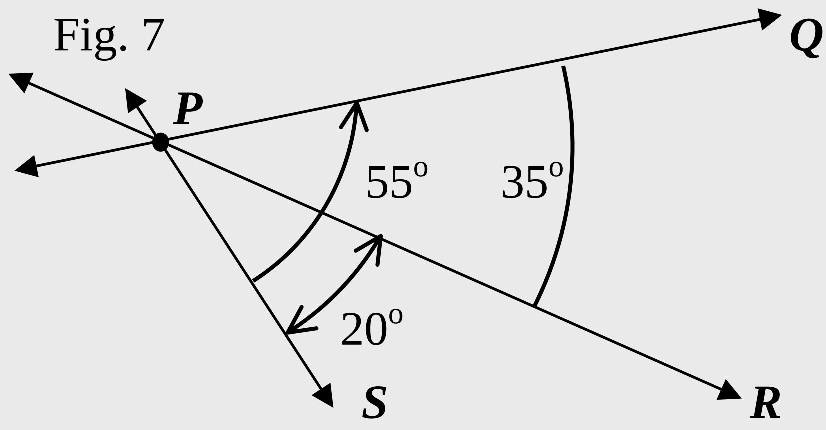


What if now, the line ***PR*** has a non zero thickness?

Is it true then that we have this: $\angle \boldsymbol{QPS} = 55^{\circ}$?

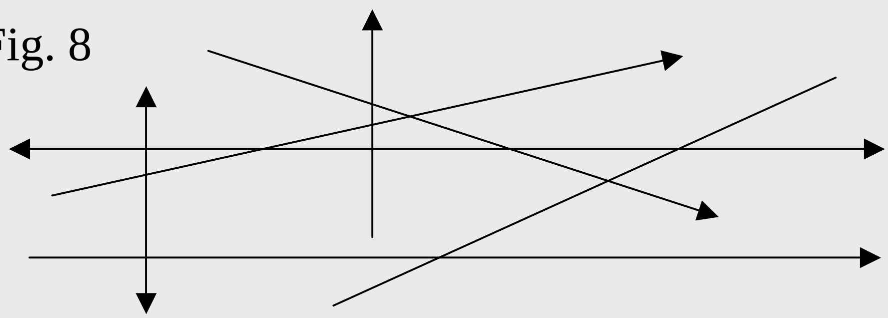
If the line ***PR*** has a thickness, $\angle$***QPS*** is less than 55^o.

Fig. 7



Though sounds unreasonable or strange, a line in math is defined to have no thickness, that is, by definition, the thickness of a line is 0.

Fig. 8



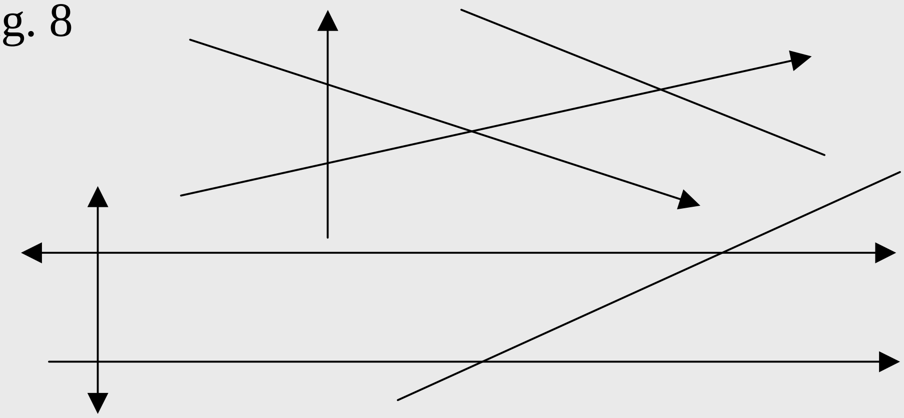
We use however, those streaks of ink as lines in math. Doing math, we use those streaks as if they were actual lines in math. Why then do we use them as lines in math?

It’s not a good idea to keep lines in thoughts only when doing math, simply because it’s too hard to do so. Also, it’s very difficult to communicate or share ideas on lines in math just explaining them in words only.

So we use visual aids. We use those streaks of ink for visual aids. Thus, we just use the streaks as lines in math as if they were actual lines in math.

When doing math, therefore, showing a line in math, we just use a long thin streak of ink as one of those shown below.

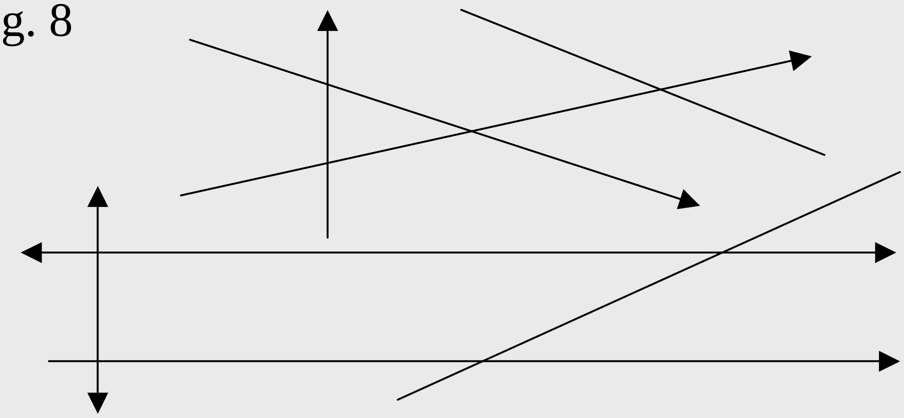
Fig. 8



Why arrows, though?   Why does a line have an arrowhead or arrowheads?

Mentioned earlier that the length of a line in math is infinite. So in some of those in the figure below, both ends are arrowheads indicating that they lengthen forever both ways.

Fig. 8

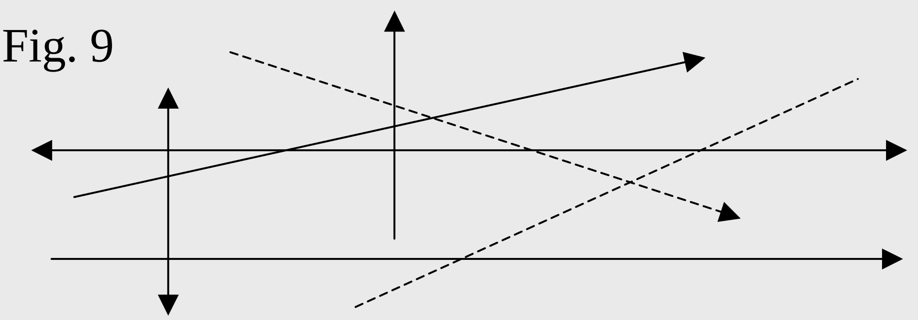


Oftentimes though, they are omitted, or just one is put at one end, which is usually on the right hand side if it’s not vertical. If it’s vertical, the arrowhead is usually at the top.

So when doing math, for visual aids, we just use those streaks of ink as the ones in the figure as if they were actual lines in math. Just don’t get fooled by the look.

They are not limited in length, and have no thickness. A line in math is an idea, a concept. So if understanding it getting the concept, we get to see what it is. What then can we do with the understanding?

Fig. 9



We can see what a line is about and how it works, so we can make use of it solving problems. How then can we understand it? How do we get the concept?

As stated earlier, if understanding its nature, characteristics or properties, and how it gets made, we get the conceptual understanding, the concept. And it begins with the definition. The definition of the math object called a line.

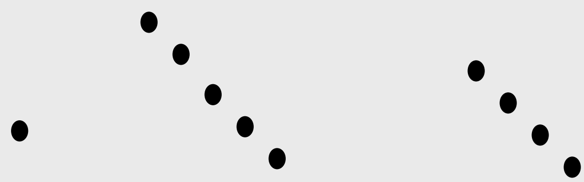
///In fact, math begins with definitions, so we may want to begin with the definition for lines. So now the question is, what is the definition? What do we mean by ***a line in math***?

It's a collection of points, and if in a line, no matter what two points we may connect, we get a line segment that has the ***same direction***. What is a point, though, in math?

Fig. 10

Though we often use a dot as a point when doing math, a dot is not an actual point in math; it's only a visual aide.

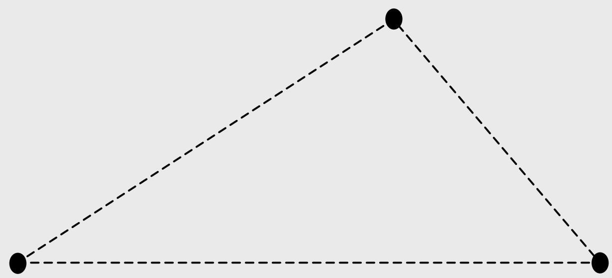
Fig. 11



A mathematical point, that is, ***a point in math*** isn't actually a dot, but is ***a concept*** that indicates ***a position only***. It has no size as width or thickness. Neither does ***a line in math***. That is, a mathematical line has no width or thickness, either. It's because a line is a collection of points continuously put together side by side.

Doing math, though, we often use a dot as a point, which is however, just a visual aid that can help us think of and handle ***the concept called a point*** with ease.

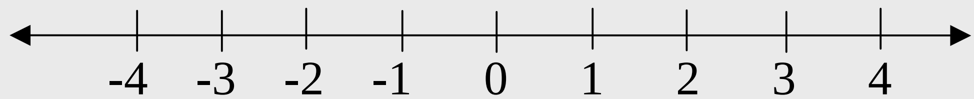
Fig. 12



A point in math is not a material object, and indicates a position only, and will tell you the exact location.

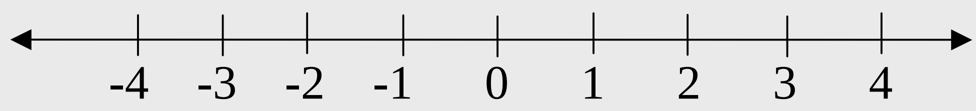
For instance, doing math, we often use a number line, which is a line that shows a sequence of numbers as shown below.

Fig. 13



So a number line is a line where we place numbers at marks called graduations, short thin vertical bars. And we can use those graduations as points.

Fig. 13



For example, we can say that the distance from the point where 2 is put to the point where 0 is put is 2.

However, the short vertical bars above are not actual points in math. Those vertical bars, the graduations are just visual aids that can help us think of mathematical points with ease.

A point in math is an idea, a concept.

And the same is true of all other math objects, too, as a line, circle, etc. It’s a math idea. So we need to get the concept, which means, understanding the nature and how it’s made.

So if getting the concept, we understand it and can use it.

We’ll continue this in the next lesson.

9.3. Angles and Lines 3

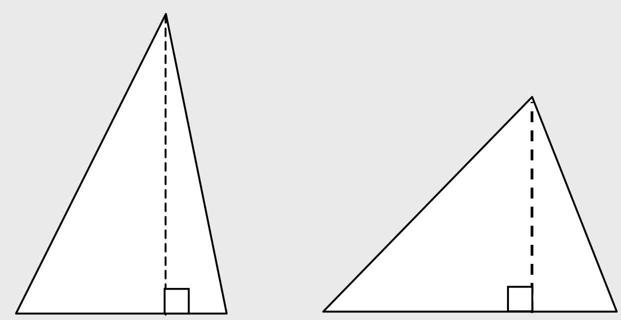
A line in math is an idea, a concept, and so is a point. It's not a material thing, but a concept. So understanding it getting the concept; we can see it, and can use it.

We can understand it doing examples, right examples. Doing those examples, we get the message, the idea, and can get the hang of it. We get the fluency with the concept.

And that's what we do learning things in the world of math, which is an idea world, other than the real world. What then in the real world, do we do with it, what we learn in math?

In the real world, we apply the ideas. Applying, for instance, those math ideas as triangles, along with the formulas, rules, or laws related, we can solve problems.

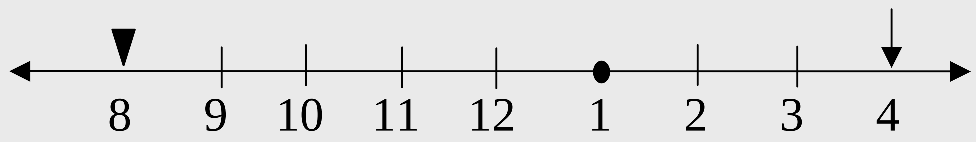
Fig. 1



So, understanding matters. It does a lot.

Now, getting back to the point we left off, we were talking about points. We often use dots for visual aids, and use them as points. And we can use others as points, too. For instance, we can use a small triangle, arrowhead, etc., as well as a small dot or short vertical bar, and if using it as a point in a time line, we mean it's a moment.

Fig. 2



A point in math can be used in many ways in real world. Math for the general public is a thinking business, and is about how to think. How to think to do what? For what?

How to think logically and tactically to get solutions. Math is about solutions. So math is not just about numbers and calculations, but it's about how to think the way stated above. Its use depends on how we think, our understanding and creativity.

A point in math has no shape and size as width or thickness. Neither does a line in math, since it's a collection of points.

And saying a line in math, we mean a straight line, which is like a long thin unbent string with no bump or dent, that is, no unevenness.

Fig. 3

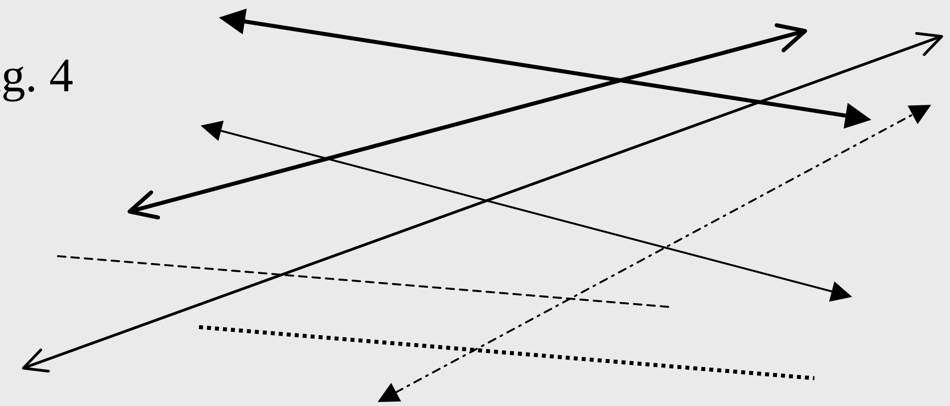
The first two in the figure above are not lines.
A line has its length, though, but the length is unlimited, that is, infinite. And it lengthens and proceeds in two opposite directions, and is assumed to grow on and on, infinitely.

So it never stops growing in length, and exists in our imaginations only. It is an ***idea***, a ***concept***, and thus, we need to get the concept, the conceptual understanding.

As mentioned earlier, a line is like a long thin unbent string with no bump or dent, that is, no unevenness. And it's length is unlimited, that is, infinite, but has no width or thickness.

Talking about a line doing math though, we use a visual aid as a pencil line, and show a part of a line. Sometimes, we use double arrowheads to indicate a line in math, but often omit the arrowheads for simplicity or speedy expression.

Fig. 4

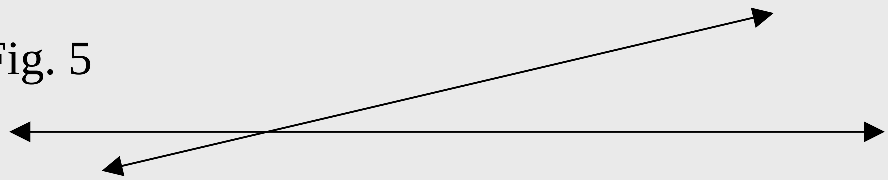


And, for identification purposes, we use a thicker line using more ink, or a thinner one using less ink, and sometimes, we use lines dashed or dotted. No matter what lines they may be though, the ***lines themselves are all the same***, and their ***lengths are assumed to be all infinite***.

So in the figure above, all the lines themselves are the same, though they look different. And all their lengths are infinite, though the ones in the figure look all limited in length, finite.   What makes then lines different from each other?

The two lines in the figure below look the same, but are taken as two different lines in some math. The math is geometry, of course, but is not just geometry. It's called analytic geometry.

Fig. 5



In analytic geometry, we put lines in a plane, but it's not just a plane. It's called a coordinate plane, full of addresses called coordinates, and in such a plane, locations matter.

So the two lines are different, only because their locations are different. A line at a different location is a different line.

Thus, in geometry analytic, we can say these:

> Lines are different if they have different locations.
>
> Lines have different locations if they are different lines.

In short, ***lines are different if and only if they have different locations***.

Also, in analytic geometry, we can put a line in an equation, too. That is to say that we can use an equation to indicate a line, and can say that a line has its equation.

So, in analytic geometry, we can say these, too:

Lines are different if they have different equations.

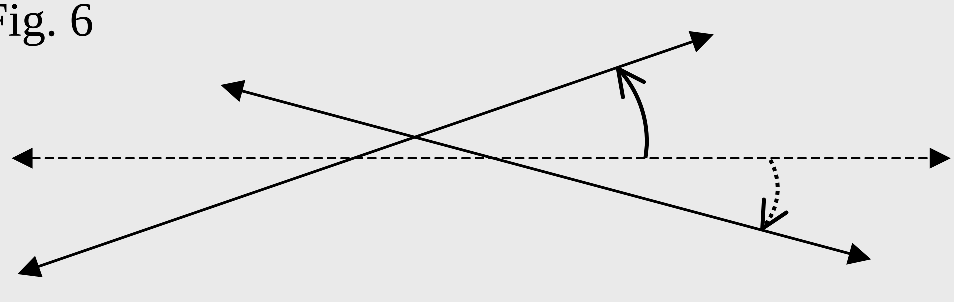
Lines have different equations if they are different lines.

We may not want to get into more detail at this time.

If interested in more on this, refer to the example section. In the section, some basics on a coordinate plane are covered.

Let’s now move on to the next.

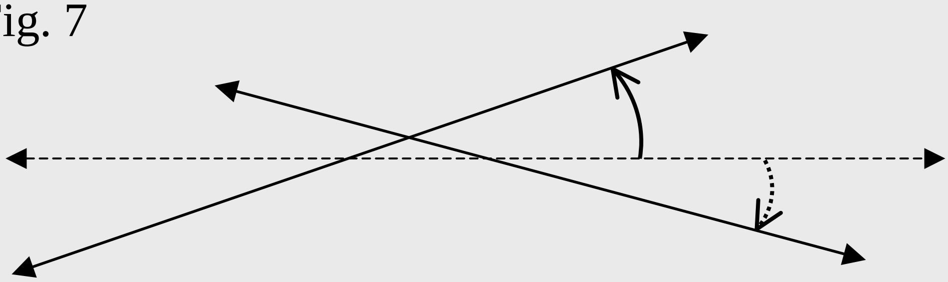
Fig. 6



The next is the direction of a line.

A line lengthens in two directions opposite of each other, but conventionally, we assume that the arrowhead on the right hand side indicates its direction.

Fig. 7



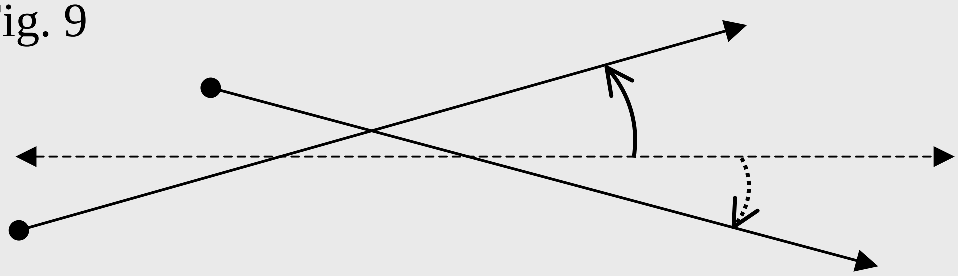
When measuring the direction, we can take the angle between the line and a horizontal line. And by convention, a vertical line is assumed to have no direction, since it has no right or left hand side.

Fig. 8

Next, what do we mean by ***a ray in math***?

A ray can be called a ***half line***, so it has the characteristics of a line, but grows in one direction only. So it is assumed to have a single direction. And we use one arrow to show that we are talking about a ray, and do not omit the arrow.

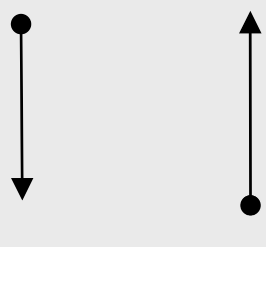
Fig. 9



As in the case of a line, when measuring the direction, we can take the angle between the ray and a horizontal line.

And by convention, too, a vertical ray is assumed to have no direction, since it has no right or left hand side.

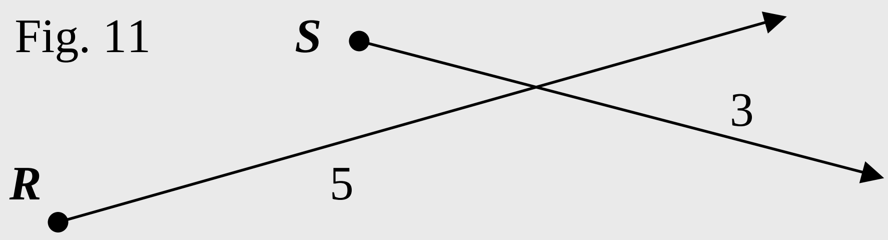
Fig. 10



What then about the length of a ray?

When just saying a ray, we mean a ray of infinite length. We can however, define a ray with a finite length, too, as a ray of length 3 or 5, and can call it a finite ray.

Fig. 11



In the figure above, ***R*** is a finite ray, and can be called a ray of length 5, and ***S*** is another finite ray, and can be called a ray of length 3.

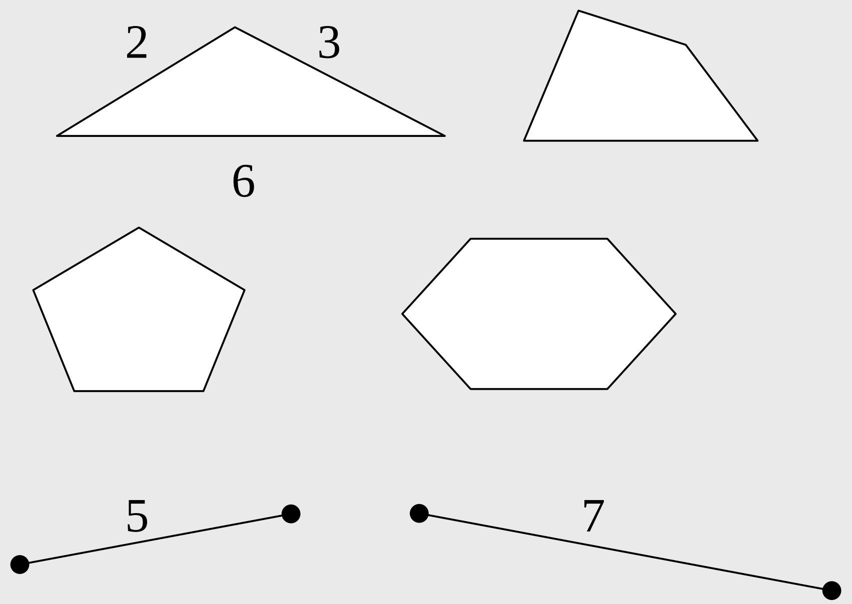
And if a ray has a finite length, it is similar to another object called a line segment.

So next, what do we mean by ***a line segment in math***?

A line segment is a part of a line or ray, and has a ***finite*** length. We use it as a side of a polygon as a triangle, which is three-sided, and as a quadrangle or tetragon, four-sided.

Also, we can use it when connecting two points showing the ***shortest distance*** between the two points.

Fig. 12

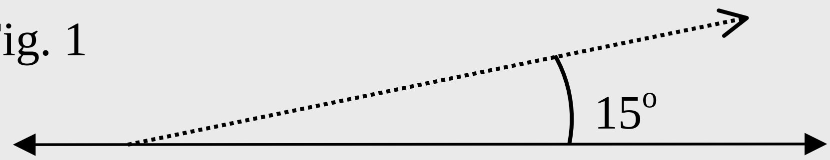


And as shown above, when saying a side of a polygon as a hexagon, six-sided, we often mean the length of it, as well as the line segment itself. So for instance, saying that a side in a triangle is 3, we mean that the length of the side is 3.

And summing up, for now, doing math, we often use lines, rays, and line segments, together with angles.

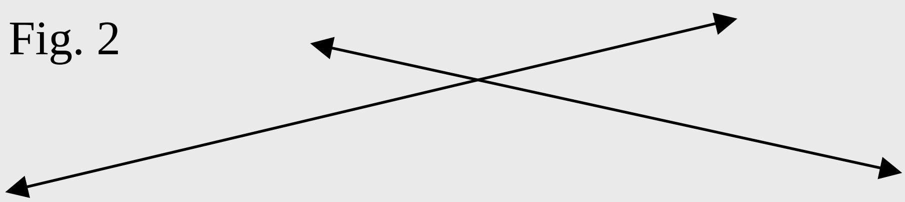
An angle can mean an amount of turning or an amount of difference in direction. For instance, you can open that door turning the knob more than only 40^o clockwise. And the direction of the flight is now 15^o against a horizontal line taken as the runway.

Fig. 1



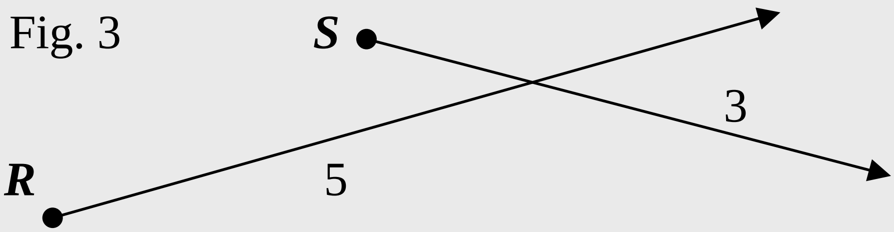
A line is a collection of points, and if in a line, connecting any two points away from each other, we get a line segment that has the same direction. It's like a long unbent string with no unevenness, and lengthens infinitely in two directions opposite of each other.

Fig. 2



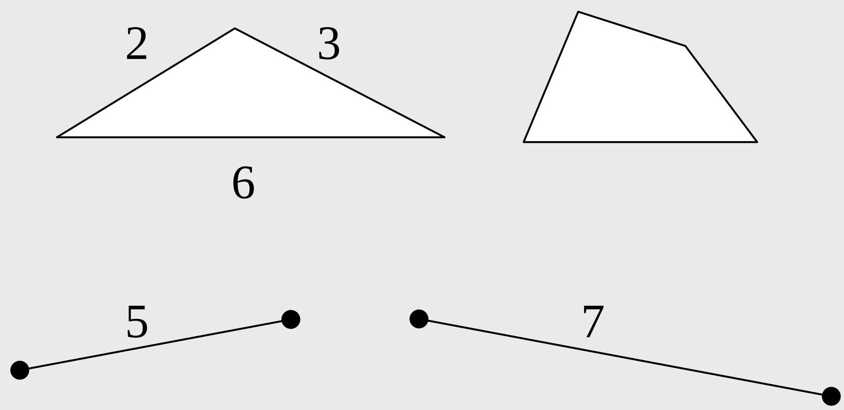
A ray can be called a half line meaning a half of a line, so its length is infinite, but can be finite, too, if we define a ray that way. For instance, we can define ***R*** to be a ray of length 5.

Fig. 3



And a line segment is a part of a line or ray, has a finite length, and is often used as a side of a polygon. We can also, use it when we connect two points showing the shortest distance between the two points.

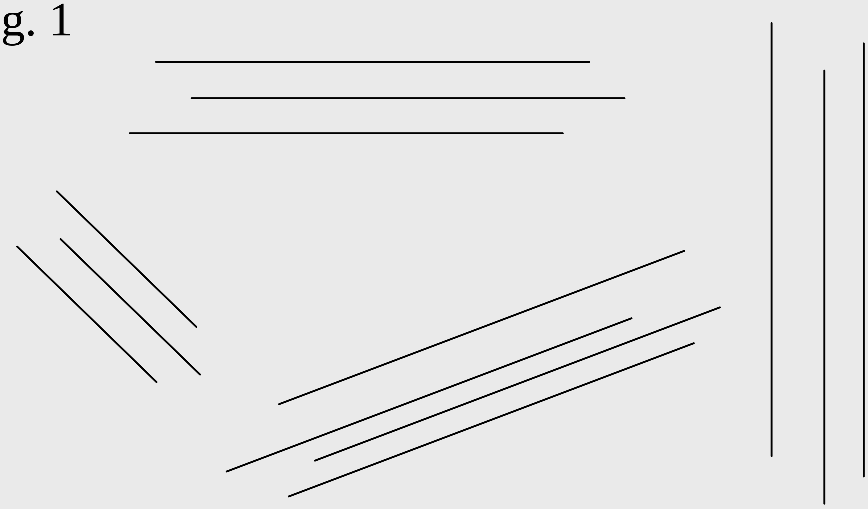
Fig. 4



9.4. Angles and Lines 4

Next, lines don't meet if they are parallel.

Fig. 1



So if parallel, the lines don't cross each other. In the figure above, there are four groups of lines parallel to each other, where the arrows are omitted for simplicity.

And all the lines in each group share the same direction.

What then is the angle between two parallel lines?

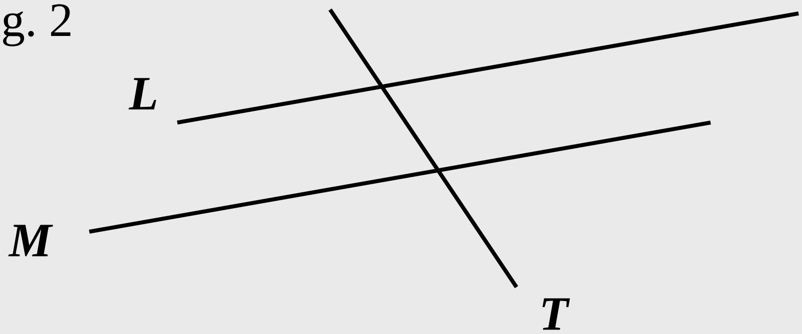
If two lines are parallel, we often say that they make no angle. It means, though, the angle between the two is 0^{o}.

So it's not no angle. There is an angle that is 0^{o}.

Mathematically, the angle between parallel lines is 0^{o}.

And for instance, if a line ***L*** is parallel to a line ***M***, using a math symbol, //, we can put the idea this way: ***L*** // ***M***.

Fig. 2



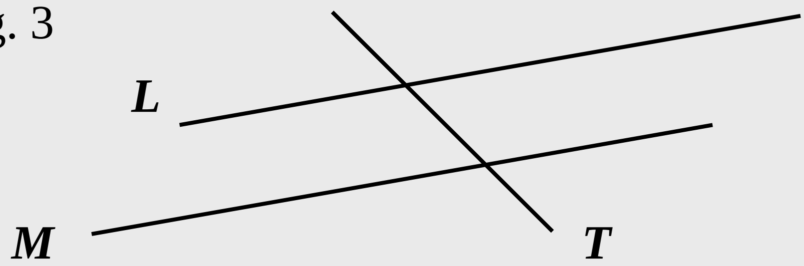
Then, the angle between the two lines ***M*** and ***L*** is 0^{o}.

What then, can be the same if some lines are parallel?

If some lines are parallel, they share the same direction. There is no difference in their directions.

An angle can mean ***an amount of difference in direction***, so 0^o can mean no difference in direction. Thus, if the angle between two lines is 0^o, the two lines share the same direction, and are parallel to each other.

Fig. 3



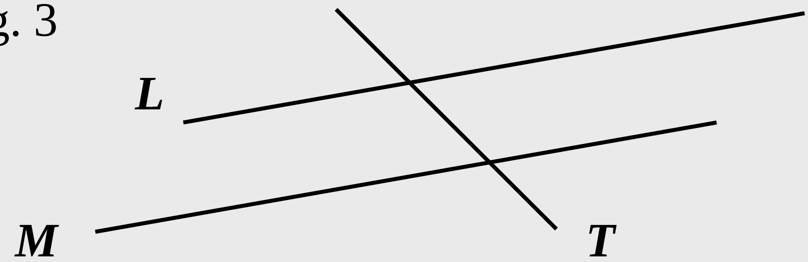
So what about it?

Is there anything, any big deal about it?

Yes, there is. And it matters.

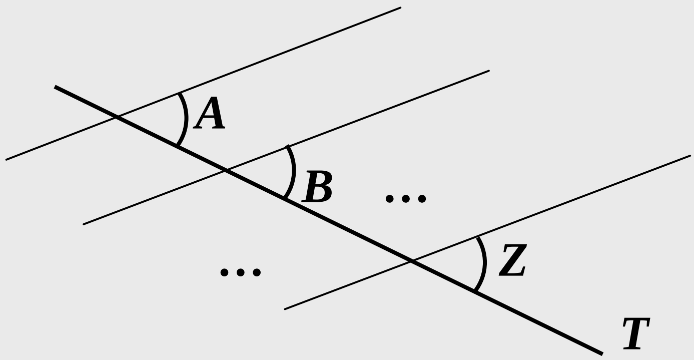
The angle between any two parallel lines is 0^o.

Fig. 3



A line therefore, crossing parallel lines makes the ***same angle*** with each and every one of all those parallel lines, and that's what matters. We often use it solving problems.

Fig. 4



Assuming all the lines crossed by ***T*** are parallel, we get this:

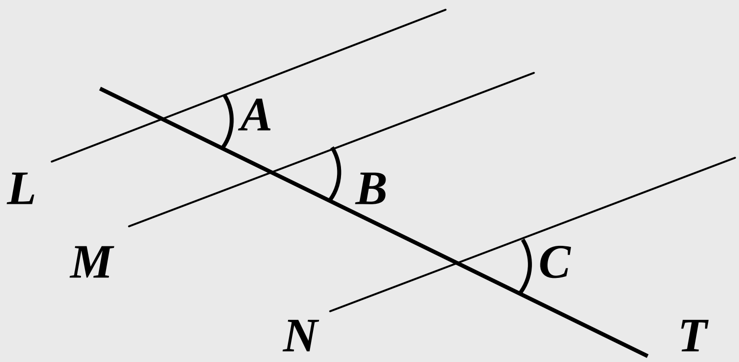
$\angle \boldsymbol{A} = \angle \boldsymbol{B} = \ldots = \angle \boldsymbol{Z}$.   The angles are all the same.

Why the same, though?

It's because the angle between any two parallel lines is 0^o, so there is ***the same difference in direction*** between the line T and each of all the parallel lines. Thus, ***the angle*** between the line T and each parallel line is ***the same***, and that's what matters.

We often use the fact above solving problems.

Fig. 5



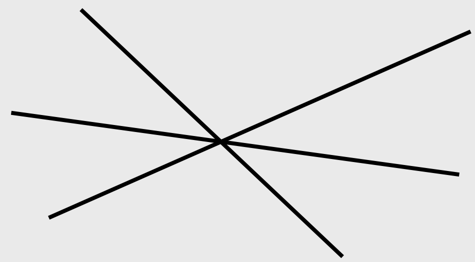
If we get this: $L \parallel M \parallel N$, we get this: $\angle A = \angle B = \angle C$.

And note that, as the line T above, if a line intersects two or more lines, parallel or not, the line is called a ***transversal***. So a transversal is a line crossing more than one line. Is it?

No line is however, the transversal if all the lines cross each other at one point. So a transversal intersects two or more lines, and crosses each line at a different point.

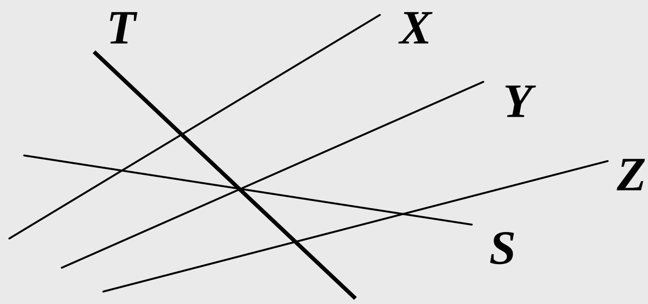
So we don't call a line a transversal if the line meets all the other lines at one point only, that is, if all the lines share the same one point as shown in the figure below.

Fig. 6



The line ***T*** in the figure below, is the transversal for the lines ***X***, ***Y***, and ***Z***, is the transversal for ***X*** and ***S***, and is the transversal for ***Z*** and ***S***, but is not for ***S*** and ***Y***.

Fig. 7



Next, how many angles do two lines make if they are not parallel, that is, if they meet at a point?

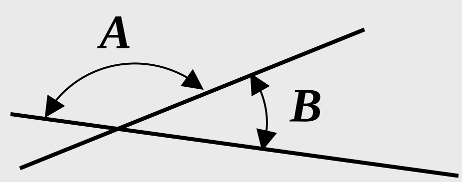
Two lines can make an angle or two different angles.

Speaking more precisely though, they can make four same angles or two different pairs of same angles.

We will cover soon ***why the same four*** or ***two different***.

So for now, we will see what the two angles are first.

Fig. 8



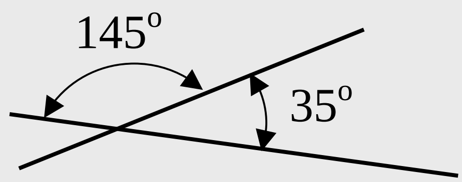
The two lines in the figure above make two angles, which are these: $\angle A$ and $\angle B$, and we have this: $\angle A + \angle B = 180^o$.

If adding up to 180^o, the two angles can be called ***supplementary*** angles. So $\angle A$ and $\angle B$ can be said to be supplementary to each other.

For instance, 70^o and 110^o are supplementary angles, and so are 30^o and 150^o.

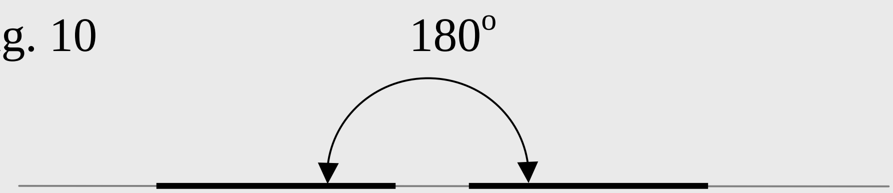
And we can say that 60^{o} is supplementary to 120^{o}, and that 145^{o} is supplementary to 35^{o}. Those two angles make 180^{o}.

Fig. 9



And the angle of 180^{o} can be called a ***straight angle***, which reflects a ***straight line***, often just called a ***line***.

Fig. 10



So the angle between two line segments in a line is a ***straight angle***, which is 180^{o}. And if an angle is bigger than 180^{o} and less than 360^{o}, it is called a ***reflex angle***.

By the way, 360^{o} is called a ***perigon***, round angle, complete angle, full angle, etc., and it seems often called a perigon.

And 90^{o} is called a ***right angle***.

Now, getting back to the number of angles two lines make if they intersect, you might wonder why not one angle.

We often say that two lines make an angle, don't we? For instance, we can say that two lines make an angle of 30^o, or can say that the angle between two lines is 30^o, can't we?

Granted, we often say so, but if two lines intersect, they can actually make one angle or two different angles, and the two angles can be called supplementary angles.

Usually, two intersecting lines make two different angles.

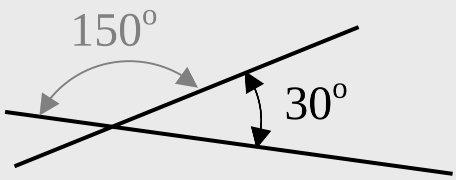
Normally though, we don't specify two angles the two lines make, but specify one of the two angles, because we focus our attention on or lay stress on the one angle specified.

Mathematically, two lines can make two angles.

And we just focus on one of the two.

So for instance, if we say that two lines make an angle of 30^o, the two lines make in fact, not one but two angles, one is 30^o, and the other is 150^o, which is supplementary to 30^o, which is the angle we focus on in this case.

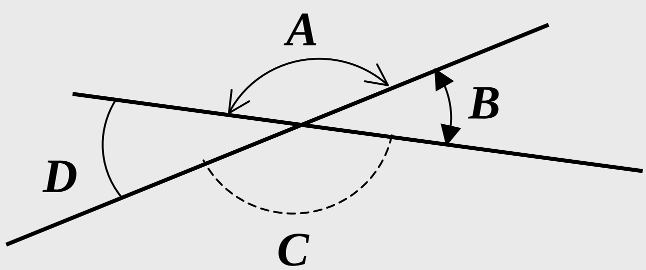
Fig. 11



Seemingly though, the two lines make more than two angles, and actually, the number itself of the angles made is four.

So why not four angles as shown below?

Fig. 12

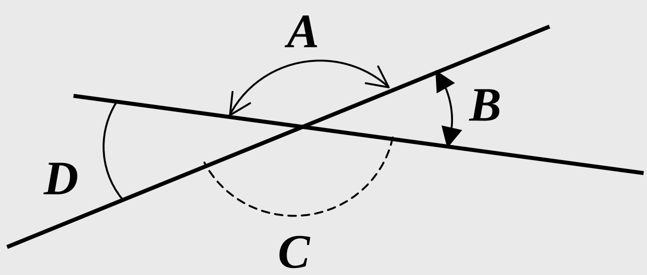


In the figure below, we can see four angles, and they are

$\angle A$, $\angle B$, $\angle C$, and $\angle D$.

So the number of the angles themselves is four.

Fig. 12

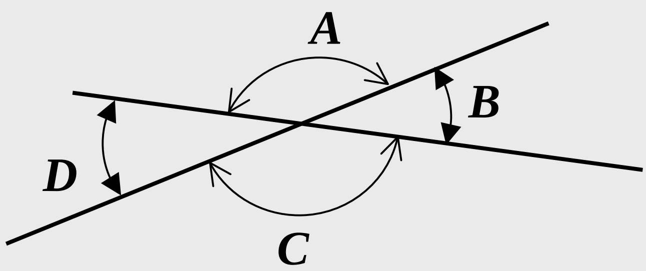


Is it not the case thus, two lines can make four angles?

Granted, normally though, saying four angles, we mean four different angles. Two lines can make not four but two different angles if they intersect. Why, though?

It's because we have this: ∠***B*** = ∠***D***, and these two angles are called ***vertically opposite angles***, simply just called ***vertical angles***, often used when we solve problems.

Fig. 13



And by the same token, we have this, too: ∠***A*** = ∠***C***, since these two are vertical angles, too, which are always the same. And, why the same, that is, the proof is as follows.

First, we have this: ∠***A*** + ∠***B*** = 180^{o}, a straight angle.

Next, we have this, too: ∠***C*** + ∠***B*** = 180^{o}, so we get this:

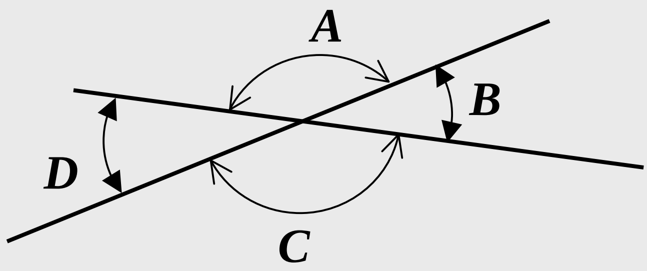
∠***A*** + ∠***B*** = ∠***C*** + ∠***B***, and looking at both sides of this: =,

we can see ∠***B*** is common; thus, we get this: ∠***A*** = ∠***C***.

What then, about this: ∠***B*** = ∠***D***?

Now again, we have this: $\angle A + \angle B = 180^o$.

Fig. 13



And next, we have this: $\angle A + \angle D = 180^o$, so we get this:

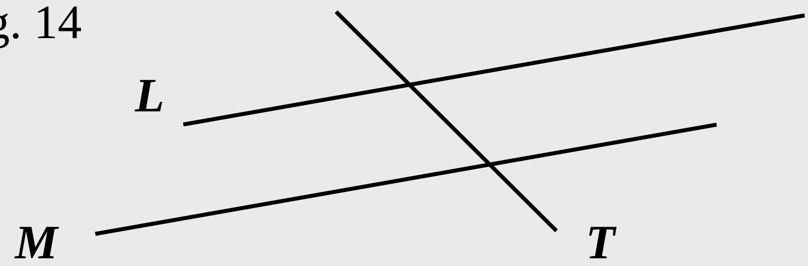
$\angle A + \angle B = \angle A + \angle D$, and in this case, $\angle A$ is common;

thus, we get this: $\angle B = \angle D$.

Now, what is the case where two lines actually make one angle only?

We have two cases where one angle gets made, and one of the two is, as mentioned earlier, the case where the two lines are parallel, and in that case, the angle is 0°, which is not no angle but an angle in math.

Fig. 14



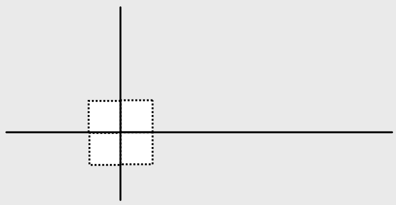
So in the figure above, if we get this $L \parallel M$, that is, if L is parallel to M, the angle between the two lines M and L is 0°, which can mean no difference in direction; that is to say that the two lines share the same direction.

However, in the case above, two lines don't meet, that is, they are parallel.

So what is the case where two lines meet and make one angle only?

Two lines can meet at 90^o, a ***right angle***, that is, they can be perpendicular to each other. Then, the two lines make four same angles, 90^o each, and thus, one angle only.

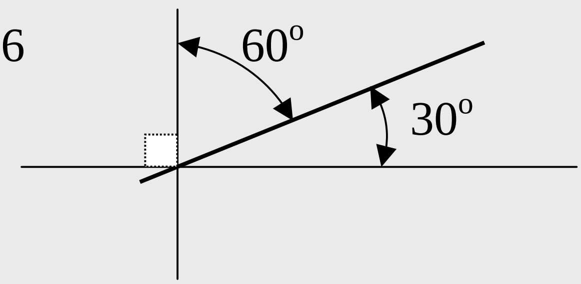
Fig. 15



By the way, if adding up to 90^o, the two angles can be called ***complimentary*** angles. So one of two angles can be said to be complimentary to the other if the two add up to 90^o.

So for instance, the angle complimentary to 30^o is 60^o, which is, of course, also complimentary to 30^o.

Fig. 16

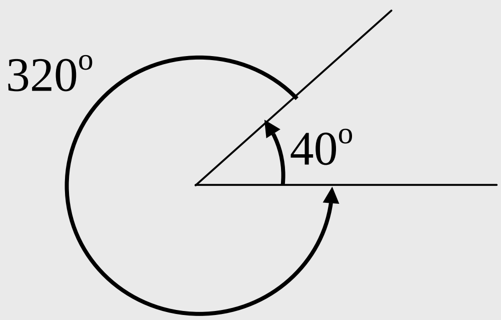


What if two angles add up to 360^o, i.e., a perigon?

If adding up to a perigon, 360^o, the two angles are said to be ***explementary angles***, and one is called an ***explement*** of the other.

So for instance, the angle explementary to 40^o is 320^o, which is, of course, also, the explement of the angle 40^o.

Fig. 17



The angle of 320^o is a reflex angle, and the angle 40^o is an ***acute angle***, which is actually defined to be bigger than 0^o and less than 90^o. That is to say that it's between 0^o and 90^o.

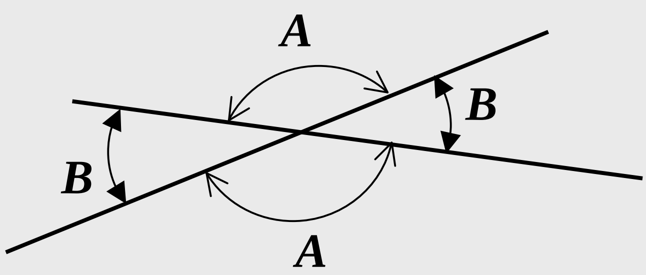
And if an angle is bigger than 90^o and less than 180^o, it is an ***obtuse angle***, which is therefore, between 90^o and 180^o. For instance, 140^o is an obtuse angle, and the same is true of these angles: 92^o, 91^o, and 175^o.

And for now, the bottom line is as follows.

First off, we have ***vertical angles***, ***two same angles***.

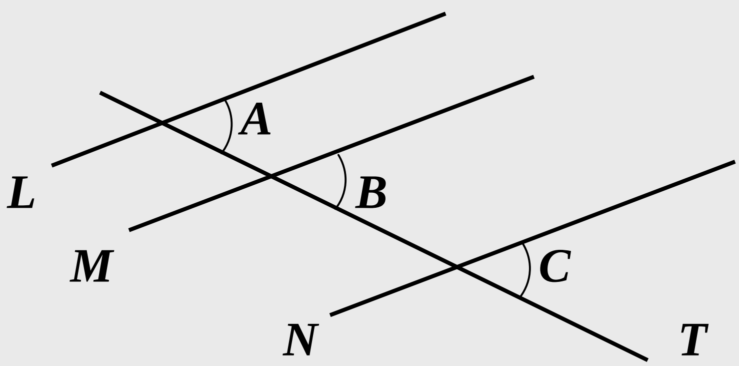
Vertical angles are two angles, ***always the same***.

Fig. 1



Next, if lines are parallel, the transversal makes the same angle with each and every one of the parallel lines.

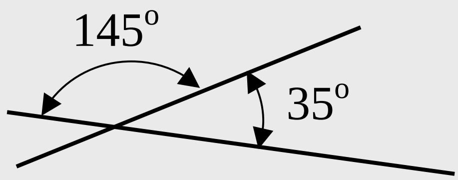
Fig. 2



So if we get this: ***L*** // ***M*** // ***N***, we get this: $\angle A = \angle B = \angle C$.

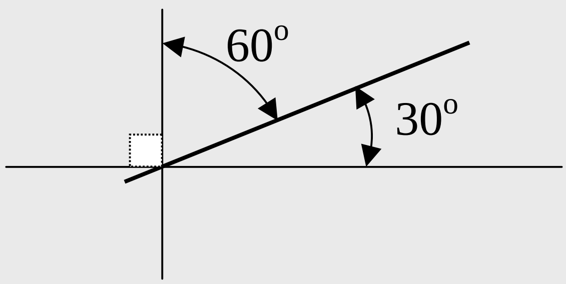
Next, if adding up to 180^o, called a ***straight angle***, the two angles can be called ***supplementary*** angles.
So for instance, 60^o and 120^o are supplementary angles.
And we can say that 145^o is supplementary to 35^o.

Fig. 3



Next, if adding up to 90^o, called a ***right angle***, the two angles can be called ***complimentary*** angles. So for instance, the angle complimentary to 30^o is 60^o.

Fig. 4

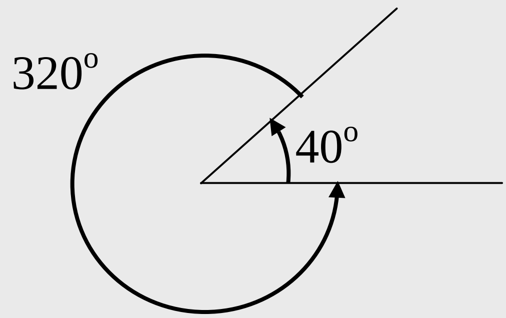


Also, if two lines make a right angle 90^o, the two are said to be perpendicular to each other.

Next, 360^o is called a ***perigon***, or a round, complete, or full angle. And if adding up to a perigon, 360^o, the two angles are said to be ***explementary angles***, and one is called an ***explement*** of the other.

So for instance, the angle explementary to 40^o is 320^o, which is, of course, also, the explement of the angle 40^o.

Fig. 5



Next, the angle of 320^o is a ***reflex angle***, and the angle 40^o is an ***acute angle***, which is bigger than 0^o and less than 90^o, that is, between 0^o and 90^o .

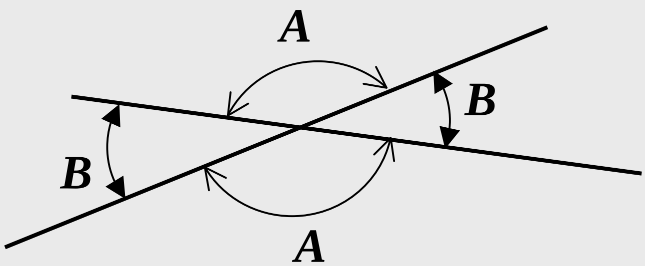
And if an angle is bigger than 90^o and less than 180^o, it is an ***obtuse angle***. For instance, among obtuse angles, we have these: 140^o, 92^o, 91^o, and 175^o.

And again, the most important are the two below.

First, we have ***vertical angles***, ***two same angles***.

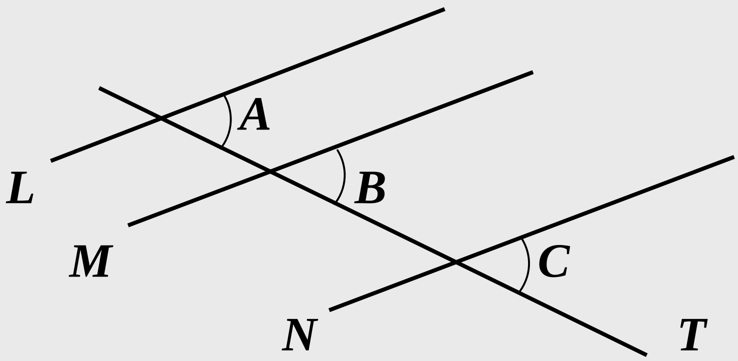
Vertical angles are two angles, ***always the same***.

Fig. 1



And next, if lines are parallel, the transversal makes the same angle with each and every one of the parallel lines.

Fig. 2

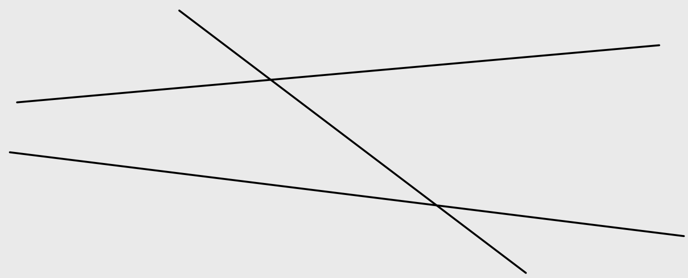


So if we get this: $L \parallel M \parallel N$, we get this: $\angle A = \angle B = \angle C$.

9.5. Angles and Lines 5

Next, with two other lines, one line can make one angle or some different angles.

Fig. 1



Looking at the figure above though, we can see more than one angle. So we may want to put it this way:

“With two other lines, one line can make some same angles only, or can make some different angles.”

So the point is not the number of angles. It’s what angles that can be made when a line meets two other lines.

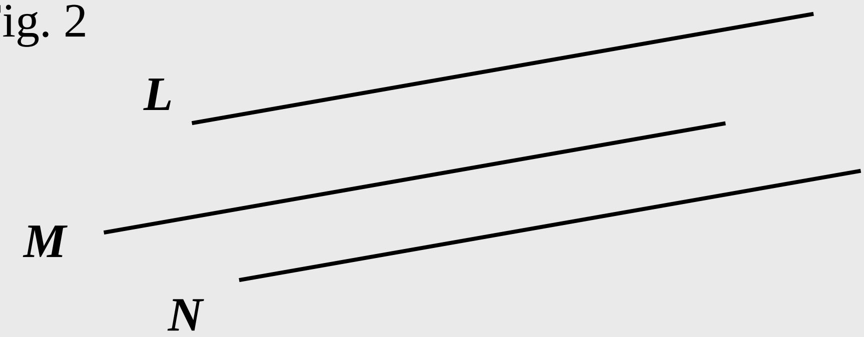
Saying that one angle gets made when more than one are expected, we mean that some same angles get made. There are three cases where the angles made are all the same. And one of the three is Case 1, which is as follows.

Case 1

Three same angles, 0^o each, if all the three lines are parallel, because the angle between two parallel lines is 0^o.

Note that it wasn't mentioned that a line has to meet two other lines. So it doesn't have to meet the other two lines, and can be parallel to the other two both as shown below.

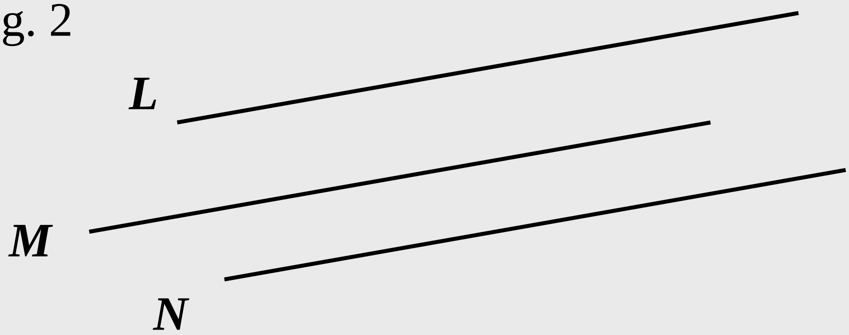
Fig. 2



Why three same angles, though, and not just one?

We can make three pairs of two lines that are parallel.

Fig. 2



And the three pairs are as follows.

(***L***, ***M***) (***M***, ***N***) (***N***, ***L***)

So between the two lines in each pair, the angle is 0^o, and therefore, we get three same angles, 0^o each.

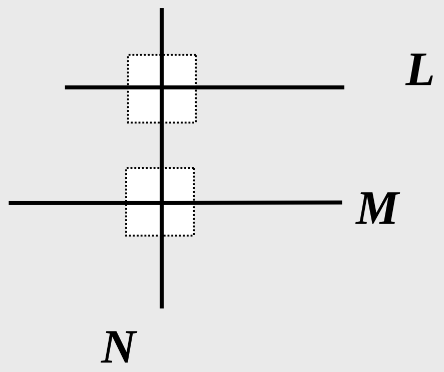
And after all, other than 0^o, no angle is made.
So one angle is made.

And it's not the only case where we get some same angles.
We have two other cases, and one of the two is Case 2, which is as follows.

Case 2

Eight same angles, 90^o each, if the two lines are perpendicular to the other line. So after all, one angle is made, and is 90^o, a right angle.

Fig. 3



And again, after all, other than 90^o, a right angle, no angle is made. So in this case, too, one angle is made,

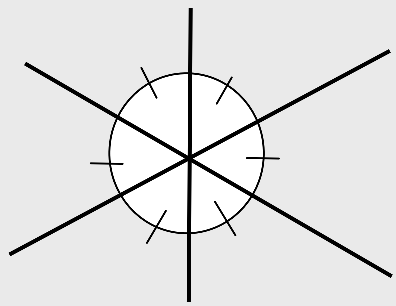
By the way, we've already covered the important fact that if lines are parallel, the transversal makes the same angle with each line, and also, if the transversal makes the same angle with each of the lines, all the lines are parallel.

And there is one more case where we get some same angles, and it is Case 3, which is as follows.

Case 3

Six same angles, 60^o each, if the three lines meet at one point only, and divide a perigon, 360^o into six equal parts. That is to say that the three lines divide a circle into six equal wedges, in each of which, the angle at the vertex is 60^o. And this time, also, after all, other than 60^o, no angle is made. So in this case, too, one angle is made.

Fig. 4



Let's now, move on to the case where a line makes some different angles with two other lines. There isn't just one case, and can be many cases where a line makes some different angles with two other lines.

Among those many cases, we are going to cover a particular case where a line intersects two other lines and crosses each line at a different point.

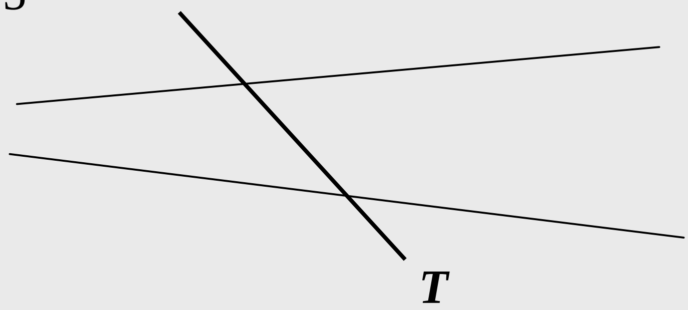
How then do we call the line intersecting the two?

It’s called a transversal, which is a line that intersects two or more lines, and crosses each at a different point.

In short, a line meeting more than a line at different points.

In the figure below, two lines are crossed by a line ***T***. If a line crosses two or more lines, parallel or not, and meets each line at a different point, it is called a ***transversal***. So we can call the line ***T*** a transversal.

Fig. 5

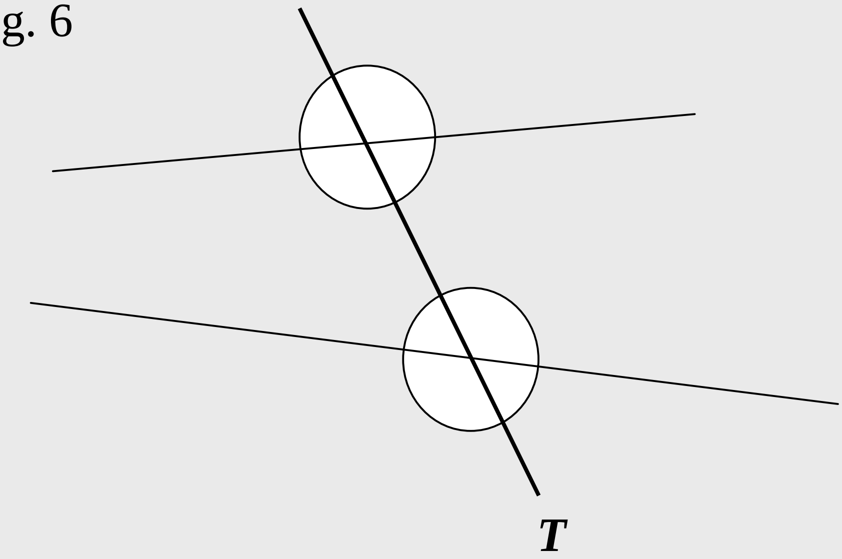


How many different angles then, do the two lines make with the line called the transversal?

And we can put the same question this way, too: How many different angles does the line ***T*** make with the two lines?

The two lines can make one, two, three, or four different angles with the transversal ***T***.

Fig. 6



Though seemingly, there are eight angles, with the two lines, the transversal ***T*** can in fact, make ***up to four different pairs*** of vertical angles.

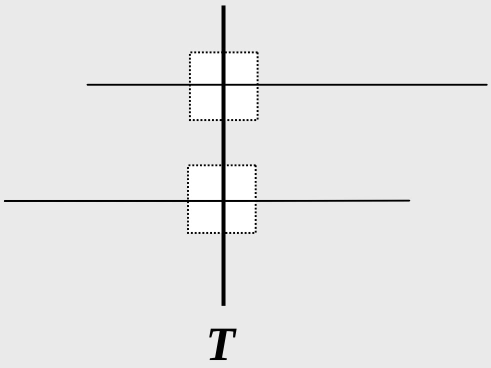
And vertical angles are equal, a pair of the same angles.
So with the two lines, the line ***T*** can make up to four different angles.

Why up to four, though?
Why not just four different angles?

The number of different angles made can be one, two, three, or four. So there are four cases, and let's see now what the four cases are, and how each case is made.

To begin with, one angle is made if the two lines are parallel and the transversal is perpendicular to the two. We've covered this already, and it was covered when we did the case called Case 1, where eight same angles are made.

Fig. 7



The one angle is 90^o, a right angle, if the two lines are parallel and the transversal is perpendicular to the two.

More specifically, as stated above, eight same angles are made, and are all right angles. And technically, there are four same pairs of vertical angles, and in each pair, the two same angles are right angles.

Next, two different angles are made if each of the two lines makes the same angle with the transversal.

There are two cases for the way the same angles are made.

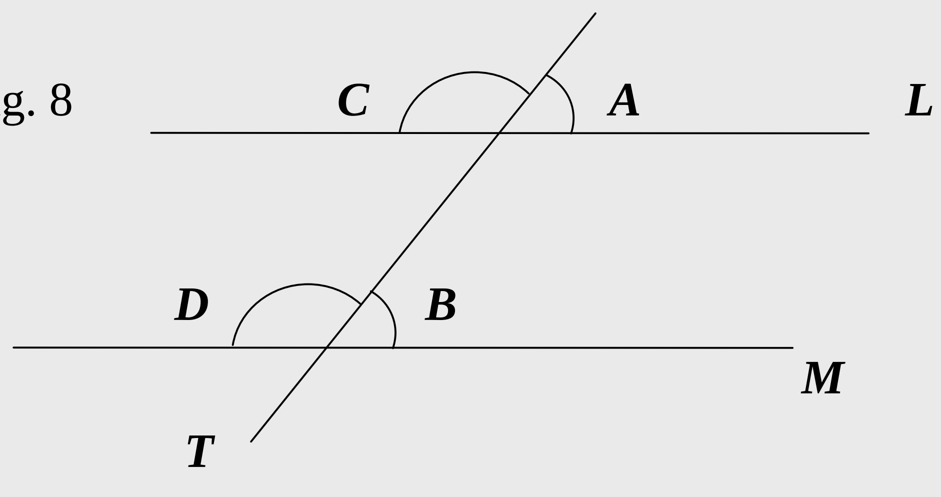
One of the two is the case where the two lines are parallel.

And the other is the case where the two lines are not.

Let's now take a look at the first of the two.

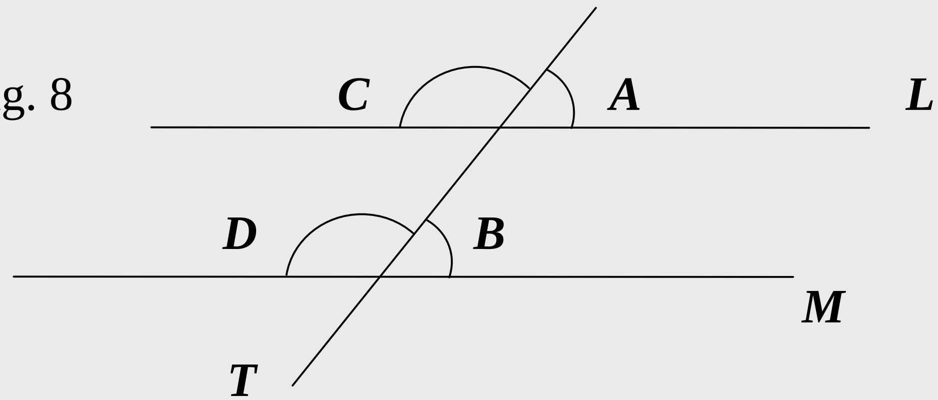
Two different angles are made if the two lines are parallel, and the transversal is not perpendicular to any of the two parallel lines. For instance, we get two different angles if the three lines are placed the way below, and $\boldsymbol{L}$ is parallel to $\boldsymbol{M}$.

Fig. 8



It's because we have the important fact that if lines are parallel, the transversal makes the same angle with each of the lines, and vice versa.

Fig. 8



So we have these: $\angle \boldsymbol{A} = \angle \boldsymbol{B}$, and $\angle \boldsymbol{C} = \angle \boldsymbol{D}$.

Also, we have this: $\angle \boldsymbol{A} \neq \angle \boldsymbol{C}$, because the transversal ***T*** is not perpendicular to the two lines ***M*** and ***L***.

And other than these two: $\angle \boldsymbol{A}$ and $\angle \boldsymbol{C}$, no angle is made.

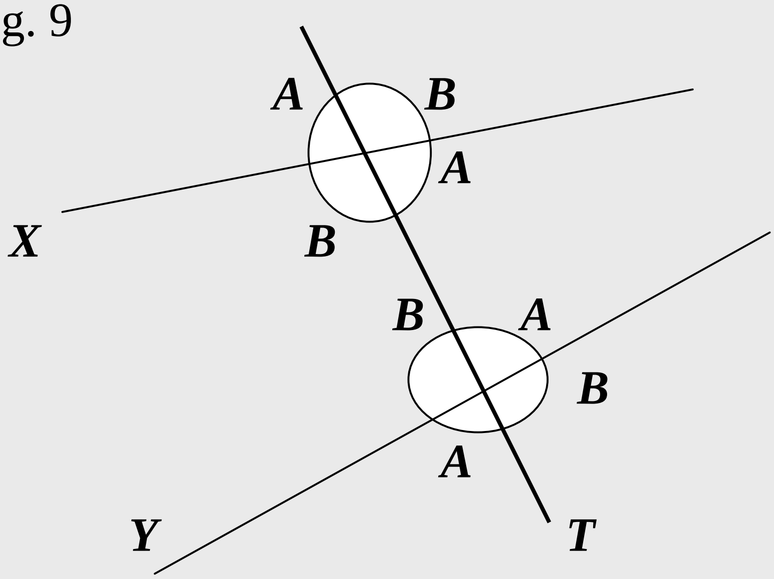
So in sum, two different angles are made.

By the way, the two angles are supplementary to each other. It's because we have this: $\angle \boldsymbol{A} + \angle \boldsymbol{C} = \angle \boldsymbol{B} + \angle \boldsymbol{D} = 180^{o}$, whether the two lines are parallel or not.

And now, we are going to see the other case where two different angles get made.

Two different angles are made if each of the two lines makes the same angle with the transversal, though the two lines are not parallel. And the way the same angles are made is shown below. In the figure below, we can see two different pairs of vertical angles. And vertical angles are two same angles. So two different angles are made.

Fig. 9



It is similar to the first case where the two lines are parallel, and the transversal is not perpendicular to any of the two.

And the two figures below are practically the same.

Fig. 9

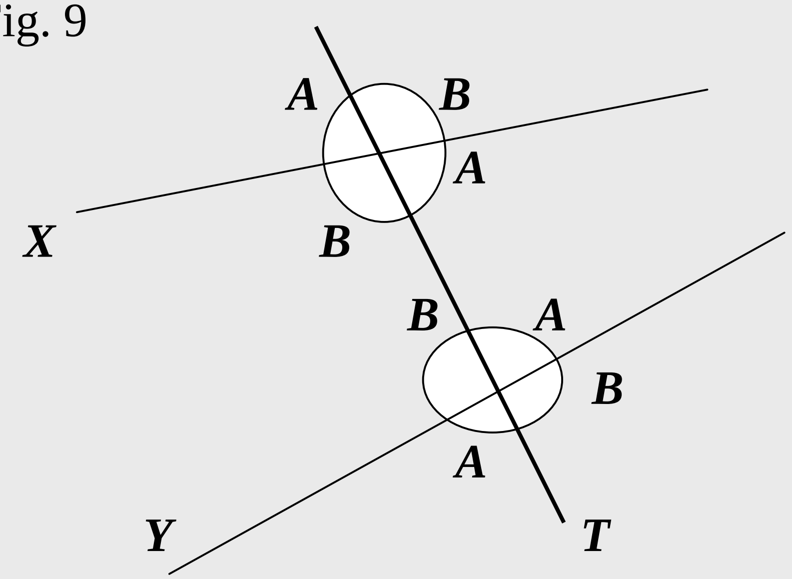
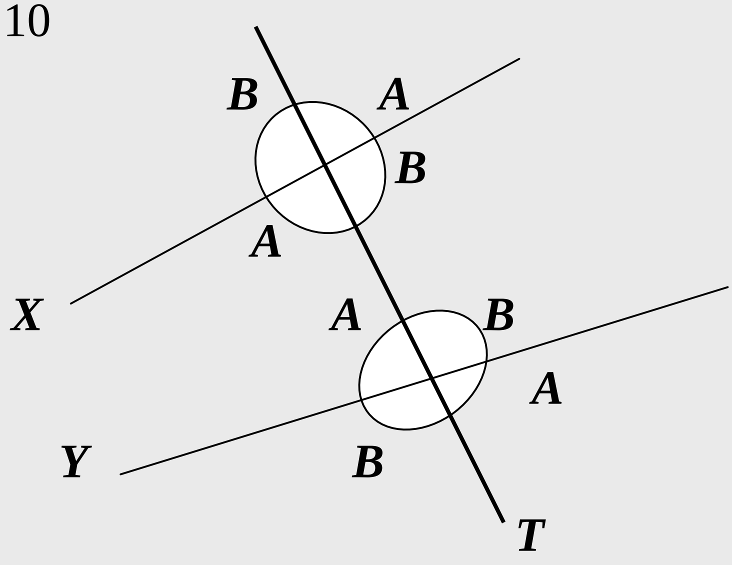


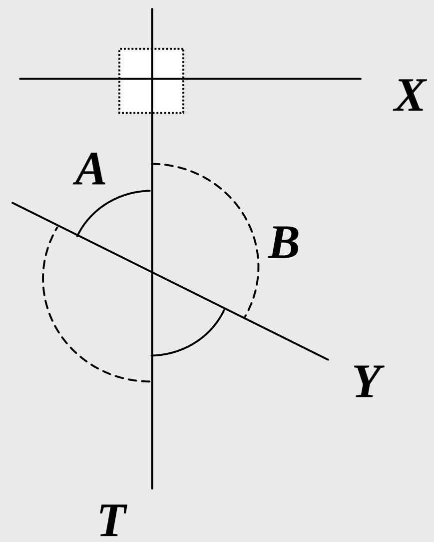
Fig. 10



Flipping each figure about the line ***T***, we get the other figure.

Next, three different angles are made if one line is perpendicular to the transversal, but the other is not.

Fig. 11



More specifically, four right angles and two different pairs of vertical angles are made, so each pair is two same angles.

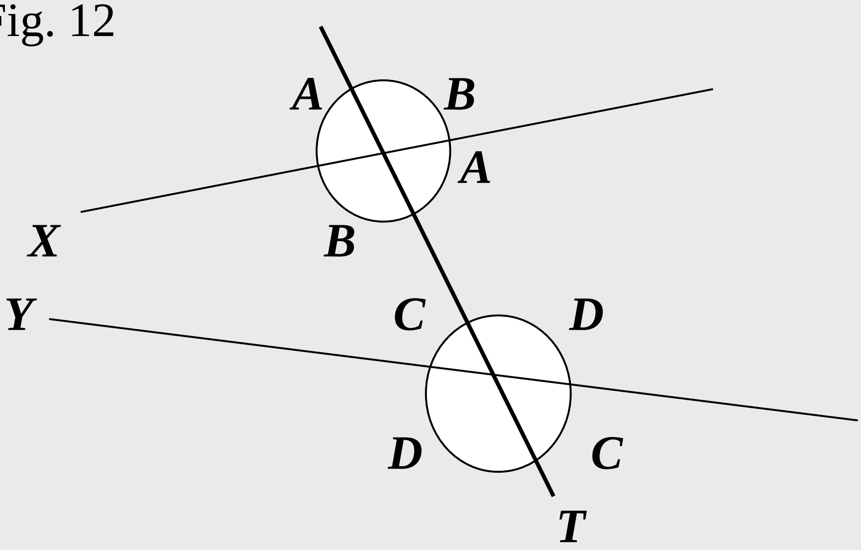
Thus, three angles are made, and are a right angle, 90^{o}, and these two angles: $\angle \boldsymbol{A}$, and$\angle \boldsymbol{B}$.

Now, the next is the last case, the last of the four.
In that case, four different angles are made.

So let's see now, how four different angles are made.

In this case, each of the two lines makes a different angle with the transversal. Then, four different pairs of vertical angles are made the way as follows.

Fig. 12



Therefore, four different angles are made, and are these: $\angle \boldsymbol{A}$, $\angle \boldsymbol{B}$, $\angle \boldsymbol{C}$, and $\angle \boldsymbol{D}$.

So in sum, the number of angles two lines can make with the transversal is one, two, three, or four.

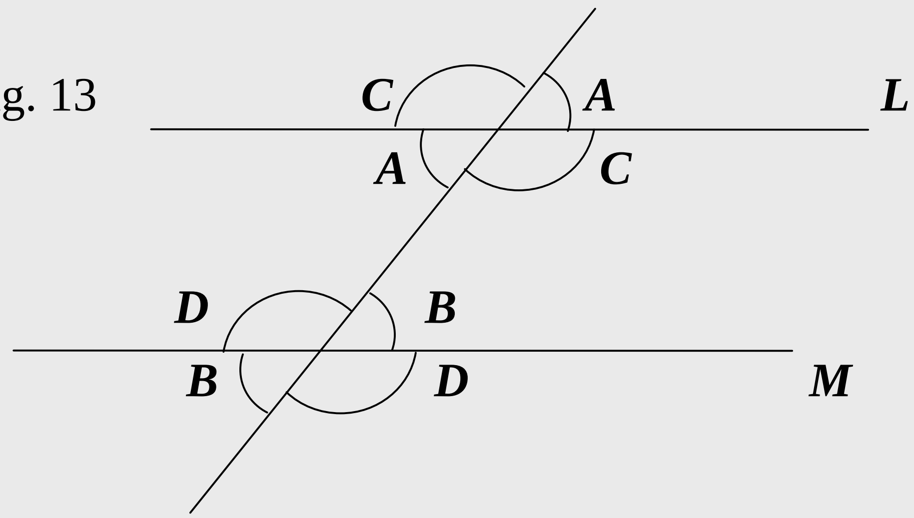
That's not it, though.

We can have some more facts that are quite interesting. And we can get the interesting facts if the two lines crossed by the transversal are ***parallel***.

Not only interesting, but important and useful, very useful.

We have in fact, already covered it, which is as follows.

Fig. 13



In the case above, we have $\angle A + \angle C = \angle B + \angle D = 180^\circ$.

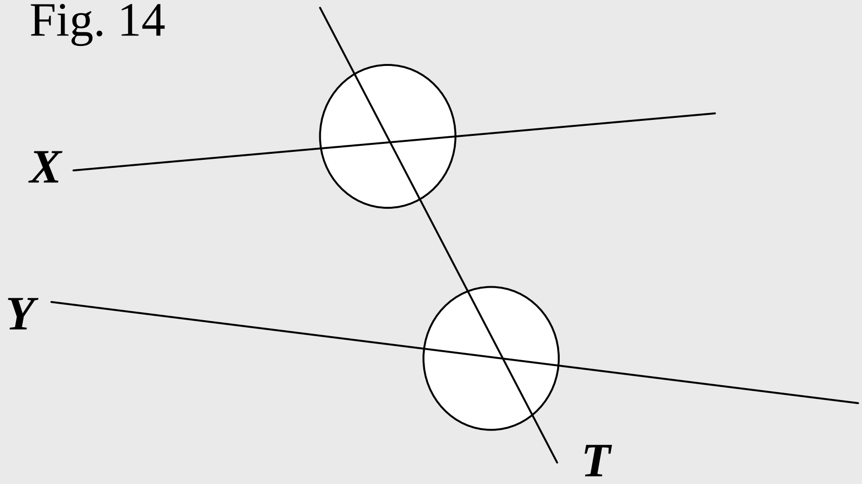
And more importantly, if M and L are parallel, we get these:

$\angle A = \angle B$, and $\angle C = \angle D$.

That is to say that we can see pairs of the same angles that are not vertical angles if two lines crossed by the transversal are ***parallel***.

So in the figure below, if the two lines ***X*** and ***Y*** are ***parallel***, we can see pairs of the same angles, though they are not vertical angles.

Fig. 14



And for now, in the figure above, same or not, we call some of the eight angles ***corresponding angles***, and call some others ***alternate angles***. What are those angles, though?

Corresponding angles? Alternate angles?

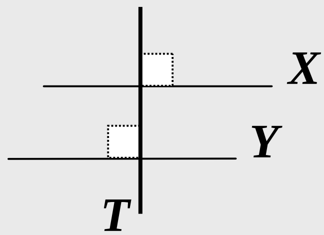
What do we mean by those angles?

So in the next section, with lines or line segments parallel or not, we will cover what those corresponding and alternate angles are. And for now, the summary is as follows.

The number of different angles two lines can make with the transversal can be one, two, three, or four.

First, one angle is made if the two lines are parallel and the transversal is perpendicular to the two. And the angle is 90^{o}, a right angle. Other than that, we get no angle.

Fig. 1



Next, two different angles are made if each of the two lines makes the same angle with the transversal. And there are two cases.

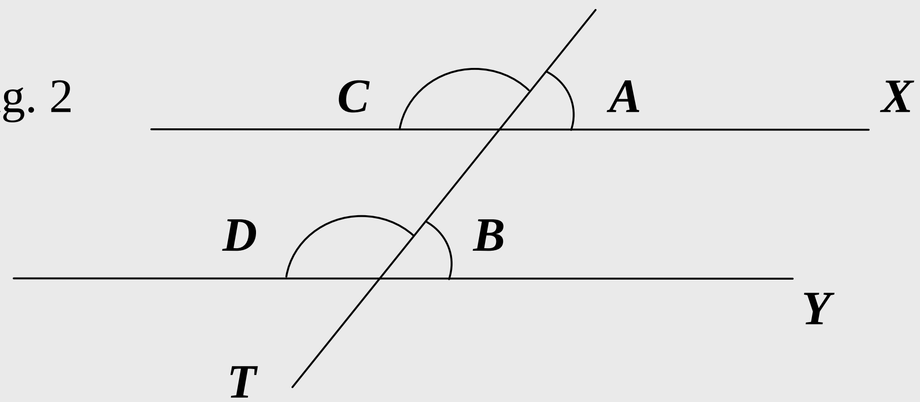
One of the two is the case where the two lines are parallel.

The other is the case where the two lines are not.

And the first of the two above is as follows.

Two different angles are made if the two lines are parallel, and the transversal is not perpendicular to any of the two parallel lines. Suppose for instance, in the figure below, ***X*** is parallel to ***Y***. Then, two different angles are made.

Fig. 2



It’s because of the important fact that if lines are parallel, the transversal makes the same angle with each of the lines, and vice versa.

So we have these: $\angle \boldsymbol{A} = \angle \boldsymbol{B}$, and $\angle \boldsymbol{C} = \angle \boldsymbol{D}$.

Also, we have this: $\angle \boldsymbol{A} \neq \angle \boldsymbol{C}$, because the transversal ***T*** is not perpendicular to the two lines ***M*** and ***L***.

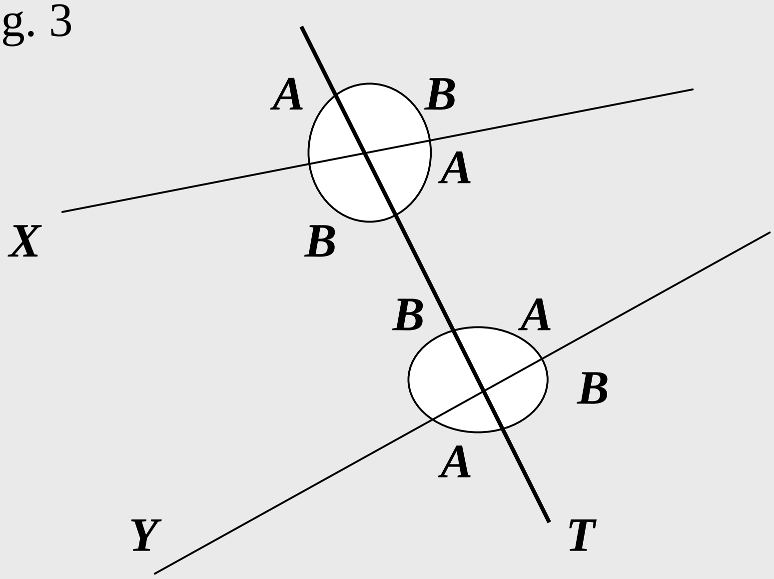
And other than these two: $\angle \boldsymbol{A}$ and $\angle \boldsymbol{C}$, no angle is made.

So in sum, two different angles are made.

And now, the next is the other of the two cases where two different angles get made.

Two different angles are made if with the transversal, each of the two lines makes the same angle, though the two lines are not parallel. And the way the same angles are made is shown below. In the figure below, we can see two different pairs of vertical angles, which are two same angles. So two different angles are made.

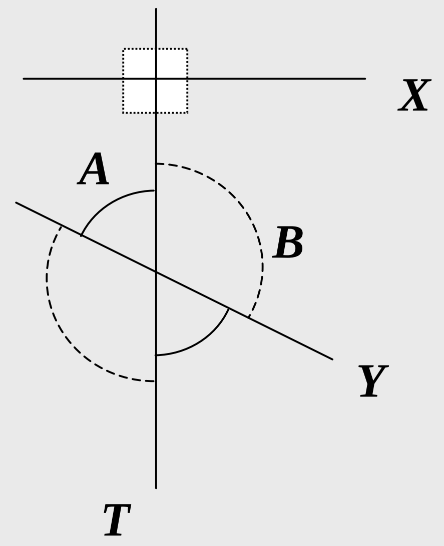
Fig. 3



It is similar to the first case where the two lines are parallel, and the transversal is not perpendicular to any of the two.

Next, three different angles are made if one line is perpendicular to the transversal, but the other is not.

Fig. 4



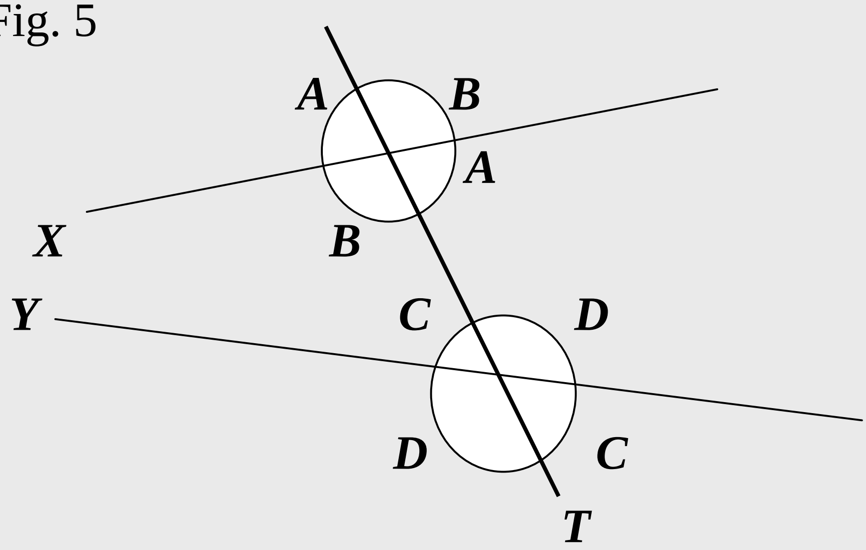
More specifically, four right angles and two different pairs of vertical angles are made, so each pair is two same angles.

And thus, three different angles are made and are a right angle, 90^{o}, and these two angles: $\angle \boldsymbol{A}$, and$\angle \boldsymbol{B}$.

And the next is the case where the number of different angles made is four.

In this case, each of the two lines makes a different angle with the transversal. Then, four different pairs of vertical angles are made the way as follows.

Fig. 5



Therefore, four different angles are made, and are these: $\angle A$, $\angle B$, $\angle C$, and $\angle D$.

So in sum, the number of different angles two lines can make with the transversal can be one, two, three, or four.

9.6. Angles and Lines 6

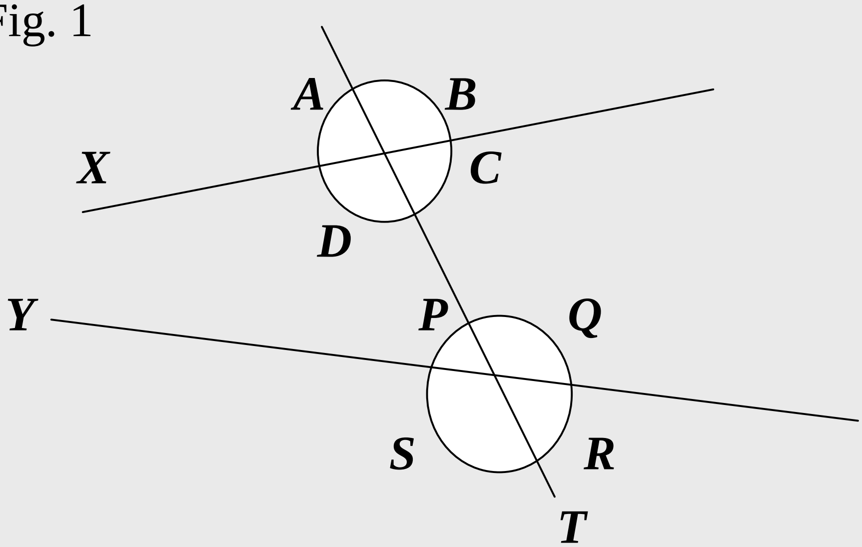
Basically, corresponding angles are on the same side of the lines involved, whereas alternate angles are on the opposite sides of the lines involved. And in either kind, the angles are in a pair, so we take and consider two angles at a time.

When considering angles corresponding or alternate, we consider three lines, and one of the three is called the transversal, which intersects or crosses the other two lines.

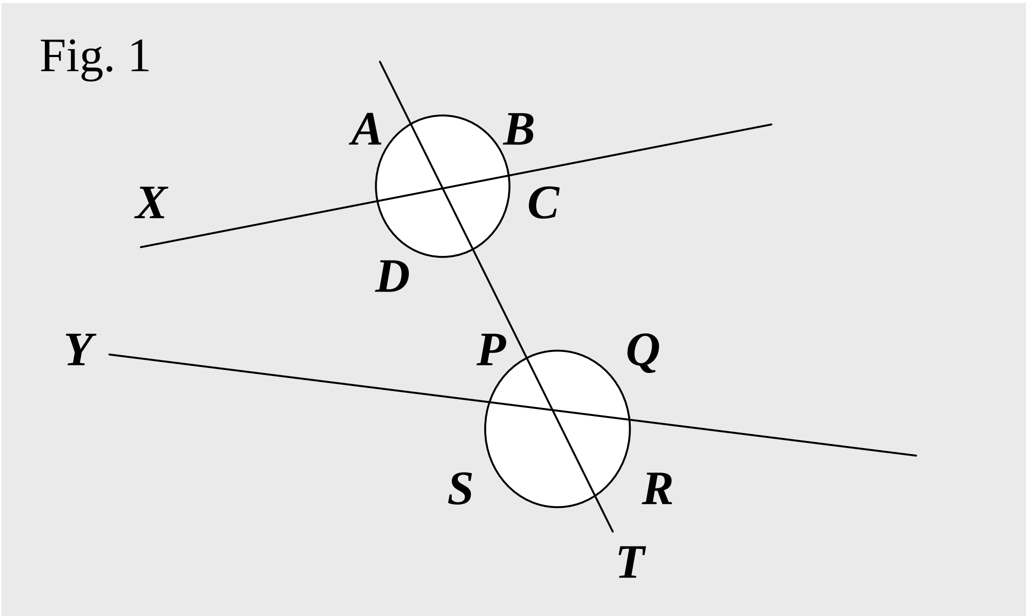
Let's now begin with corresponding angles.
We will be working with the eight angles in the figure below, and the angles are as follows.

$\angle A$, $\angle B$, $\angle C$, $\angle D$, $\angle P$, $\angle Q$, $\angle R$, and $\angle S$.

Fig. 1



Corresponding angles are in a pair, and both angles are on the same side of the transversal and on the same sides of the two lines crossed by the transversal, which is the line ***T*** in the figure below.

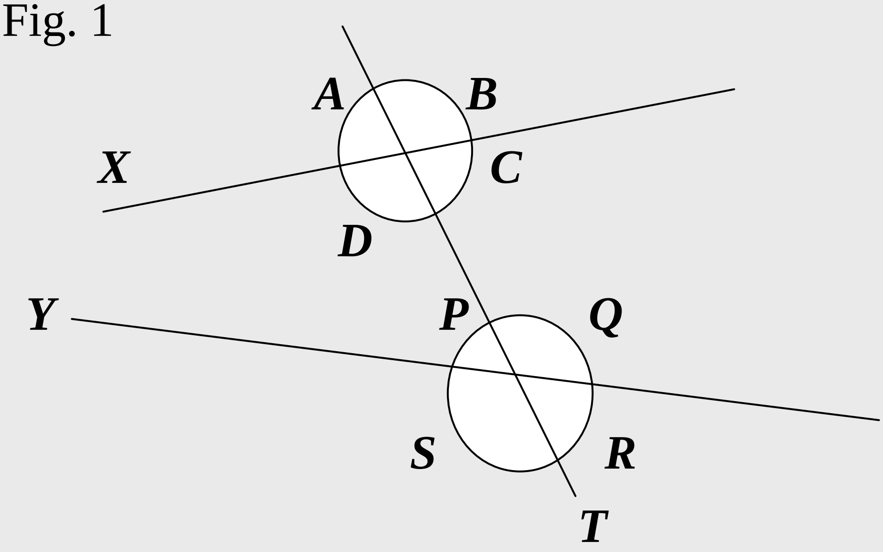


If therefore, for instance, one angle is on the ***right*** of the transversal ***T*** and ***above*** one of the two lines crossed by ***T***, the other angle is also, on the ***right*** of ***T*** and ***above*** the other line crossed by ***T***.

So in the figure above, ∠***B*** and ∠***Q*** are corresponding angles. And the same is true of these pairs, too: (∠***A*** and ∠***P***), (∠***C*** and ∠***R***), and (∠***D*** and ∠***S***). Why, though?

It's because as shown below, both angles $\angle A$ and $\angle P$ are on the left of the transversal T, $\angle A$ is above the line X, and $\angle P$ is above the line Y, so both angles are above the two lines crossed by the transversal.

Fig. 1



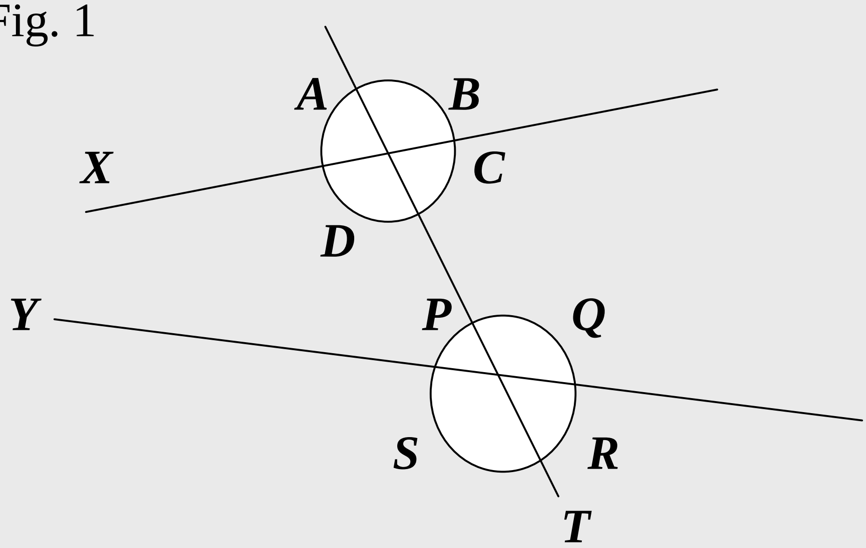
So both are on the ***same*** side of the transversal and on the ***same*** sides of the two lines crossed by the transversal. Thus, the two, $\angle A$ and $\angle P$ are corresponding angles.

And the same is true of the pairs below, too.
($\angle C$ and $\angle R$), and ($\angle D$ and $\angle S$).

What then about ($\angle D$ and $\angle P$)?

They are not corresponding angles.

Fig. 1

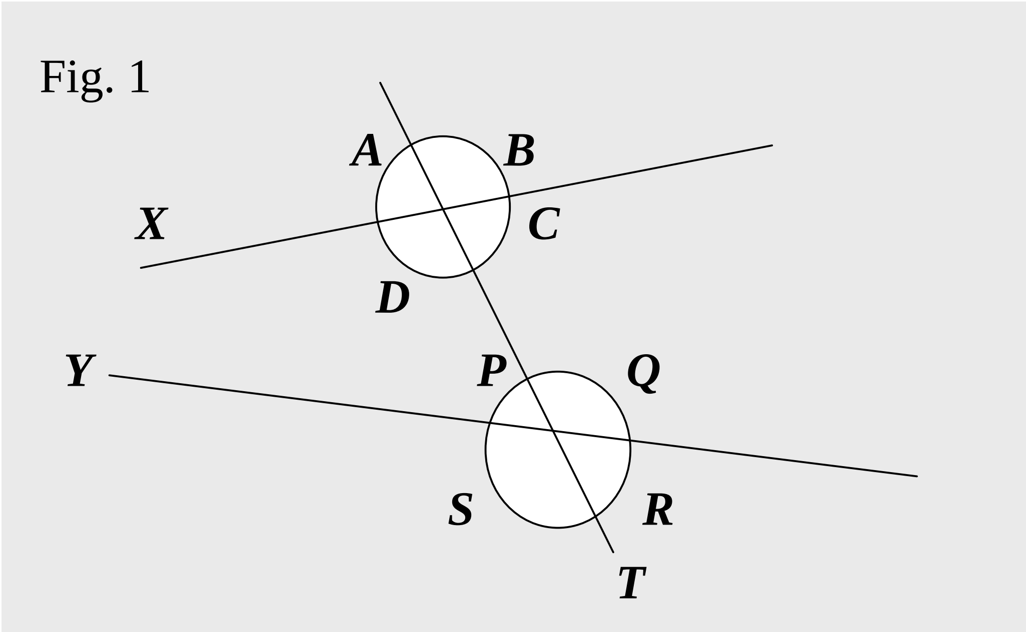


It's because $\angle D$ is below one line crossed by T, but $\angle P$ is above the other line crossed by T. Though both angles are thus, on the same side of the transversal, they are not on the same sides of the two lines crossed by the transversal.

Corresponding angles are on the same side of the transversal and on the same sides of the lines crossed by the transversal. So $\angle D$ and $\angle S$ are corresponding angles, and the same is true of $\angle C$ and $\angle R$, too.

What then about ($\angle B$ and $\angle P$)?

They are not corresponding angles, either.



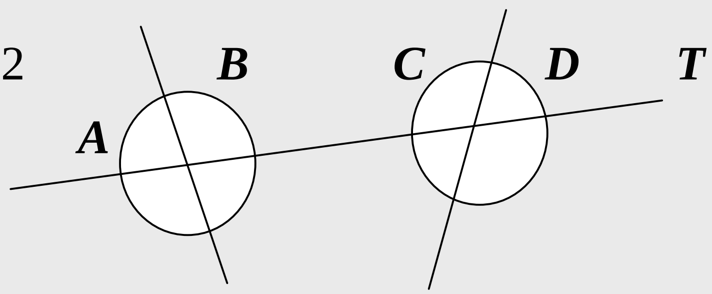
It's because $\angle B$ is on the right of the transversal T, but $\angle P$ is on the left of the transversal T. That is, though both angles are on the same sides of the lines crossed by the transversal T, they are not on the same side of T.

And, for now, the bottom line is as follows.

Corresponding angles are in a pair, and are on the same sides of all the three lines involved, the line called the transversal and the two lines crossed by the transversal.

What then about the two angles, $\angle \boldsymbol{B}$ and $\angle \boldsymbol{D}$ shown in the figure below?

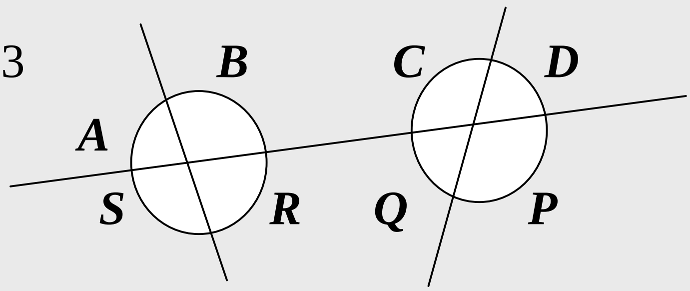
Fig. 2



Are the two angles, $\angle \boldsymbol{B}$ and $\angle \boldsymbol{D}$ corresponding angles?

Yes, they are.

Fig. 3



It’s because of the two facts as follows.

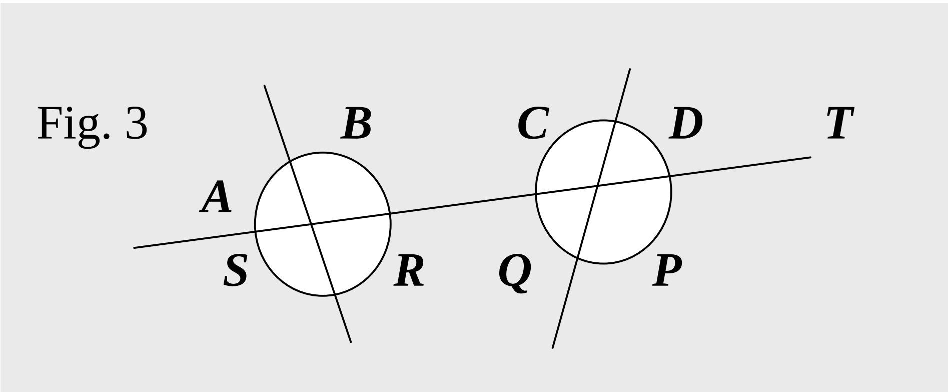
First off, both $\angle B$ and $\angle D$ are on the right of the two lines crossed by the transversal T, so both angles are on the same sides of the two lines crossed by the transversal.

And next, both angles are above the transversal T, that is, both are on the same side of the transversal.

What then about ($\angle S$ and $\angle P$)?

They are not, because as shown below, $\angle \boldsymbol{S}$ is on the left of one line crossed by the transversal $\boldsymbol{T}$, but $\angle \boldsymbol{P}$ is on the right of the other line crossed by the line $\boldsymbol{T}$.

That is to say that both angles are not on the same sides of the two lines crossed by the transversal.



And again, corresponding angles are in a pair, on the same side of the transversal, and also, on the same sides of the two lines crossed by the transversal.

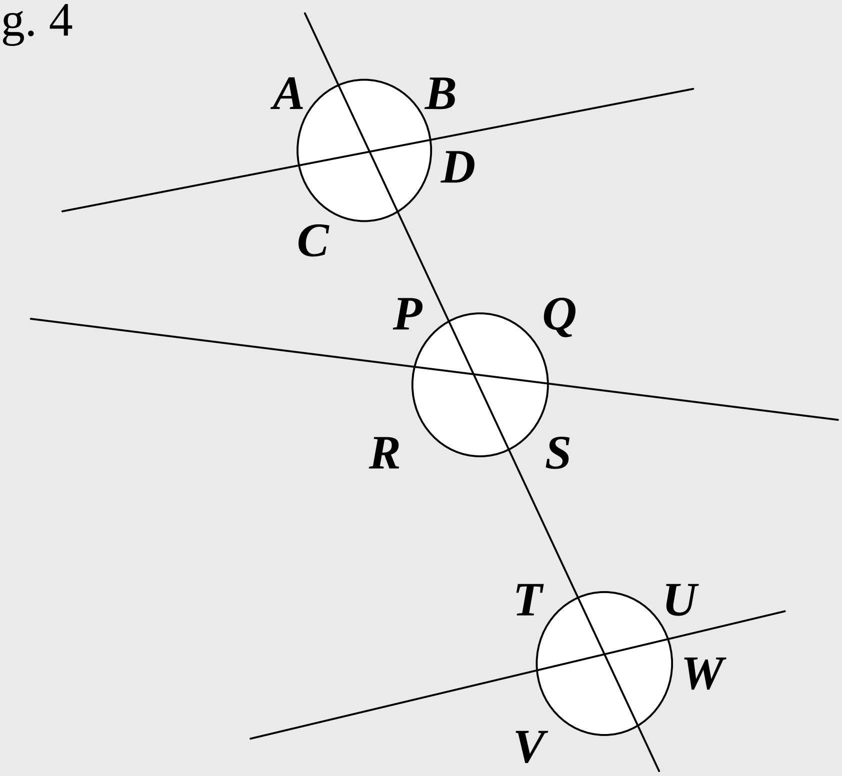
So for another instance, $\angle \boldsymbol{A}$ and $\angle \boldsymbol{C}$ are corresponding angles.

And the same is true of ($\angle \boldsymbol{S}$ and $\angle \boldsymbol{Q}$) and ($\angle \boldsymbol{R}$ and $\angle \boldsymbol{P}$).

And since corresponding angles are in a pair, when finding corresponding angles, we consider two lines at a time crossed by the transversal.

So for instance, in the figure below, $\angle A$ and $\angle P$ are corresponding angles, $\angle P$ and $\angle T$ are corresponding angles, and $\angle A$ and $\angle T$ are corresponding angles.

Fig. 4



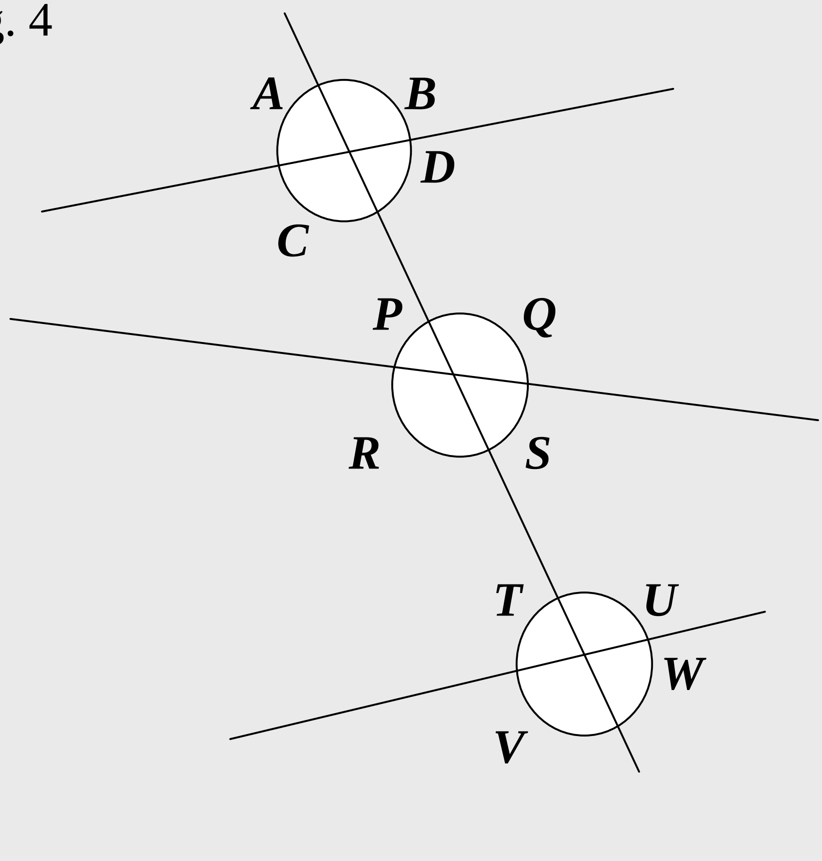
And the same is true of all the pairs below, too.

(∠***C*** and ∠***R***), (∠***C*** and ∠***V***), (∠***R*** and ∠***V***),

(∠***B*** and ∠***Q***), (∠***B*** and ∠***U***), (∠***Q*** and ∠***U***),

(∠***D*** and ∠***S***), (∠***D*** and ∠***W***), and (∠***S*** and ∠***W***).

Fig. 4

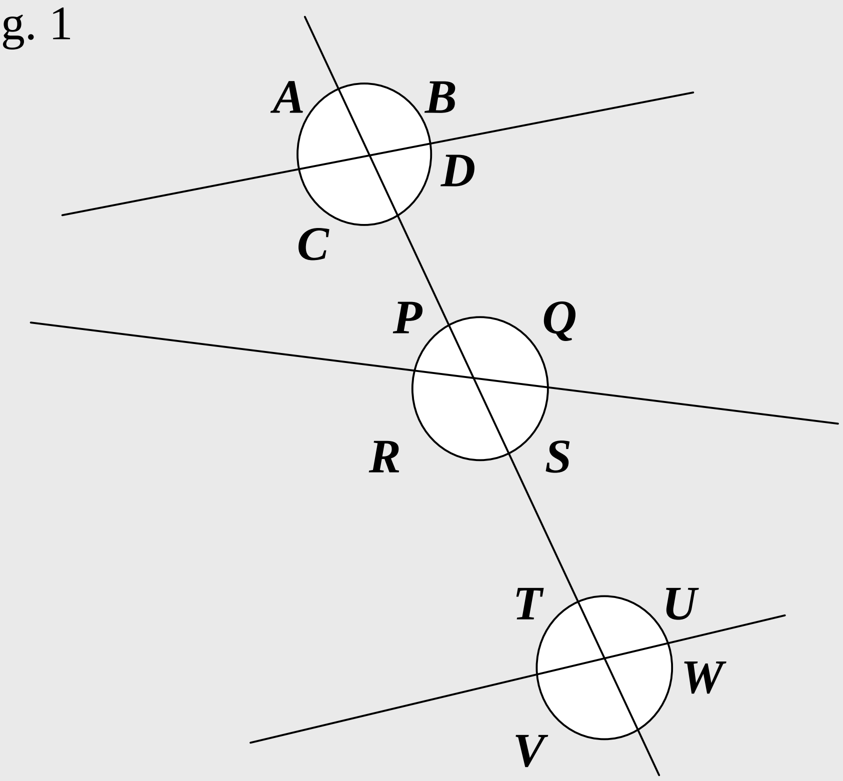


And the bottom line is as follows.

Corresponding angles are in a pair, on the same side of the transversal, and also, on the same sides of the two lines crossed by the transversal.

So for instance, in the figure below, $\angle A$ and $\angle P$ are corresponding angles, $\angle P$ and $\angle T$ are corresponding angles, and $\angle A$ and $\angle T$ are corresponding angles.

Fig. 1



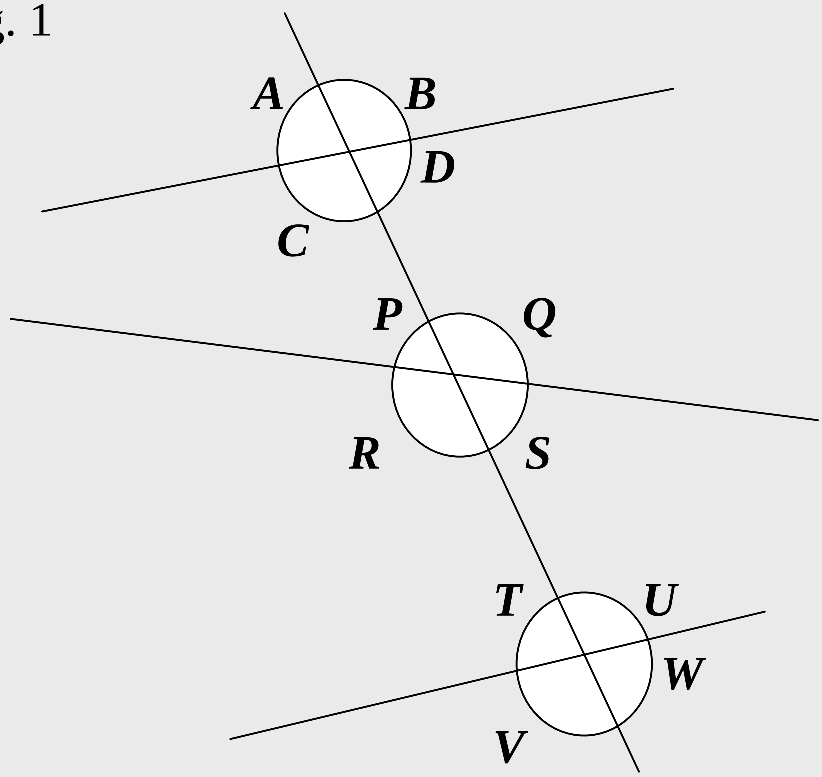
And the same is true of all the pairs below, too.

(∠***C*** and ∠***R***),   (∠***C*** and ∠***V***),   (∠***R*** and ∠***V***),

(∠***B*** and ∠***Q***),   (∠***B*** and ∠***U***),   (∠***Q*** and ∠***U***),

(∠***D*** and ∠***S***),   (∠***D*** and ∠***W***), and (∠***S*** and ∠***W***).

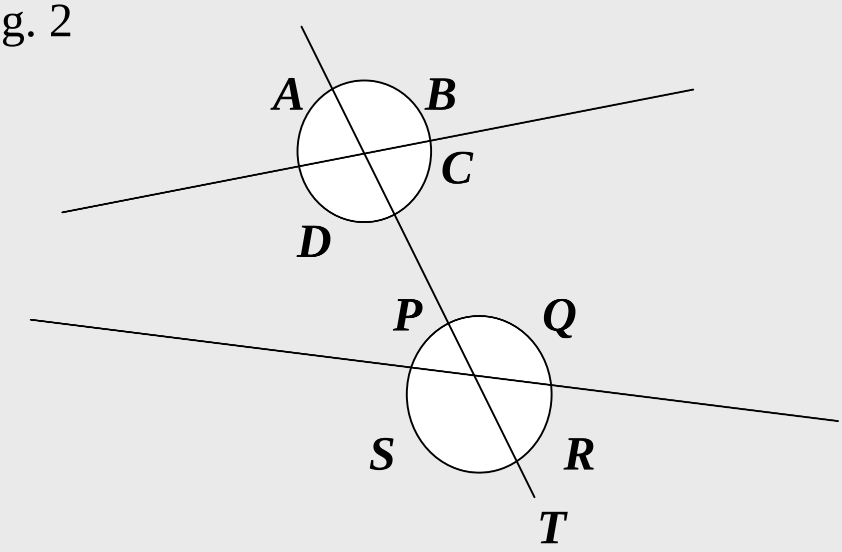
Fig. 1



What then about alternate angles?

Alternate angles are in a pair, on the opposite sides of the transversal, and also, on the opposite sides of the lines crossed by the transversal, the line ***T*** in the figure below.

Fig. 2



If one angle is on the right of the transversal ***T***, and is below one of two lines crossed by ***T***, the other angle is on the left of ***T***, and is above the other line crossed by ***T***.

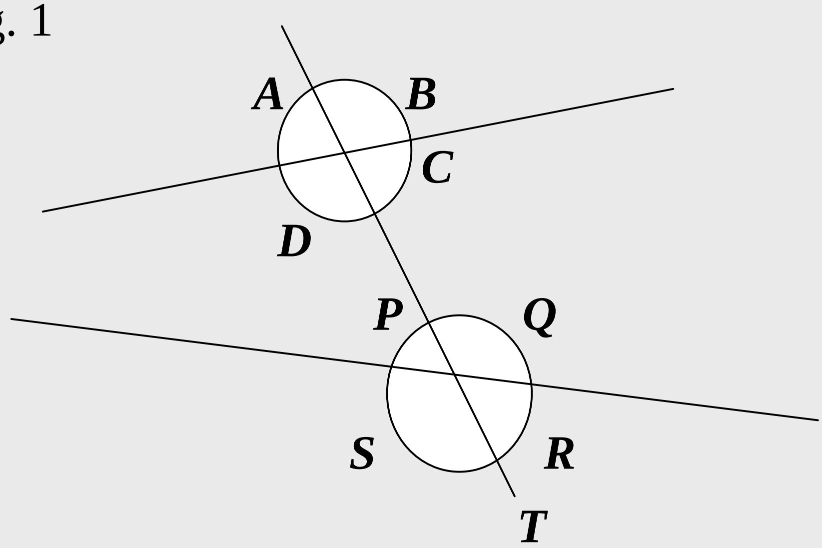
So for instance, in the figure above, $\angle C$ and $\angle P$ are alternate angles. And the same is true of the pairs as follows: ($\angle D$ and $\angle Q$), ($\angle B$ and $\angle S$), and ($\angle A$ and $\angle R$).

And more examples will be covered in the next section.

9.7. Angles and Lines 7

Alternate angles are in a pair, on the opposite sides of the transversal, and also, on the opposite sides of the two lines crossed by the transversal, the line T in the figure below.

Fig. 1



If one angle is on the ***right*** of the transversal T, and is ***below*** one of two lines crossed by T, the other angle is on the ***left*** of T, and is ***above*** the other line crossed by T.

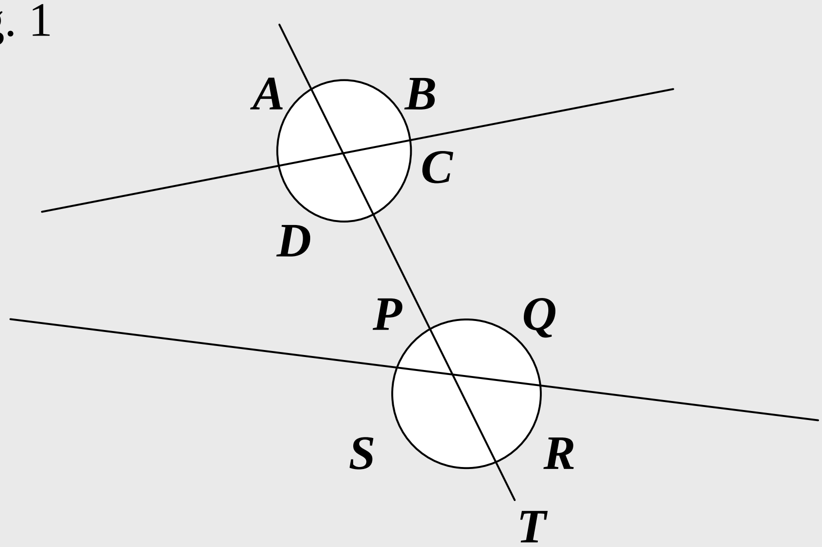
So for instance, in the figure above, $\angle C$ and $\angle P$ are alternate angles. And the same is true of the pairs as follows: ($\angle D$ and $\angle Q$), ($\angle B$ and $\angle S$), and ($\angle A$ and $\angle R$).

What then about $\angle R$ and $\angle D$? Are they alternate angles?

They are not. It’s because both angles are below the two lines; ∠***R*** is below one of the two lines crossed by ***T***, and also, ∠***D*** is below the other line crossed by ***T***.

That is to say that the two angles are not on the opposite sides of the two lines crossed by the transversal.

Fig. 1

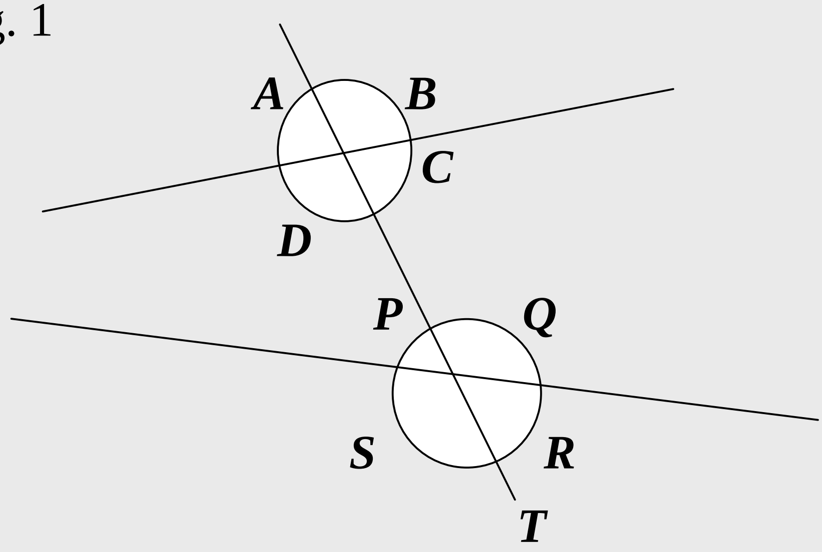


By the same token, ∠***A*** and ∠***Q*** are not alternate angles. It’s because both are above the two lines crossed by the transversal, and thus, on the same sides of the two lines, that is, not on the opposite sides of the two.

What then about ∠***B*** and ∠***P***?

They are not alternate angles, either.

Fig. 1



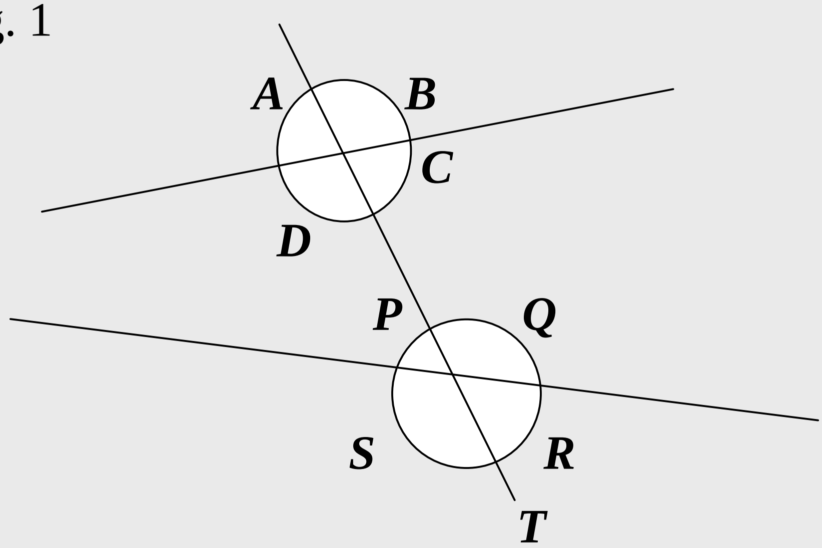
It's because $\angle B$ and $\angle P$ both are above the two lines crossed by the transversal, that is, on the same sides of the two lines, and not on the opposite sides of the two.

Alternate angles are on the opposite sides of the transversal, and also, are on the opposite sides of the two lines crossed by the transversal. So $\angle B$ and $\angle S$ are alternate angles, and the same is true of $\angle C$ and $\angle P$.

What then about $\angle D$ and $\angle P$?

They are not alternate angles, either.

Fig. 1



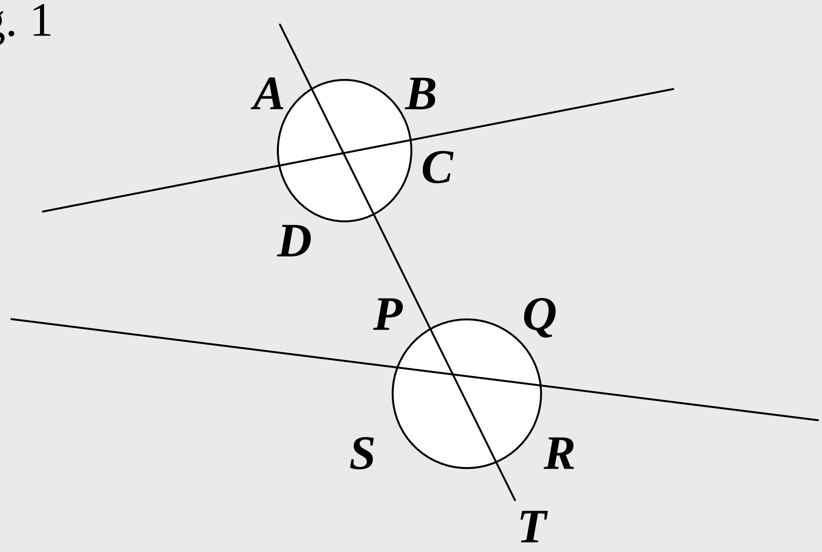
It’s because both ∠***D*** and ∠***P*** are on the left of the transversal ***T***, that is, on the same side of the transversal , and not on the opposite sides of the transversal.

And again, alternate angles are on the opposite sides of the transversal, and also, are on the opposite sides of the two lines crossed by the transversal. So ∠***D*** and ∠***Q*** are alternate angles.

What then about ∠***A*** and ∠***R***?

They are alternate angles.

Fig. 1

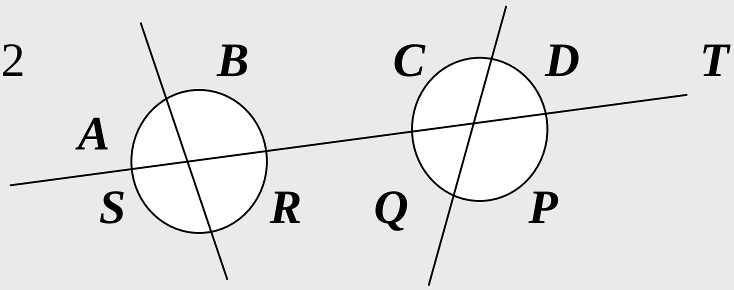


It's because ∠***A*** is above one of the two lines crossed by the transversal ***T***, but ∠***R*** is below the other line crossed by the transversal, and also, they are on the opposite sides of the transversal ***T***.

That is to say that the two angles are on the opposite sides of the transversal, and also, are on the opposite sides of the two lines crossed by the transversal.

What then about the two angles, $\angle B$ and $\angle Q$ shown in the figure below?

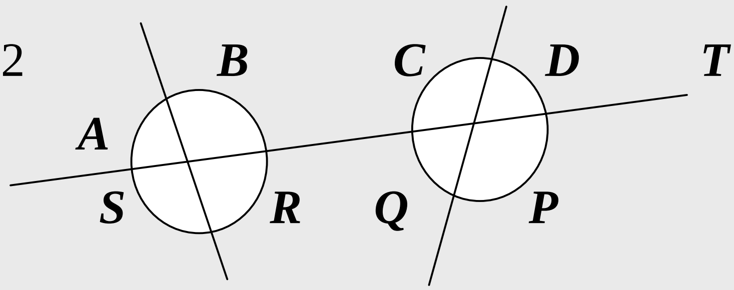
Fig. 2



Are the two angles, $\angle B$ and $\angle Q$ alternate angles?

Yes, they are.

Fig. 2



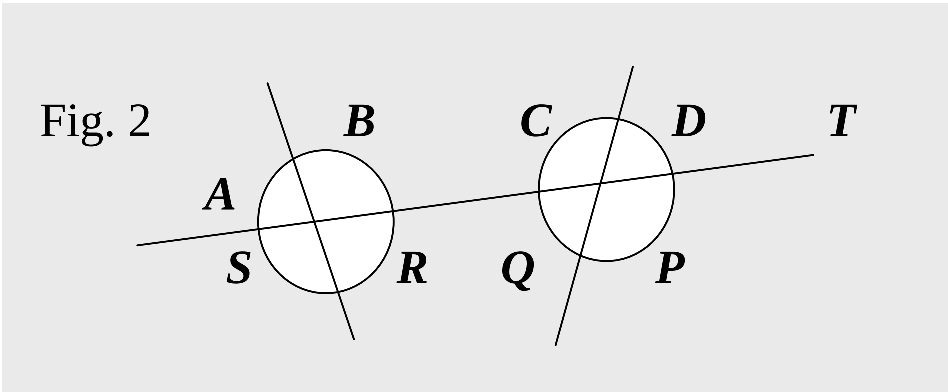
It's because of the two facts as follows.

First off, $\angle \boldsymbol{B}$ is on the right of one of the two lines crossed by the transversal $\boldsymbol{T}$, but $\angle \boldsymbol{Q}$ is on the left of the other line crossed by $\boldsymbol{T}$, that is, they are on the opposite sides of the two lines crossed by the transversal.

And next, $\angle \boldsymbol{B}$ is above the transversal $\boldsymbol{T}$, but $\angle \boldsymbol{Q}$ is below the transversal, that is, they are on the opposite sides of the transversal.

What then about $\angle \boldsymbol{S}$ and $\angle \boldsymbol{C}$?

They are not, because as shown below, both ∠***S*** and ∠***C*** are on the left of the two lines crossed by ***T***. That is, though the two angles are on the opposite sides of the transversal, both are on the same sides of the two lines crossed by the transversal.



And again, alternate angles are in a pair, on the opposite sides of the transversal, and also, on the opposite sides of the two lines crossed by the transversal.

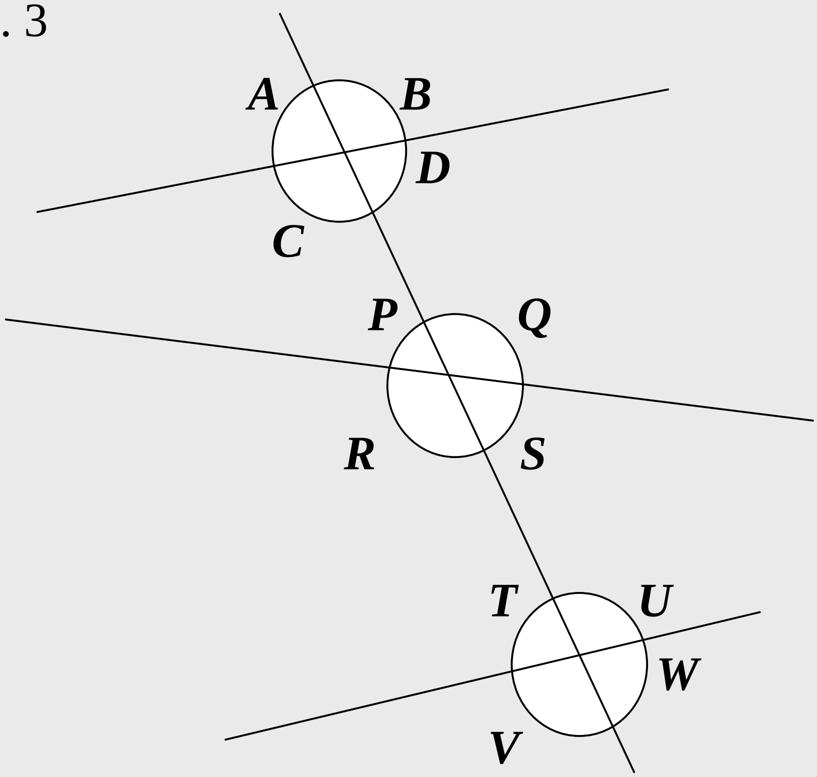
So for another instance, ∠***A*** and ∠***P*** are alternate angles.

And the same is true of (∠***S*** and ∠***D***) and (∠***R*** and ∠***C***).

And as in the case of corresponding angles, since alternate angles are in a pair, when finding alternate angles, we consider at a time two lines crossed by the transversal.

So for instance, in the figure below, $\angle A$ and $\angle S$ are alternate angles, $\angle A$ and $\angle W$ are alternate angles, and $\angle P$ and $\angle W$ are alternate angles.

Fig. 3



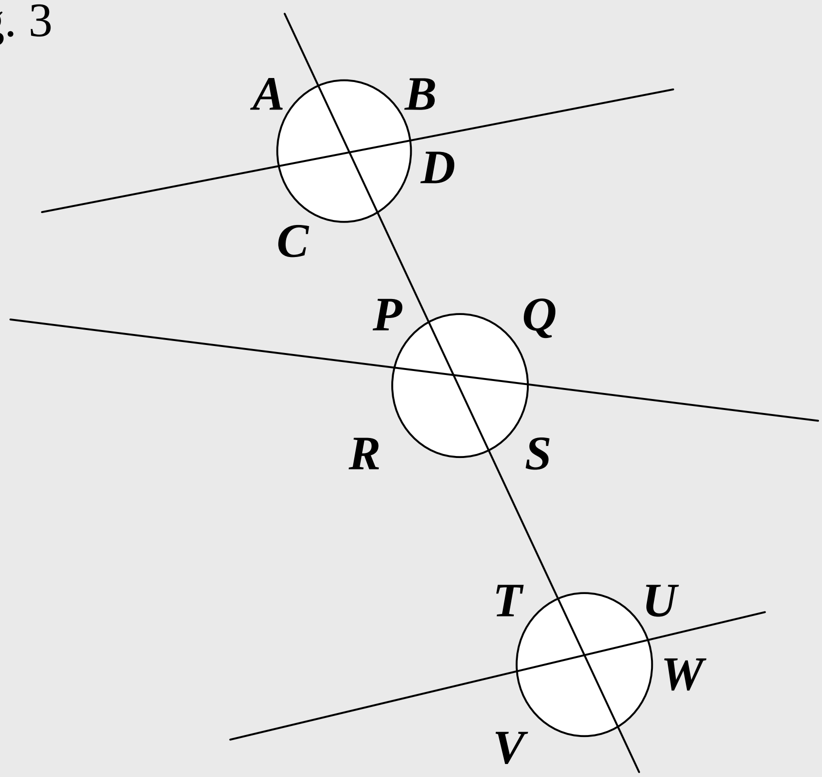
And the same is true of all the pairs below, too.

($\angle C$ and $\angle Q$), ($\angle C$ and $\angle U$), ($\angle R$ and $\angle U$),

($\angle B$ and $\angle R$), ($\angle B$ and $\angle V$), ($\angle Q$ and $\angle V$).

($\angle D$ and $\angle P$), ($\angle D$ and $\angle T$), and ($\angle S$ and $\angle T$).

Fig. 3

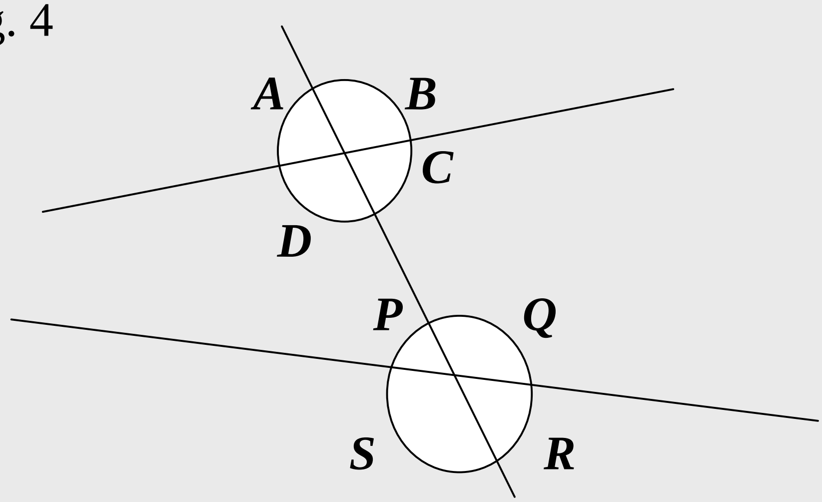


So alternate angles are in a pair, on the opposite sides of the line called the transversal, and also, on the opposite sides of the two lines crossed by the transversal.

If therefore, of two angles, one is above one line crossed by the transversal and on the left of the transversal, and the two are alternate angles, the other is below the other line crossed by the transversal and on the right of the transversal.

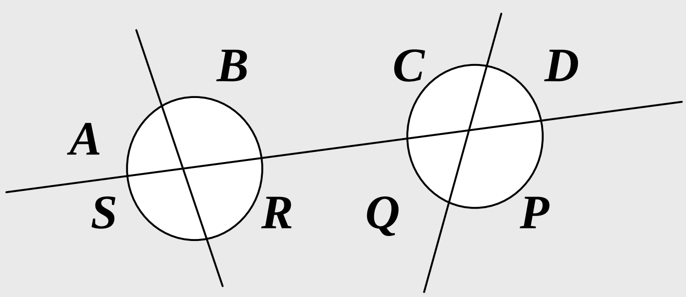
So the two angles $\angle A$ and $\angle R$ below are alternate angles.

Fig. 4



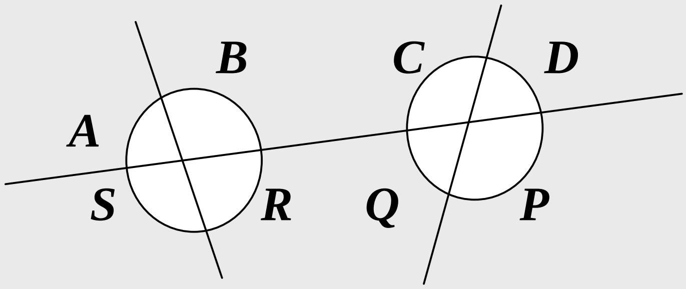
What then, about $\angle R$ and $\angle C$ in the figure below?

Fig. 5



They are alternate angles, too.

Fig. 5

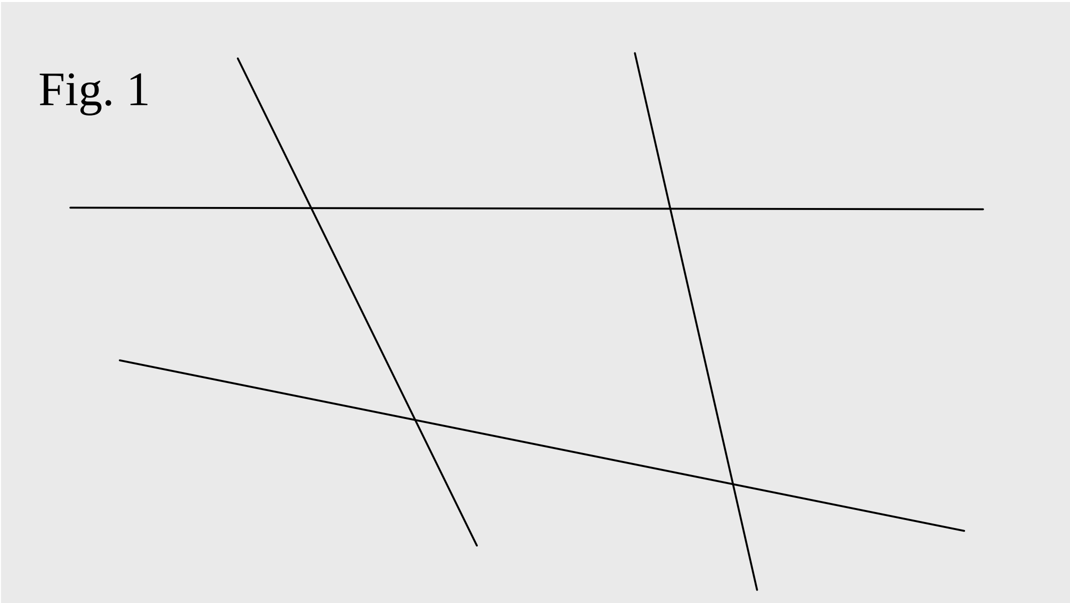


If therefore, of two angles, one is on the right of one line crossed by the transversal and below the transversal, and the two are alternate angles, the other is on the left of the other line crossed by the transversal and above the transversal. So in the figure above, the two angles $\angle \boldsymbol{R}$ and $\angle \boldsymbol{C}$ are alternate angles.

And in the next section, we will cover more examples.

9.8. Angles and Lines 8

In the figure below, there are two pairs of lines, and the two in each pair are neither intersecting nor parallel.



Note that they are lines, so their lengths are infinite, but in this case, we consider only the parts of the lines shown in the figure above, that is, we consider line segments, i.e., what's shown in the figure are all we have to work with.

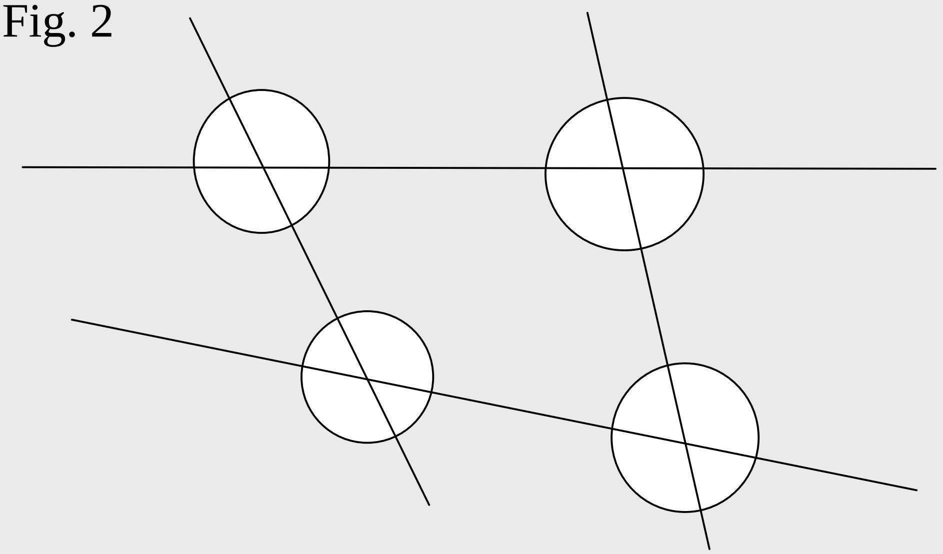
Now, the four line segments make angles with each other.

How many different angles then, do they make?

In other words, how many different angles do the four line segments make with each other?

As shown in the figure below, the four line segments can make ***up to eight different angles*** with each other.

Fig. 2



Why eight?　Why not sixteen?

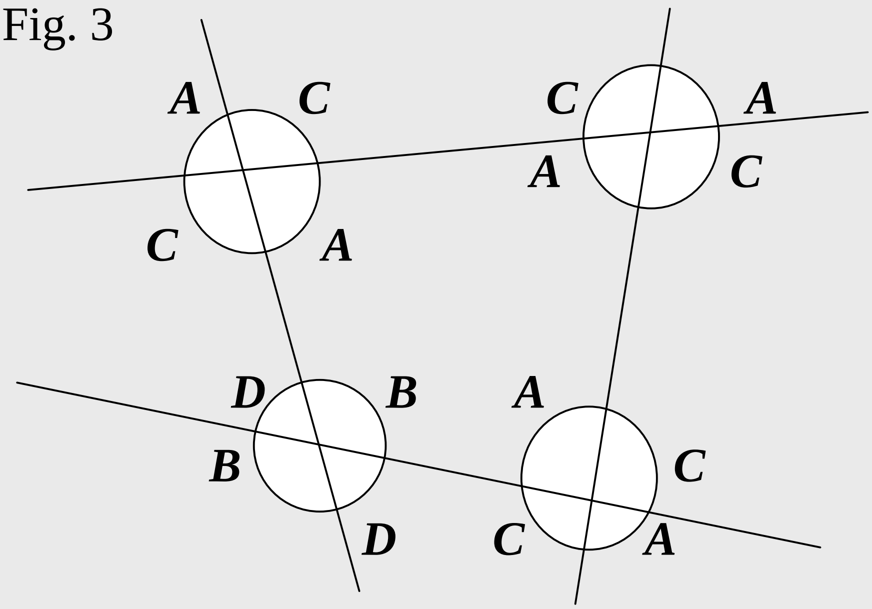
There are sixteen, of course, but they aren't sixteen different angles. The sixteen are actually eight pairs of vertical angles.

Vertical angles happen in a pair, and the two angles in each pair are equal. So eight different angles can be made. More specifically, ***up to*** eight different angles. Thus, there can be ***up to eight different pairs of vertical angles***.

Why ***up to eight***, though?　Why not just eight?

It's because the two line segments in a pair can meet a line segment in the other pair at the same angles as shown below.

Fig. 3



So in the figure above, we can see four different angles.

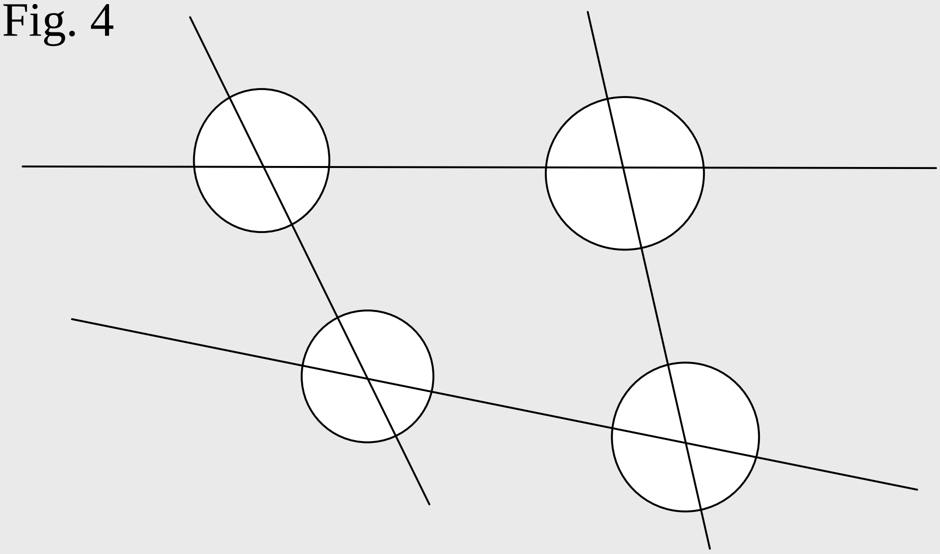
The number of different angles depends on how the lines are positioned.

So as a practice, you may want to try other possible cases where the lines can be positioned so that different number of angles can be made.

Now, in the figure below, we are going to pair up the angles. Paring up the angles though, we don't put vertical angles in each pair. When paring up, we put together two angles the way as follows.

In each of some pairs, the two are called corresponding angles, and in each of the other pairs, the two are called alternate angles.

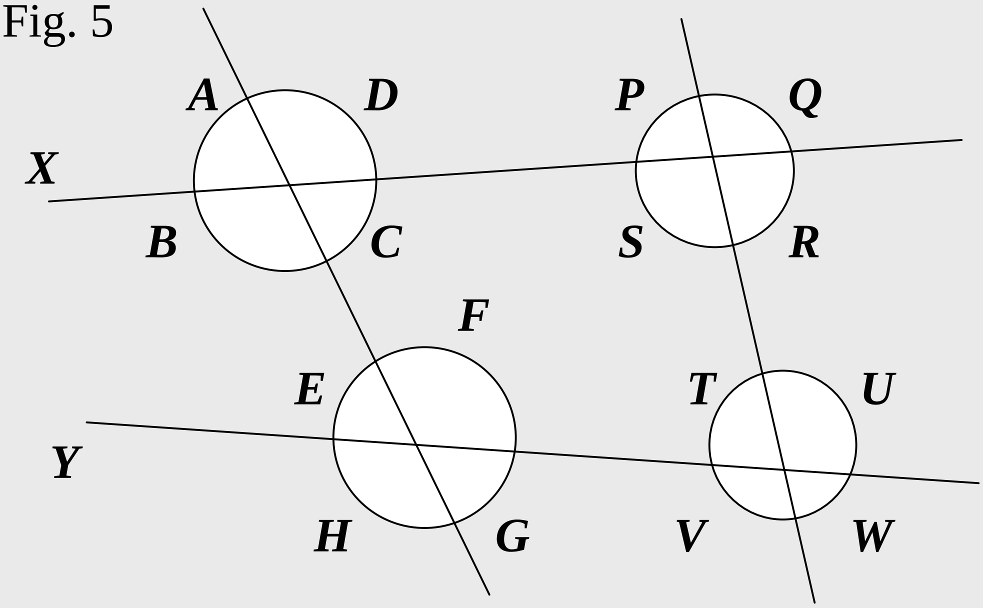
Fig. 4



Basically, corresponding angles are on the same sides of the lines involved, whereas alternate angles are on the opposite sides of the lines involved.

Note that no two lines need to be parallel in the figure below.
And we can pair up the angles the way as follows.

Fig. 5



For instance, $\angle A$ and $\angle E$ are corresponding angles.
And the same is true of the pairs as follows.

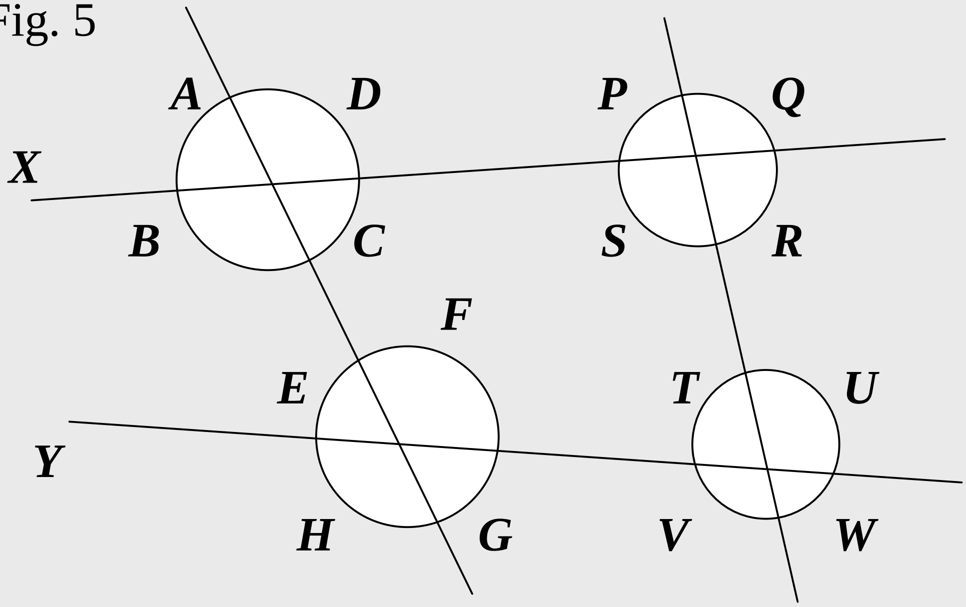
($\angle B$ and $\angle H$), ($\angle D$ and $\angle F$), ($\angle C$ and $\angle G$),

($\angle P$ and $\angle T$), ($\angle S$ and $\angle V$), ($\angle Q$ and $\angle U$),

($\angle R$ and $\angle W$), ($\angle A$ and $\angle P$), ($\angle D$ and $\angle Q$), and so on.

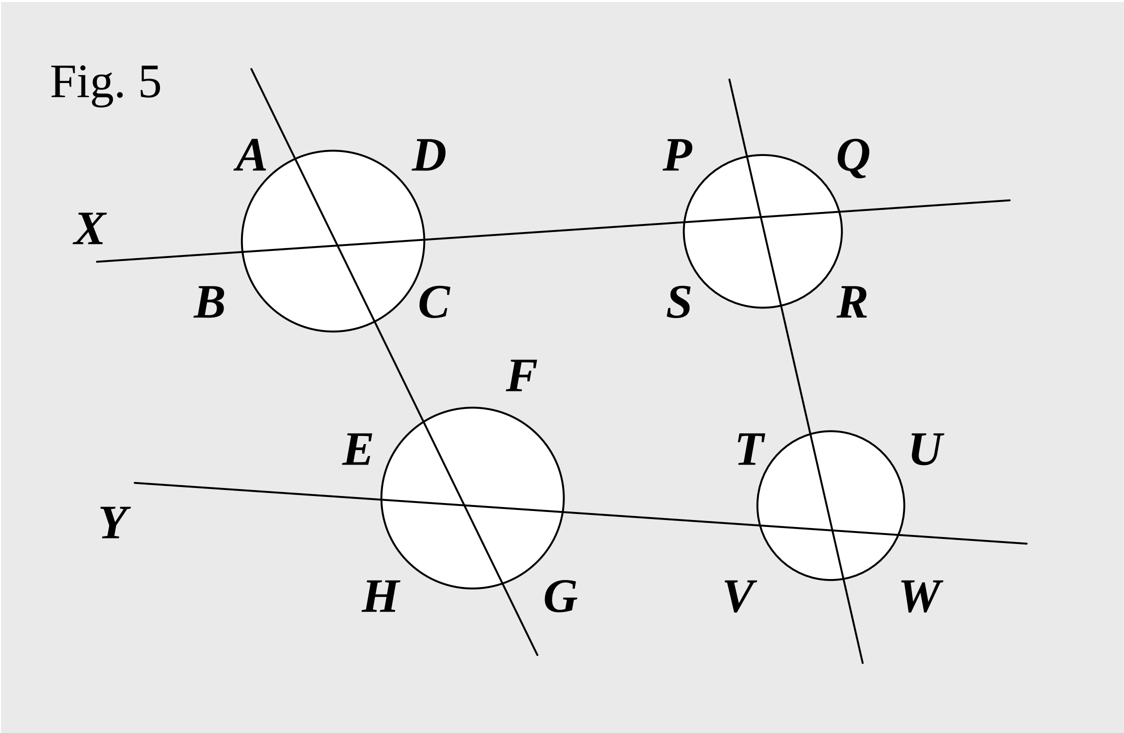
What then, about (∠***B*** and ∠***V***)?

Fig. 5



Are the two angles, ∠***B*** and ∠***V*** corresponding angles?

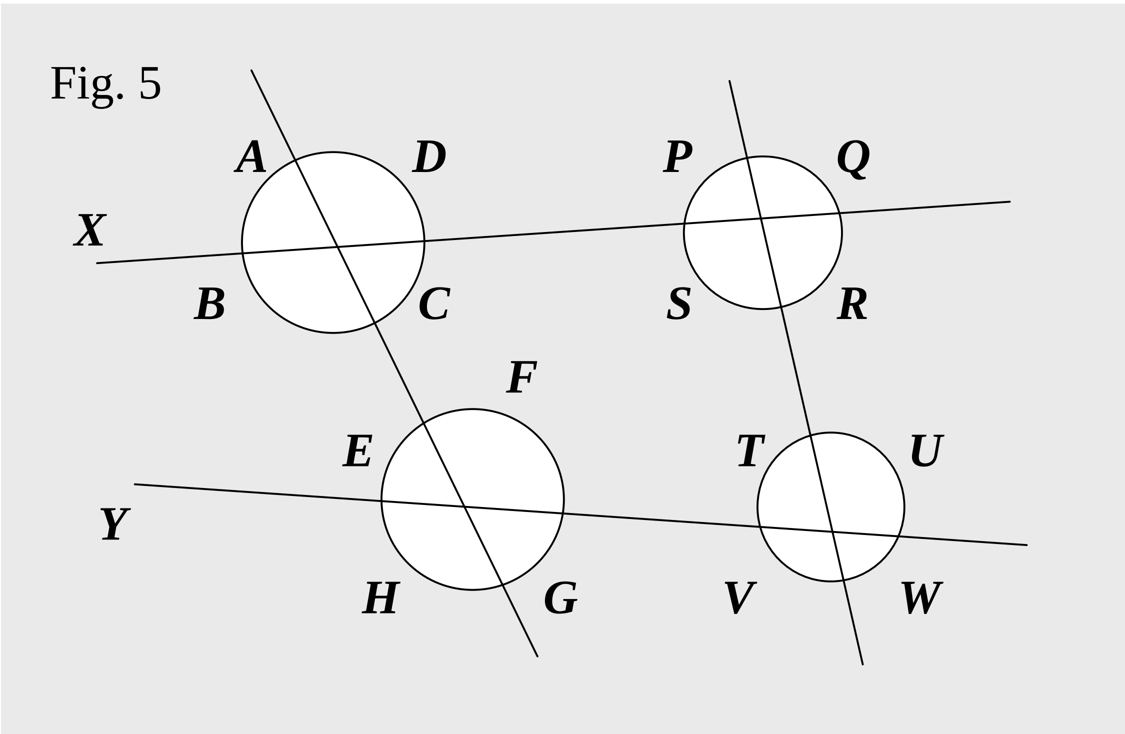
They are not. Corresponding angles are not only on the same sides of two lines crossed by the ***transversal***, but on the same side of the ***transversal*** crossing the two lines.



And the ***transversal*** matters. With no transversal, no two angles can be considered to be corresponding or alternate.

So we need to note that in either case, that is, whether corresponding or alternate, the two angles in a pair need to share a line called the ***transversal***, which is the reason that the two angles $\angle B$ and $\angle V$ cannot be corresponding angles. The two can be neither corresponding nor alternate.

In the case of the two angles, ∠***B*** and ∠***V***, there is no transversal, which means, the two angles don't share a line that can be called the transversal. So two cannot be corresponding angles.



So for instance, ∠***A*** and ∠***T*** are not corresponding angles, whereas ∠***A*** and ∠***E*** are. It's because ∠***A*** and ∠***T*** don't share a line that can be called the transversal, whereas ∠***A*** and ∠***E*** do and are on the same sides of the lines involved.

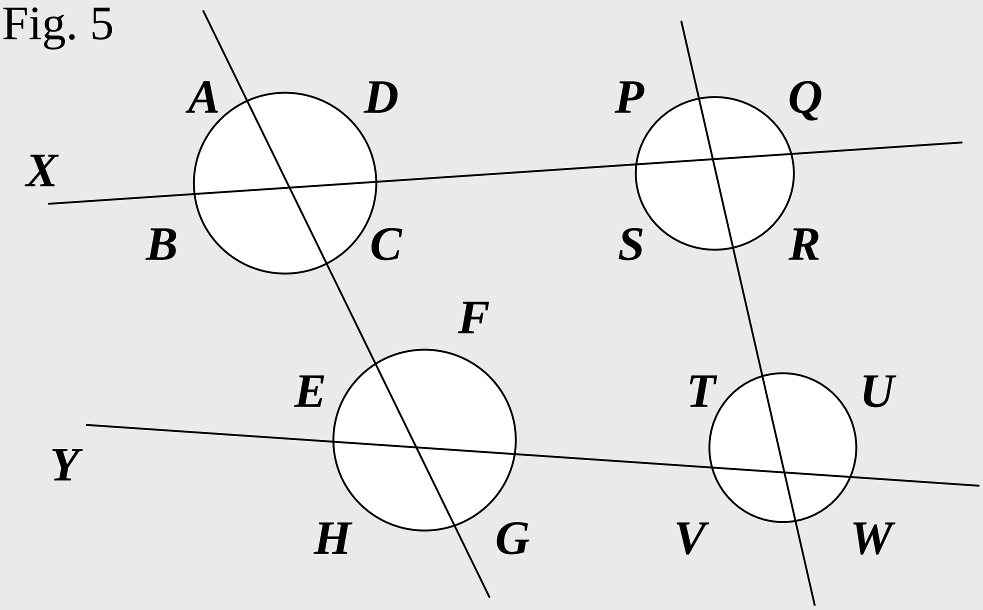
What then about the alternate angles?

The same is true of alternate angles, too.

So the two angles in a pair need to share first a line that can be called the transversal, and then, follow the rule as follows.

Alternate angles are on the opposite sides of the transversal crossing two lines, as well as on the opposite sides of the two lines crossed by the transversal.

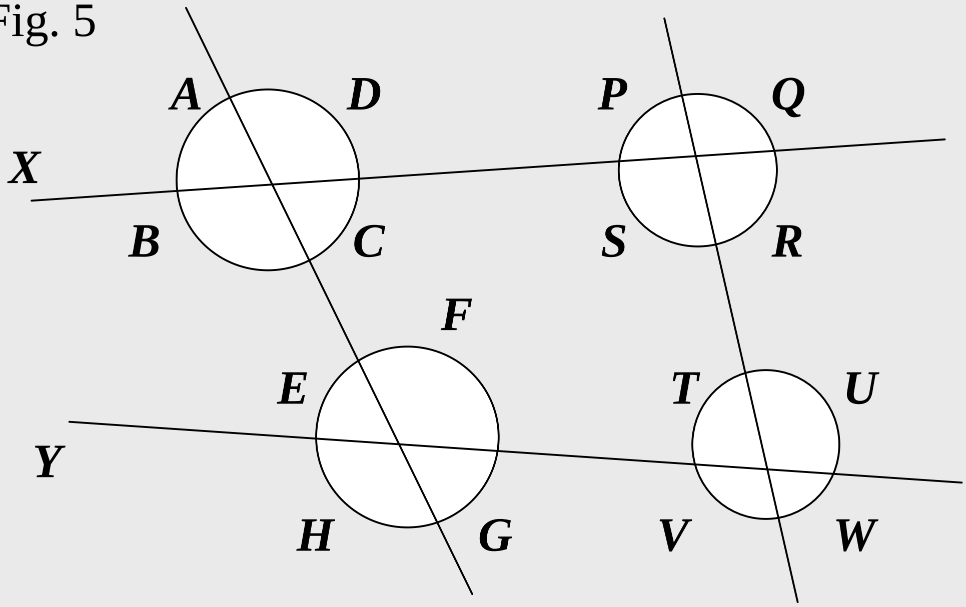
Fig. 5



So for instance, $\angle C$ and $\angle T$ are not alternate angles, whereas $\angle C$ and $\angle E$ are. Why?

It’s because $\angle C$ and $\angle T$ don’t share a line that can be called the transversal, whereas $\angle C$ and $\angle E$ do and are on the opposite sides of the lines involved.

Fig. 5



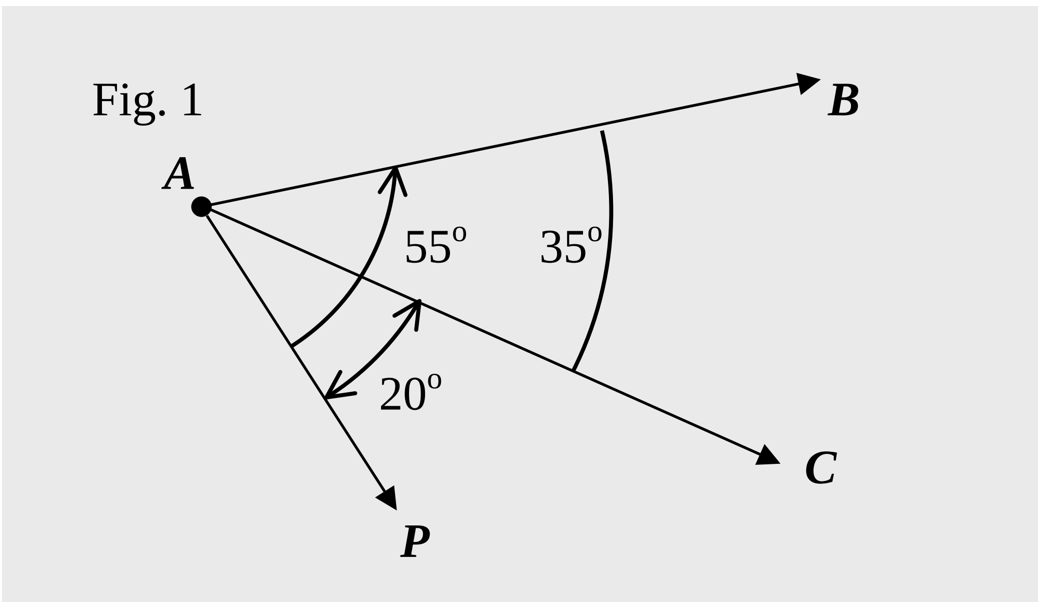
So again, we need to note that in either case, that is, whether corresponding or alternate, the two angles in a pair need to share first a line called the ***transversal***, and then, follow the rule that applies. If corresponding, both angles are on the same sides of the lines involved, and if alternate, both are on the opposite sides of the lines involved.

And we’ll continue this in the next lesson.

9.9. Angles and Lines 9

Before moving on to the next, we may want to take the summery of the basics covered.
So summing up, for now, we have covered the basics on angles and lines as follows.

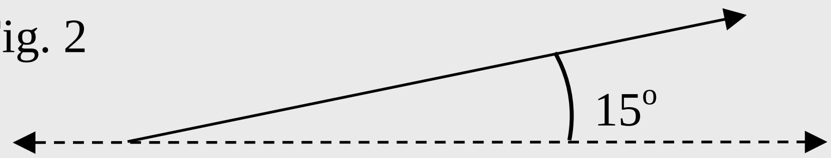
To begin with, doing math, and particularly doing geometry, we often use lines, rays, and line segments, together with angles.



An angle can mean an amount of turning or an amount of difference in direction.

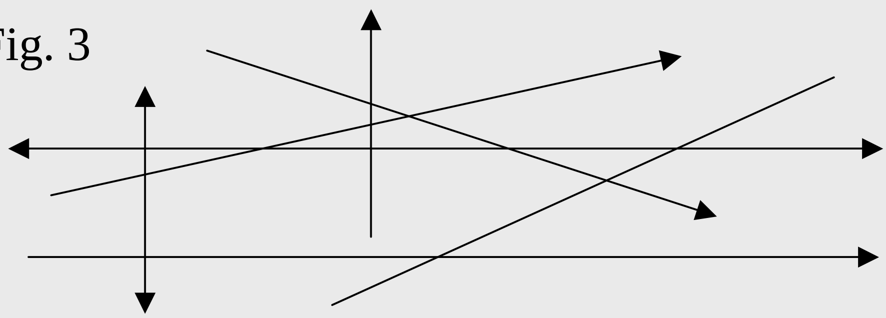
Opening a door, for instance, we need to turn the knob more than 90^o clockwise. And for another instance, the direction of the flight is 15^o against a horizontal line.

Fig. 2



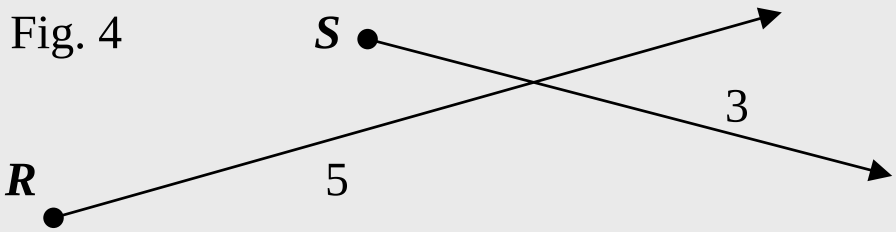
A line is not a material thing in real world but an idea, a math concept. It's like a long thin unbent string with no unevenness. And it has no thickness or width, but lengthens infinitely in two directions opposite of each other.

Fig. 3



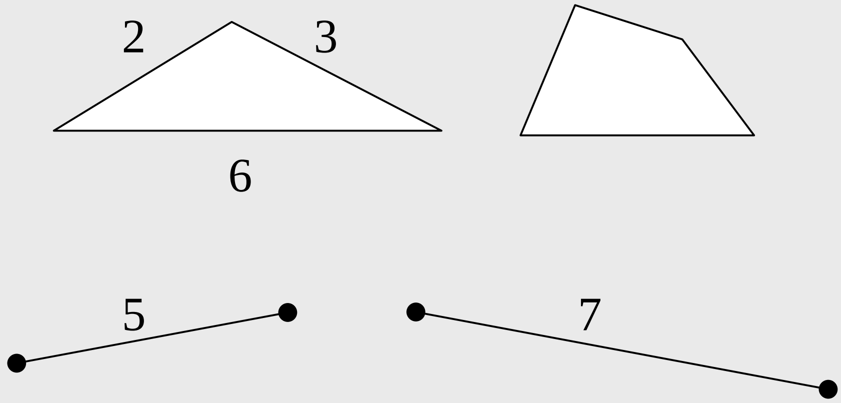
A ray can be called a half line meaning a half of a line, so its length is infinite, but can be finite, too, if we define a ray that way. For instance, we can define ***R*** to be a ray of length 5.

Fig. 4



And a line segment is a part of a line or ray, has a finite length, and is often used as a side of a polygon, and when we connect two points showing the shortest distance between the two points.

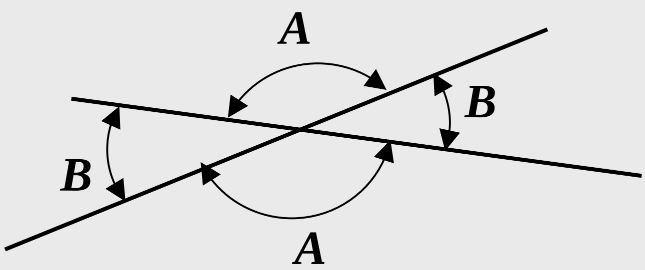
Fig. 5



Next, we have ***vertical angles***, ***two same angles***.

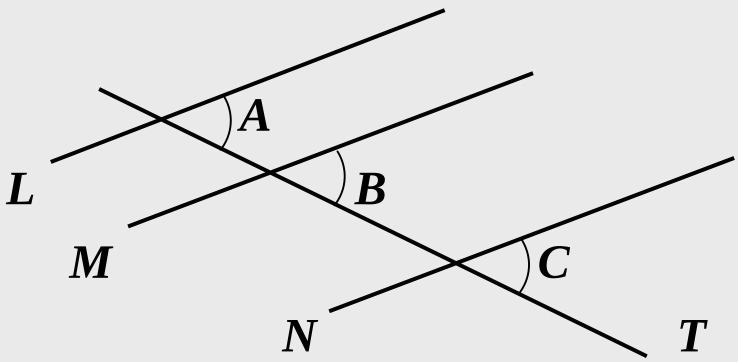
Vertical angles are two angles, ***always the same***.

Fig. 6



Next, if lines are parallel, the transverse makes the same angle with each and every one of the parallel lines.

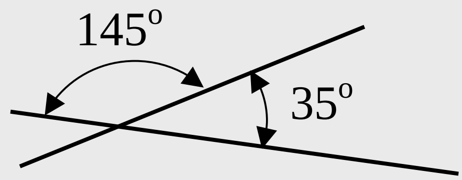
Fig. 7



If we get this: $L \parallel M \parallel N$, we get this: $\angle A = \angle B = \angle C$.

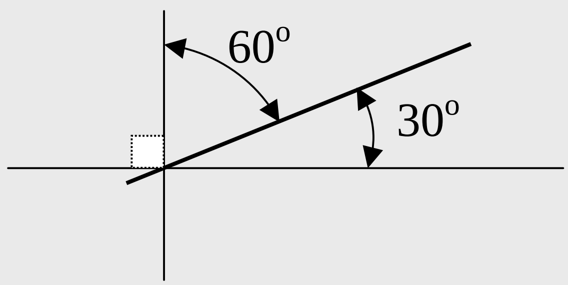
Next, if adding up to 180^o, called ***straight angle***, the two angles can be called ***supplementary*** angles. So for instance, 60^o and 120^o are supplementary angles. We can say that 145^o is supplementary to 35^o.

Fig. 8



And if adding up to 90^o, called ***right angle***, the two angles can be called ***complimentary*** angles. So for instance, the angle complimentary to 30^o is 60^o.

Fig. 9



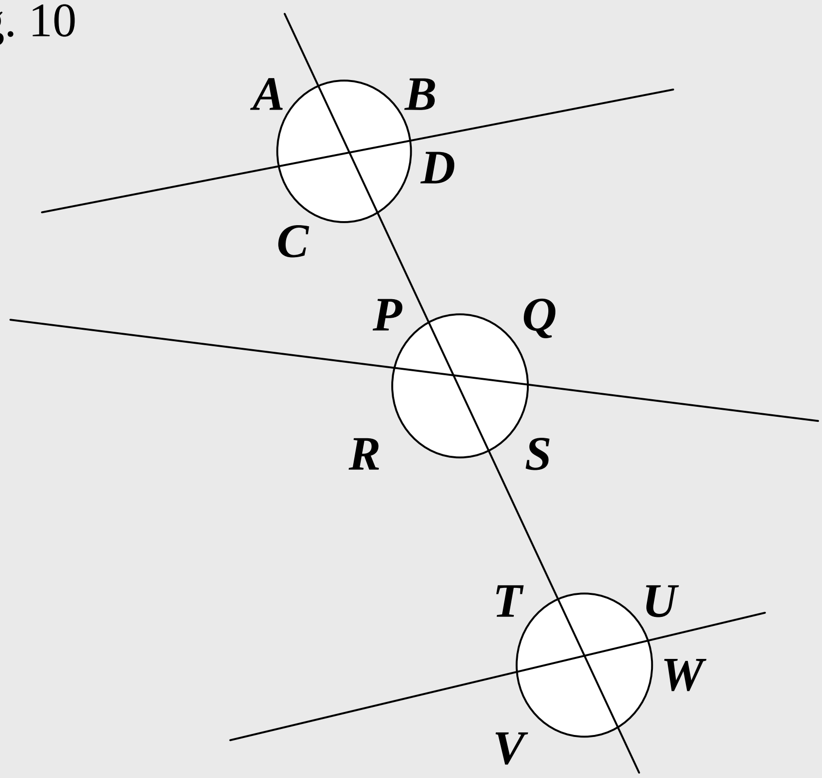
By the way, 360^o is called ***perigon*** or ***round angle***.

And we often use the facts above when solving problems.

Next, corresponding angles are in a pair, on the same side of the transverse line, and on the same sides of the two lines crossed by the transverse line.

So for instance, in the figure below, $\angle \boldsymbol{A}$ and $\angle \boldsymbol{P}$ are corresponding angles, $\angle \boldsymbol{P}$ and $\angle \boldsymbol{T}$ are corresponding angles, and $\angle \boldsymbol{A}$ and $\angle \boldsymbol{T}$ are corresponding angles.

Fig. 10



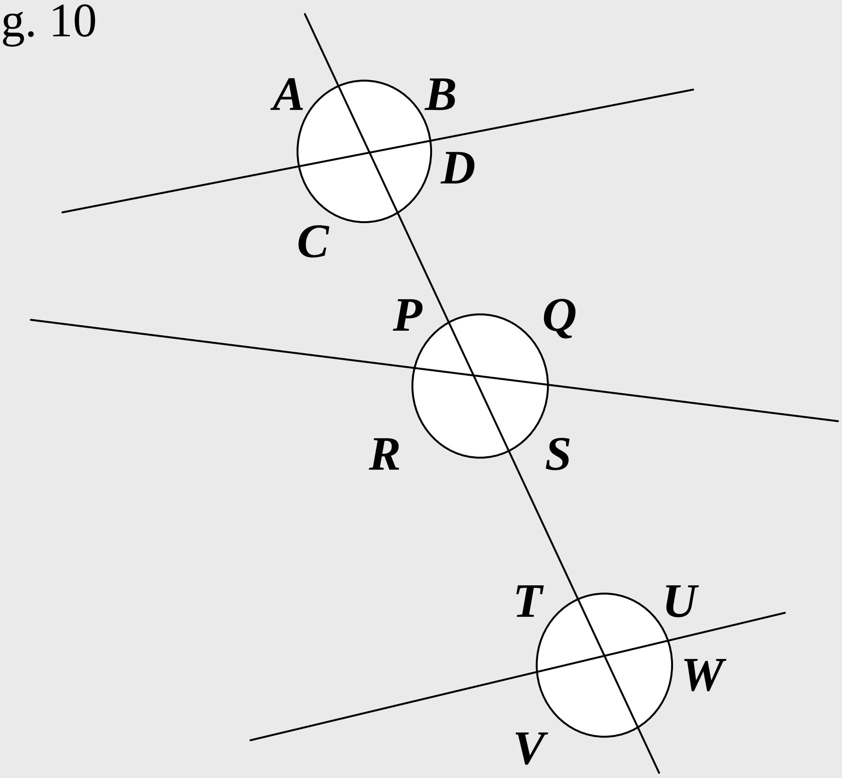
And the same is true of all the pairs as follows, too.

(∠***C*** and ∠***R***), (∠***C*** and ∠***V***), (∠***R*** and ∠***V***),

(∠***B*** and ∠***Q***), (∠***B*** and ∠***U***), (∠***Q*** and ∠***U***),

(∠***D*** and ∠***S***), (∠***D*** and ∠***W***), and (∠***S*** and ∠***W***).

Fig. 10

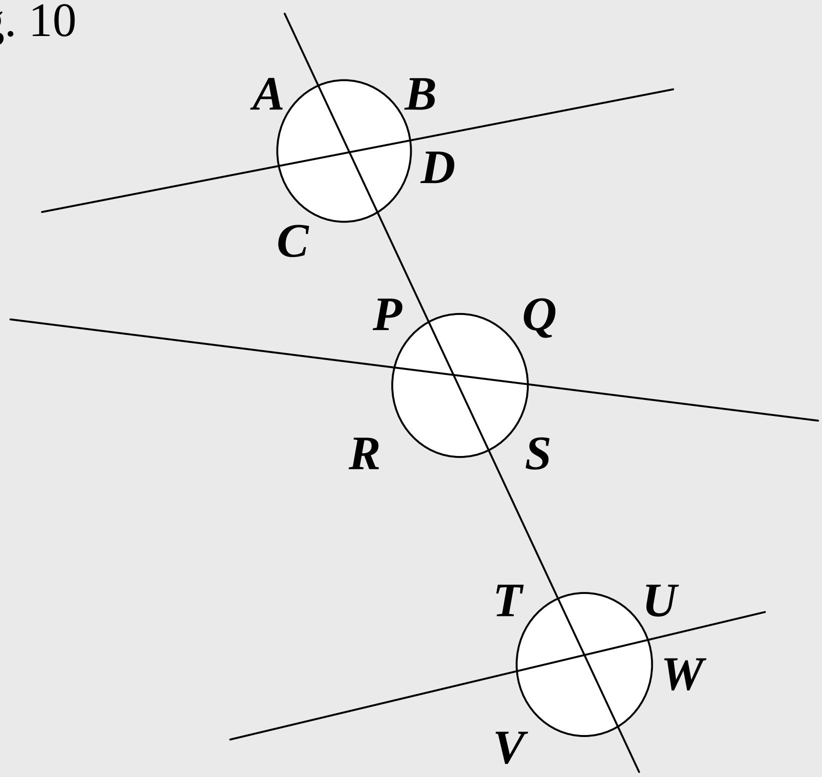


What then about alternate angles?

As in the case of corresponding angles, since alternate angles are in a pair, when finding alternate angles, we consider at a time two lines crossed by the transversal.

And they are in a pair, on the opposite sides of the transversal, and also, on the opposite sides of the two lines crossed by the transversal. So for instance, in the figure below, $\angle A$ and $\angle S$ are alternate angles, $\angle A$ and $\angle W$ are alternate angles, and $\angle P$ and $\angle W$ are alternate angles.

Fig. 10



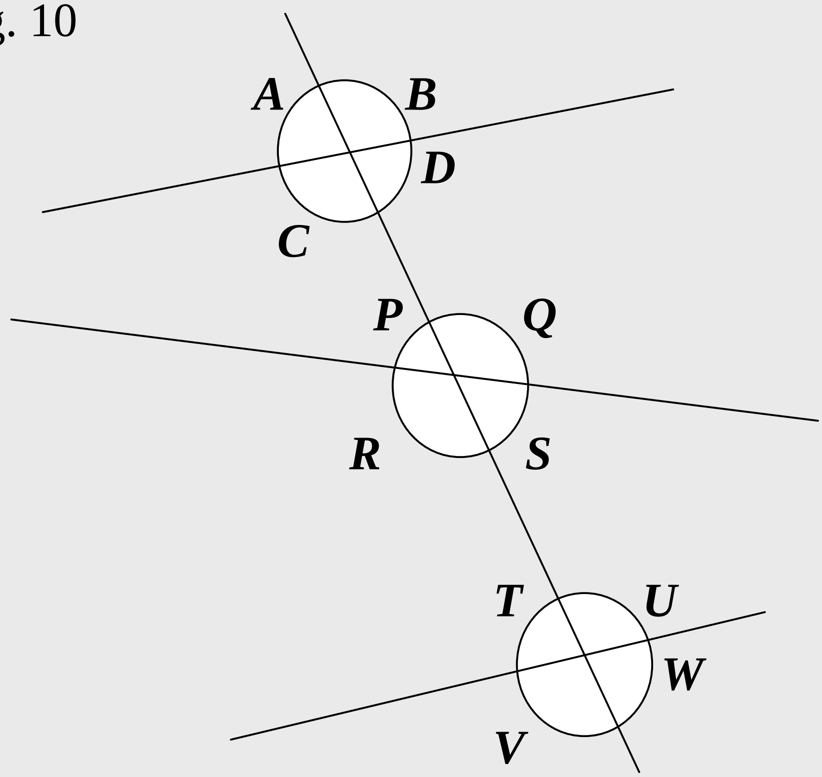
And the same is true of all the pairs below, too.

($\angle C$ and $\angle Q$), ($\angle C$ and $\angle U$), ($\angle R$ and $\angle U$),

($\angle B$ and $\angle R$), ($\angle B$ and $\angle V$), ($\angle Q$ and $\angle V$),

($\angle D$ and $\angle P$), ($\angle D$ and $\angle T$), and ($\angle S$ and $\angle T$).

Fig. 10



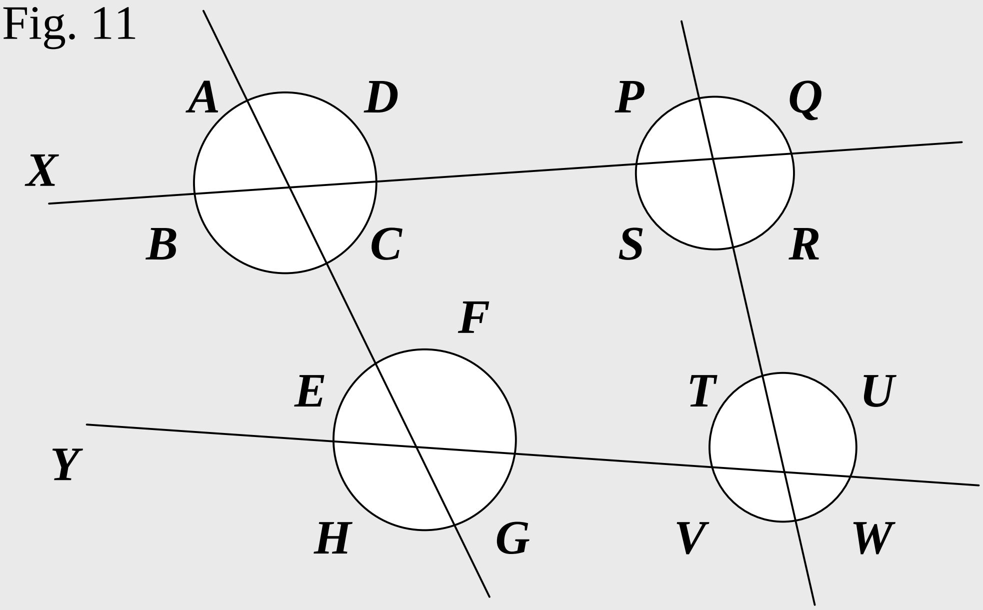
So the alternate angles are in a pair, are on the opposite sides of the transversal, and also, are on the opposite sides of the two lines crossed by the transversal.

And the ***transversal*** matters. With no transversal, no two angles can be considered to be corresponding or alternate.

So we need to note that in either case, that is, whether corresponding or alternate, the two angles in a pair need to share a line called the ***transversal***.

So for instance, $\angle P$ and $\angle E$ are not corresponding angles, whereas $\angle P$ and $\angle T$ are. It's because $\angle P$ and $\angle E$ don't share a line that can be called the transversal, whereas $\angle P$ and $\angle T$ do and are on the same sides of the lines involved.

Fig. 11

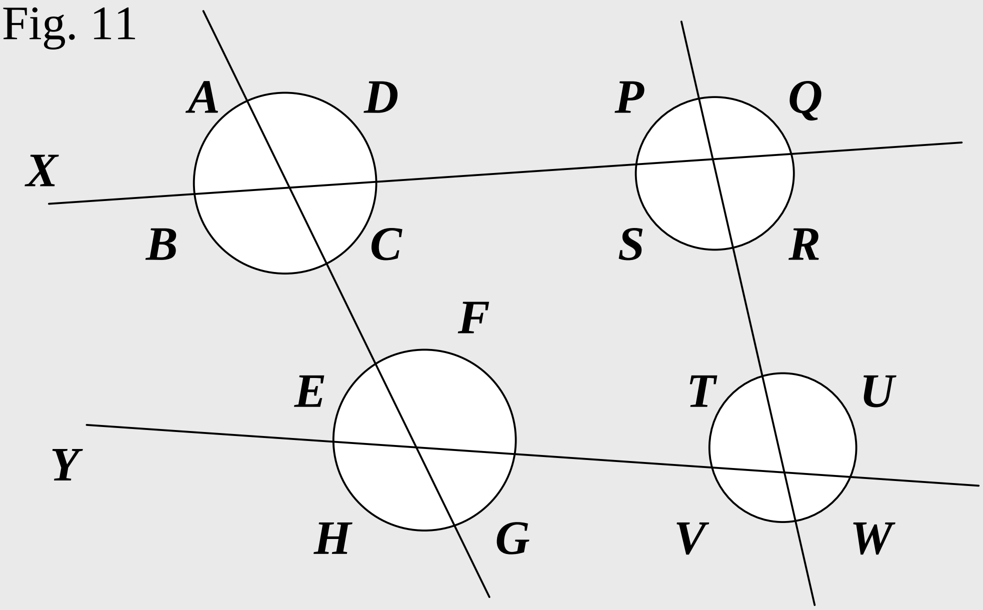


The same is true of alternate angles, too.

So the two angles in a pair need to share first a line that can be called the transversal, and then, follow the rule as follows.

Alternate angles are on the opposite sides of the transversal crossing two lines, as well as on the opposite sides of the two lines crossed by the transversal.

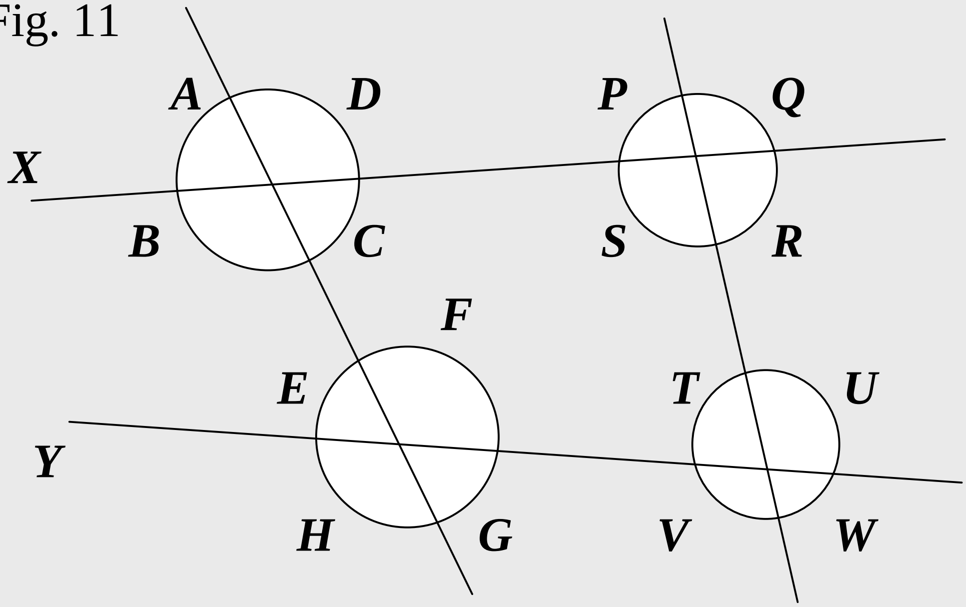
Fig. 11



So for instance, $\angle Q$ and $\angle H$ are not alternate angles, whereas $\angle Q$ and $\angle V$ are. Why?

It's because $\angle \boldsymbol{Q}$ and $\angle \boldsymbol{H}$ don't share a line that can be called the transversal, whereas $\angle \boldsymbol{Q}$ and $\angle \boldsymbol{V}$ do and are on the opposite sides of the lines involved.

Fig. 11



So whether corresponding or alternate, the two angles in a pair need to share first a line called the ***transversal***, and then, follow the rule that applies. And we can put the rule in short this way:

If corresponding, both angles are on the same sides of the lines involved, and if alternate, both are on the opposite sides of the lines involved.

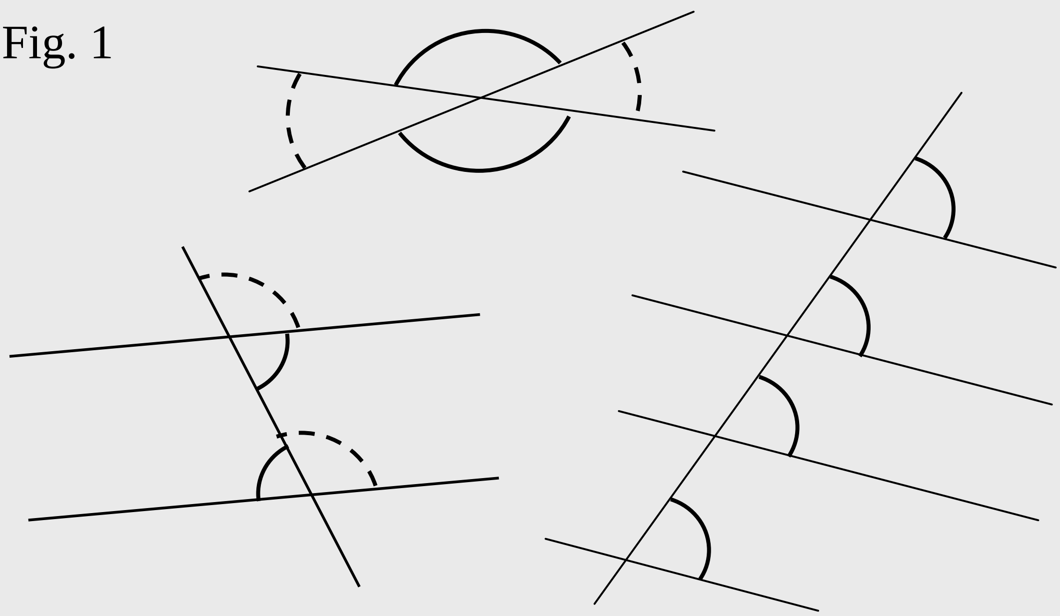
9.10. Angles and Lines 10

When solving problems in geometry, we often work with angles and lines. And knowing that ***some particular angles are the same***, we can readily solve many problems.

Of those same angles, some are made by a line and a set of parallel lines, so can be called ***angles with parallel lines***. And of course, among those the same, we have vertical angles, too, made by two lines crossing each other.

Some of the same angles are in a pair. And some are not, so more than two angles with parallel lines can be the same.

Fig. 1



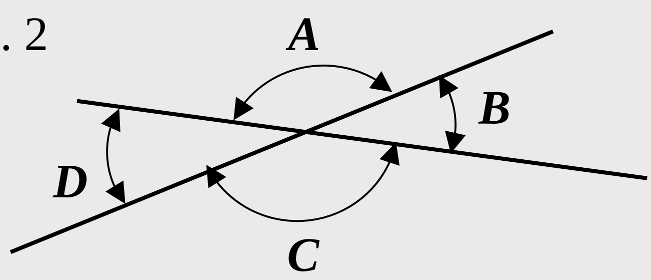
As stated earlier, we use the angles often when solving problems, so the more familiar, the better.

To be familiar, we may want to know first, how the angles are made, understand how the angles happen when lines intersect, and then, use them often doing many examples.

Let's now begin with going over two same angles called vertical angles, since they are the base of same angles.

So suppose now, two lines meet at a point as shown below.

Fig. 2

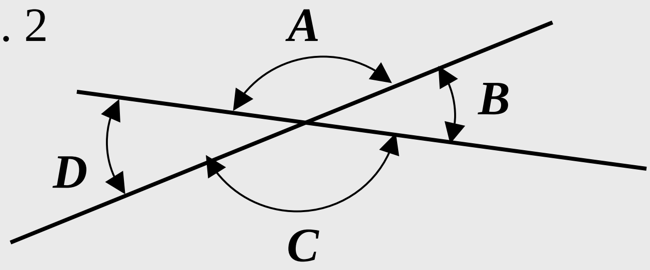


Then, we have this: $\angle B = \angle D$, and these two angles are ***vertically opposite angles***, simply called ***vertical angles***, frequently used when we solve problems.

And we have this, too: $\angle A = \angle C$, since the two angles are vertical angles, always the same.   Why the same, though?

First, we have this: $\angle \boldsymbol{A} + \angle \boldsymbol{B} = 180^{\circ}$, and next, we have this, too: $\angle \boldsymbol{C} + \angle \boldsymbol{B} = 180^{\circ}$, so we get this: $\angle \boldsymbol{A} = \angle \boldsymbol{C}$.

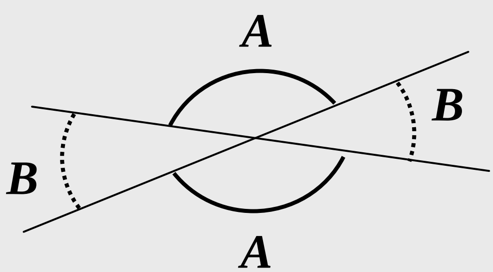
Fig. 2



Now again, we have this: $\angle \boldsymbol{A} + \angle \boldsymbol{B} = 180^{\circ}$, and next, we have this: $\angle \boldsymbol{A} + \angle \boldsymbol{D} = 180^{\circ}$, so we get this, too: $\angle \boldsymbol{B} = \angle \boldsymbol{D}$.

Vertical angles are therefore, two same angles made by two lines crossing each other. Thus, we have this:

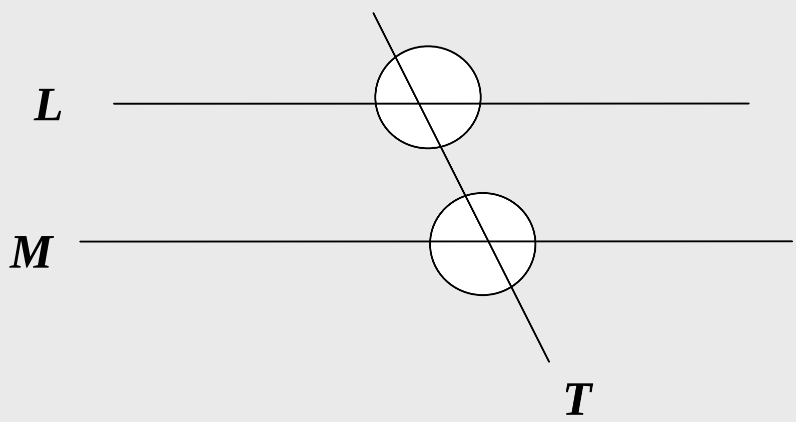
Fig. 3



What then about angles with parallel lines?

Suppose now, as shown below, two lines are parallel, and angles get made by the two and the transversal, the line ***T*** crossing the two in the figure below.

Fig. 4 The line ***L*** is parallel to the line ***M***.



Then, seemingly, eight angles get made, but there are only two angles. Only two different angles get made, since there are two different groups of ***four same angles***.

And we can put it this way, too:
Two same groups of two different pairs of vertical angles.

So either way, the number of different angles made by the two parallel lines and the transversal are just two.

What then are those ***four same angles*** stated above?

There are two kinds: corresponding angles, alternate angles.

So if the two lines crossed by another line called the transversal are parallel, the corresponding angles are equal, and so are the alternate angles.

Fig. 5

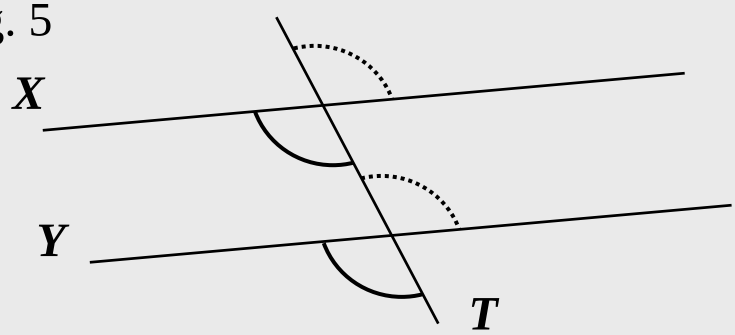
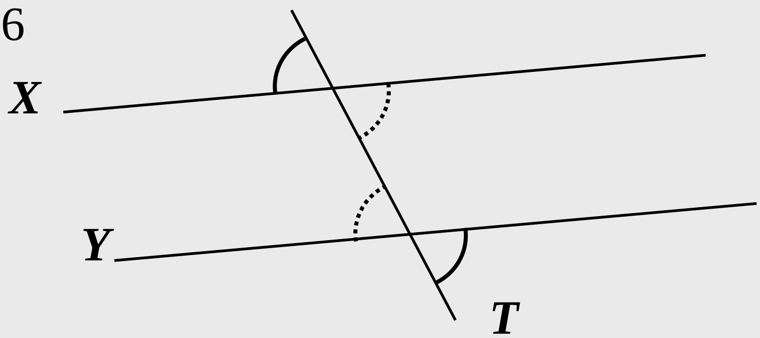


Fig. 6

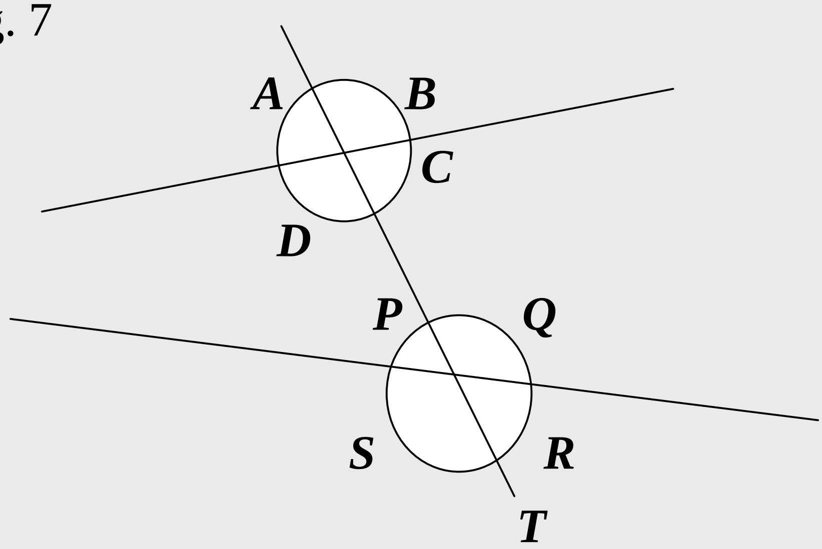


They are angles made by two parallel lines and a line crossing the parallel lines.
So let’s now, call them ***angles with parallel lines***, and see how those angles have to be the same.

First off, corresponding angles are in a pair, on the ***same side*** of the transversal, and also, on the ***same sides*** of the lines crossed by the transversal, which is the line $\boldsymbol{T}$ in the figure below.

In short, corresponding angles are on the same sides.

Fig. 7



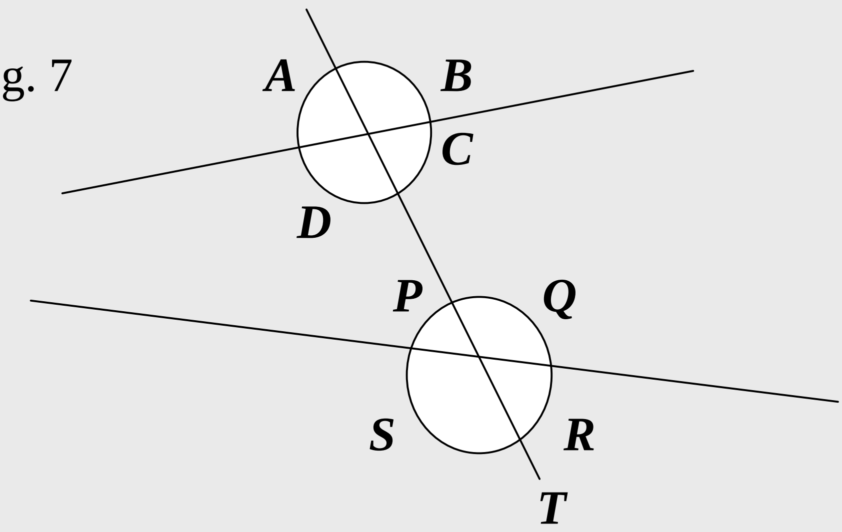
So for instance, in the figure above, $\angle \boldsymbol{A}$ and $\angle \boldsymbol{P}$ are corresponding angles. And the same is true of the pairs as follows: ($\angle \boldsymbol{B}$ and $\angle \boldsymbol{Q}$), ($\angle \boldsymbol{C}$ and $\angle \boldsymbol{R}$), and ($\angle \boldsymbol{D}$ and $\angle \boldsymbol{S}$).

What then about alternate angles?

Alternate angles are in a pair, too, but on the ***opposite sides*** of the transversal, and also, on the ***opposite sides*** of the lines crossed by the transversal, the line $\boldsymbol{T}$ shown below.

In short, alternate angles are on the opposite sides.

Fig. 7



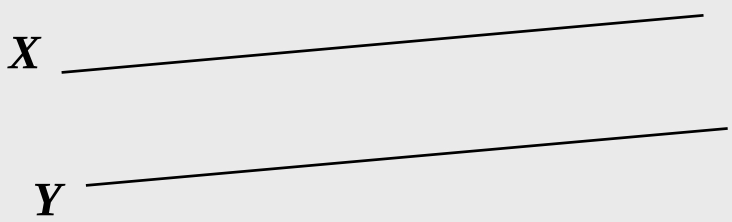
If one angle is on the ***right*** of the transverse line $\boldsymbol{T}$, and is ***below*** one of the two lines crossed by $\boldsymbol{T}$, the other angle is on the ***left*** of $\boldsymbol{T}$, and is ***above*** the other line crossed by $\boldsymbol{T}$.

So for instance, in the figure above, $\angle \boldsymbol{C}$ and $\angle \boldsymbol{P}$ are alternate angles. And the same is true of the pairs as follows: ($\angle \boldsymbol{D}$ and $\angle \boldsymbol{Q}$), ($\angle \boldsymbol{B}$ and $\angle \boldsymbol{S}$), and ($\angle \boldsymbol{A}$ and $\angle \boldsymbol{R}$).

What then do those angles have to do with parallel lines?

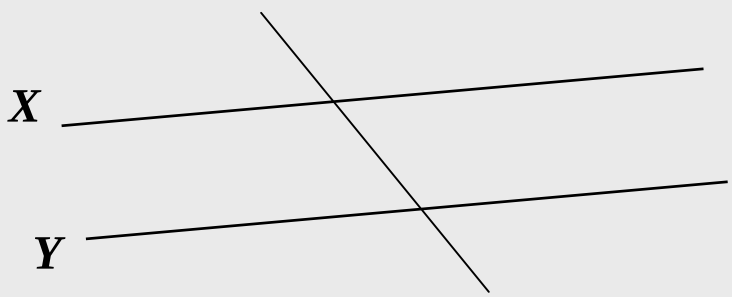
For starters, as stated in a section earlier, an angle can mean an amount of difference in direction, so 0^{o} can mean ***no difference in direction***, the same direction. Therefore, if the angle between two lines is 0^{o}, the two share the same direction, and are parallel to each other.

Fig. 8 The line ***X*** is parallel to the line ***Y***.



So there is ***no difference in direction*** between parallel lines; thus, there is ***the same difference in direction*** between each of parallel lines and a particular line crossing the parallel lines. What then is the particular line?

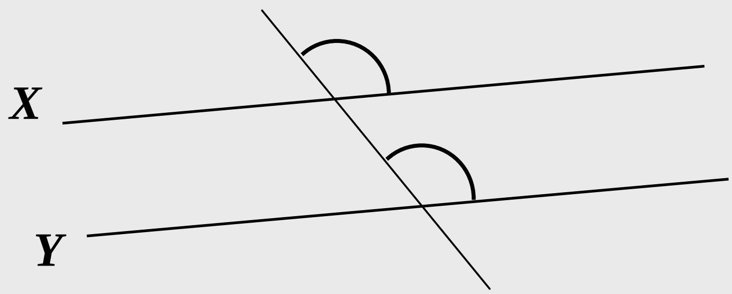
Fig. 9 The line ***X*** is parallel to the line ***Y***.



It is the transverse line, just called ***the transversal***, for short. There is thus, the same difference in direction between each of parallel lines and the transversal.

What then, can ***the same difference in direction*** mean?

Fig. 10 The line ***X*** is parallel to the line ***Y***.



Mentioned earlier that an angle can mean an amount of difference in direction. So it can mean ***the same angle***.

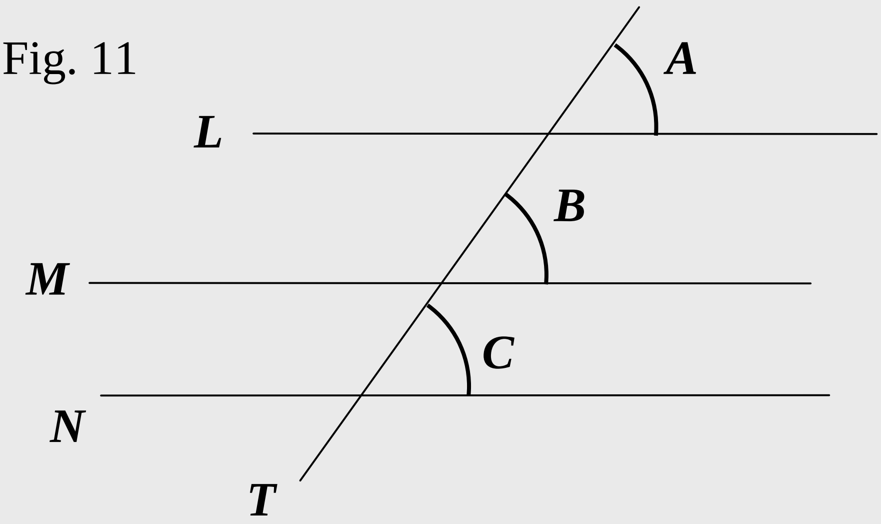
The same difference in direction can mean the same angle.

So now, since there is the same difference in direction between the transversal and each of parallel lines, the angle between each parallel line and the transversal is the same.

Parallel lines and the transversal make the ***same angles***.
In other words, the angles made by parallel lines and the transversal are all equal.

Suppose now, in the figure below, we have this: $L \parallel M \parallel N$, that is, the three lines, ***N*, *M*,** and ***L*** are parallel.

Fig. 11



Then, we get this: $\angle A = \angle B = \angle C$.

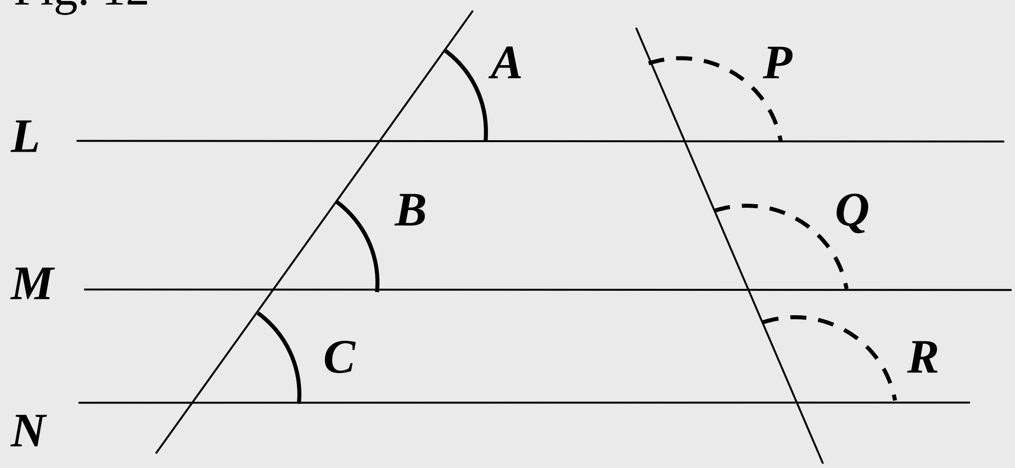
So the angles made by parallel lines and the transversal are all the same. In short, ***angles with parallel lines are equal***, which is another important fact we can use when solving many problems.

What if we add another transversal to the figure above?

Then, we can get another group of the same angles.

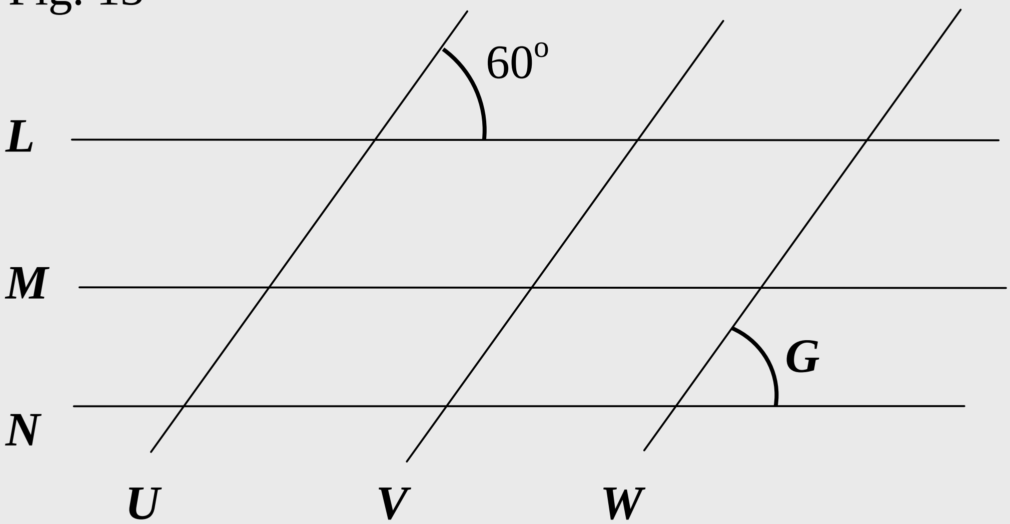
So in the figure below, if we have this: ***L*** // ***M*** // ***N***, we get these: ∠***A*** = ∠***B*** = ∠***C***, and ∠***P*** = ∠***Q*** = ∠***R***.

Fig. 12

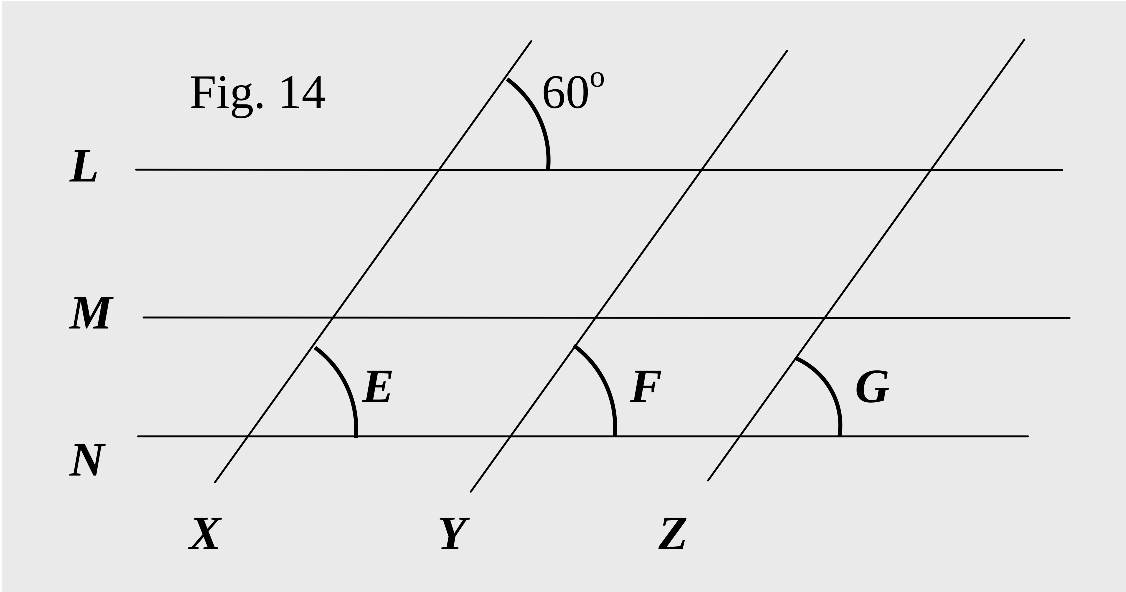


What then is $\angle G$ if we have L // M // N, and U // V // W ?

Fig. 13



Since we have this: ***L*** // ***M*** // ***N***, we get this: ∠***E*** = 60°.



And we have this, too: ***X*** // ***Y*** // ***Z***, so we get this: ∠***G*** = 60°.

That is to say that if taking the line ***N*** as the transversal crossing the parallel lines ***X, Y,*** and ***Z,*** we get this:

∠***E*** = ∠***F*** = ∠***G***.

So, once again, the angles made by parallel lines and the transversal are all the same, which is now, your handy math tool you can use when solving many problems.

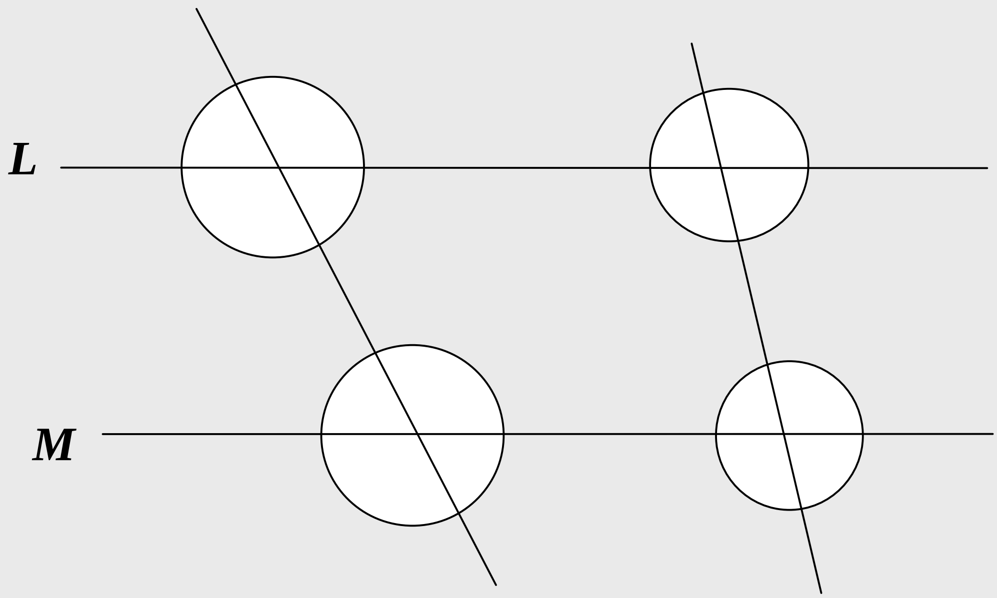
And that's not it. We can find more angles that are equal.

What angles are then those same angles?

They are corresponding angles that are equal and alternate angles that are equal, as well as vertical angles.

In the figure below, the corresponding angles are equal, and also, the alternate angles are equal.

Fig. 15 The line ***L*** is parallel to the line ***M***.



That is to say that if the two lines crossed by the transversal are parallel, the corresponding angles are equal, and so are the alternate angles.

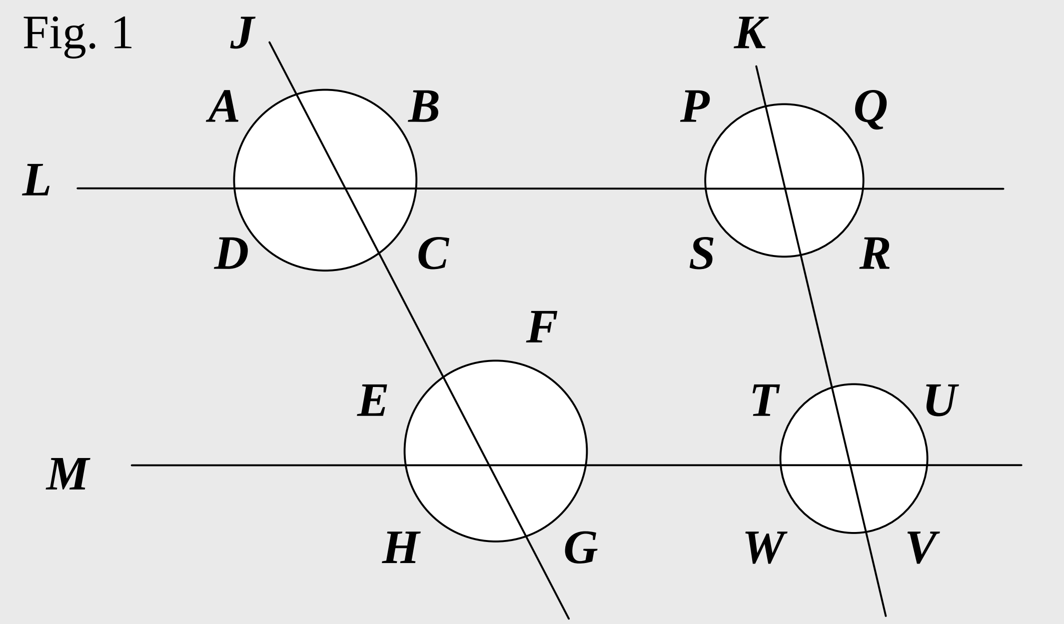
How then, do we know that those angles are equal?

We'll discuss the answer in the next lesson.

9.11. Angles and Lines 11

Suppose now, in the figure below, we have this: ***L*** // ***M***.

Fig. 1



Then, since angles with parallel lines and the transversal are equal, if taking the line ***J*** as the transversal, we get these: $\angle A = \angle E$, $\angle B = \angle F$, $\angle D = \angle H$, and $\angle C = \angle G$.

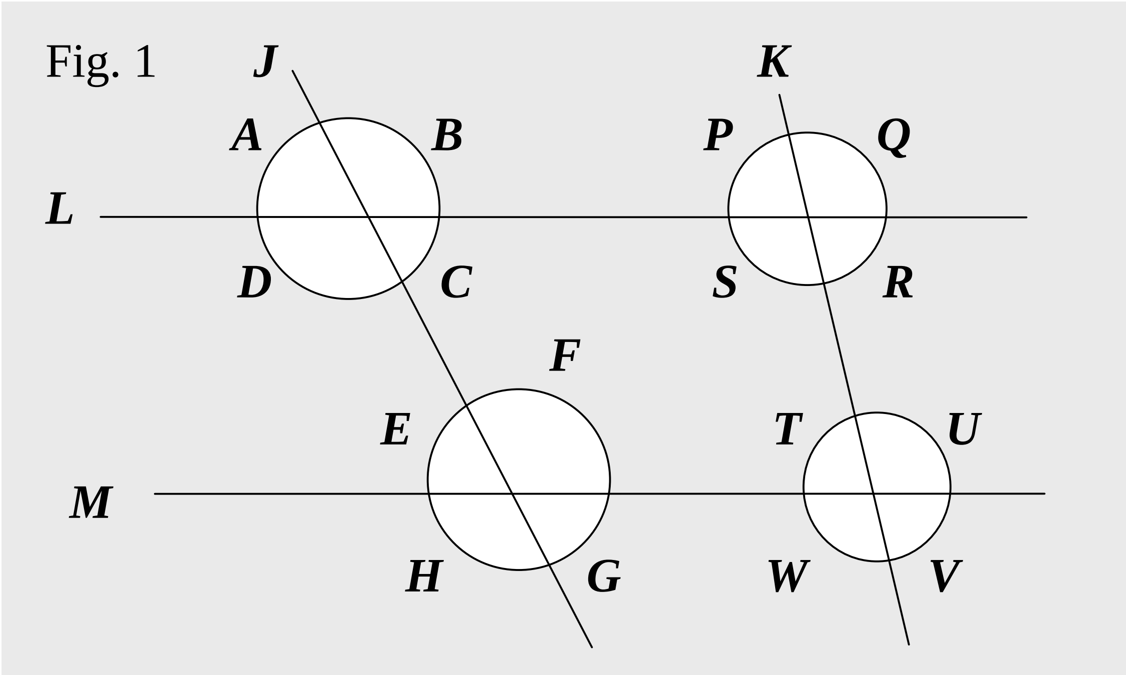
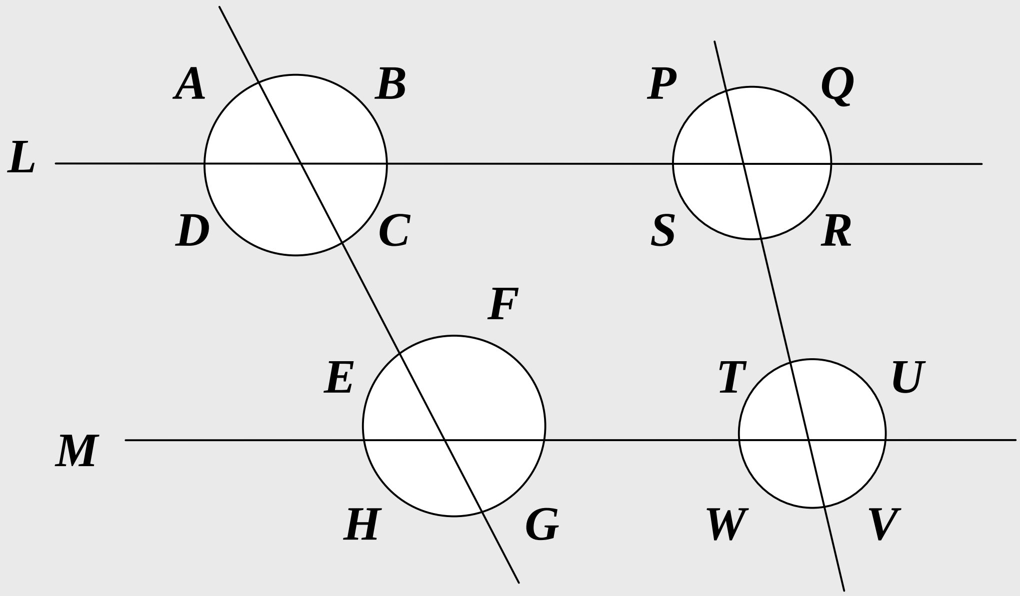


Fig. 1

And by the same token, if in the figure, taking the line ***K*** as the transversal, we get these, too: ∠***P*** = ∠***T***, ∠***Q*** = ∠***U***, ∠***R*** = ∠***V***, and ∠***S*** = ∠***W***.

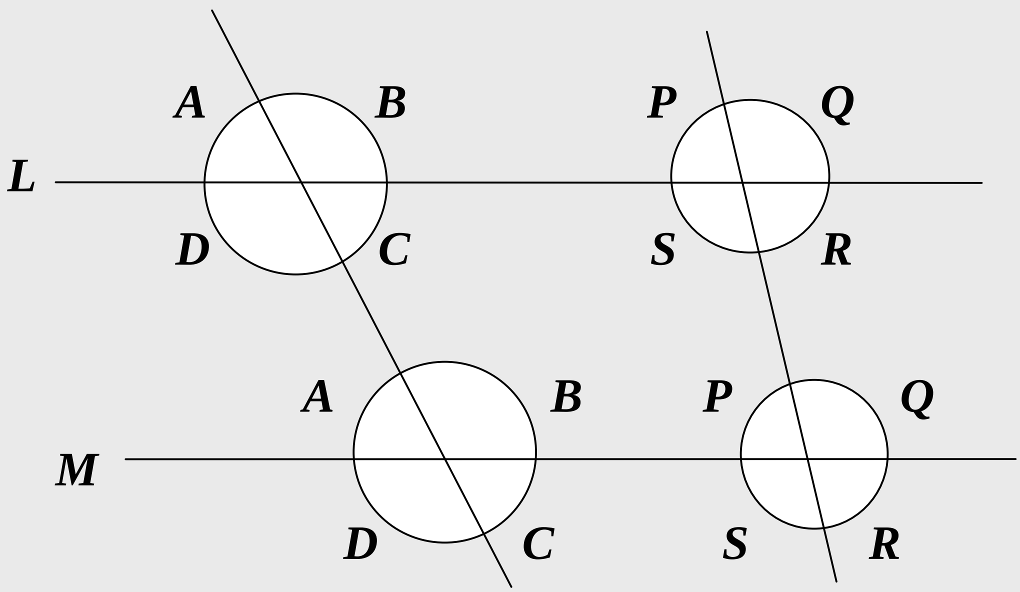
In each equality above, the two angles are corresponding angles. So the corresponding angles are equal if they are made by parallel lines and the transversal.

Fig. 2 The line ***L*** is parallel to the line ***M***.



And we can now put the figure above the way as follows.

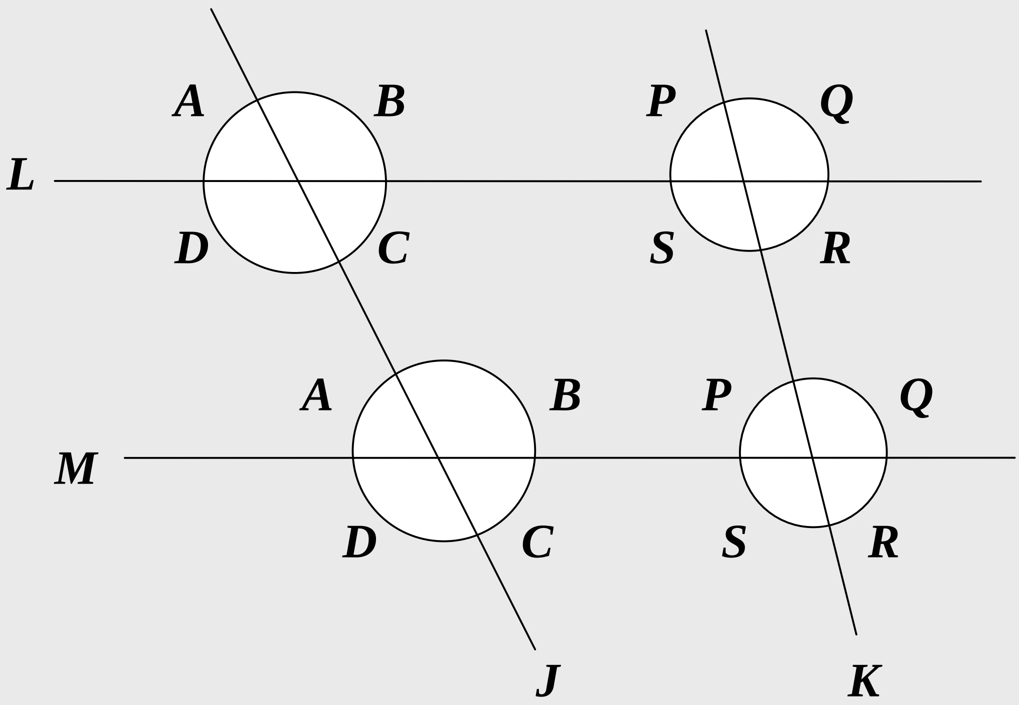
Fig. 3 The line ***L*** is parallel to the line ***M***.



Therefore, along each of the two parallel lines, we have now, the same group of eight angles. And the same group is this:

$\angle A$, $\angle B$, $\angle C$, $\angle D$, $\angle P$, $\angle Q$, $\angle R$, and $\angle S$.

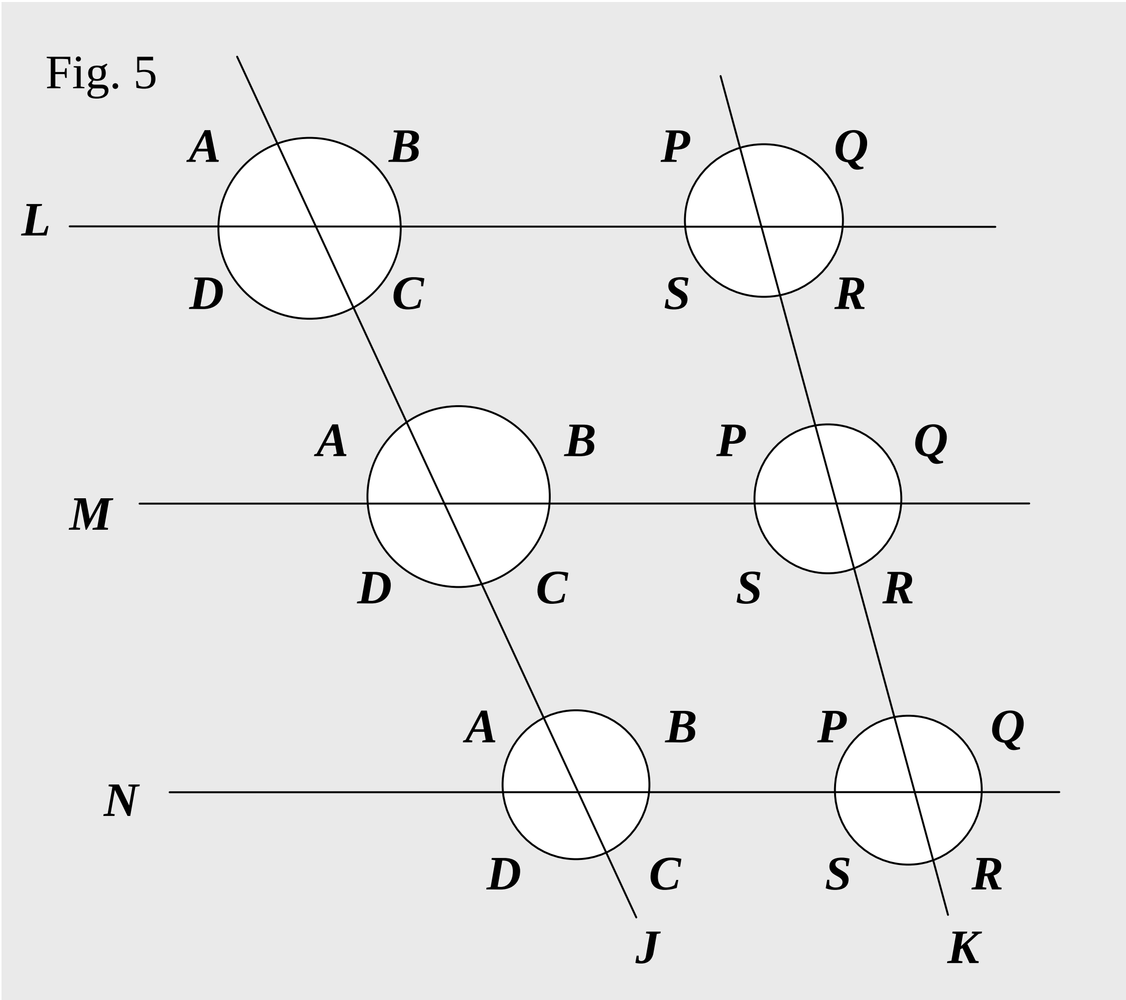
Fig. 4 The line ***L*** is parallel to the line ***M***.



The three lines ***L***, ***M***, and ***J*** make pairs of the same angles, since we have this: ***L*** // ***M***, and ***J*** is the transversal. And by the same token, so do the other three lines ***L***, ***M***, and ***K***.

Adding another line parallel to the line ***L*** or the line ***M***, we get the same group again.

In the figure below. the three lines ***N***, ***M***, and ***L*** are parallel.



The four lines ***L***, ***M***, ***N***, and ***J*** make pairs of the same angles, since we have this: ***L*** // ***M*** // ***N***, and ***J*** is the transversal. And by the same token, so do the other four lines ***L***, ***M***, ***N***, and ***K***.

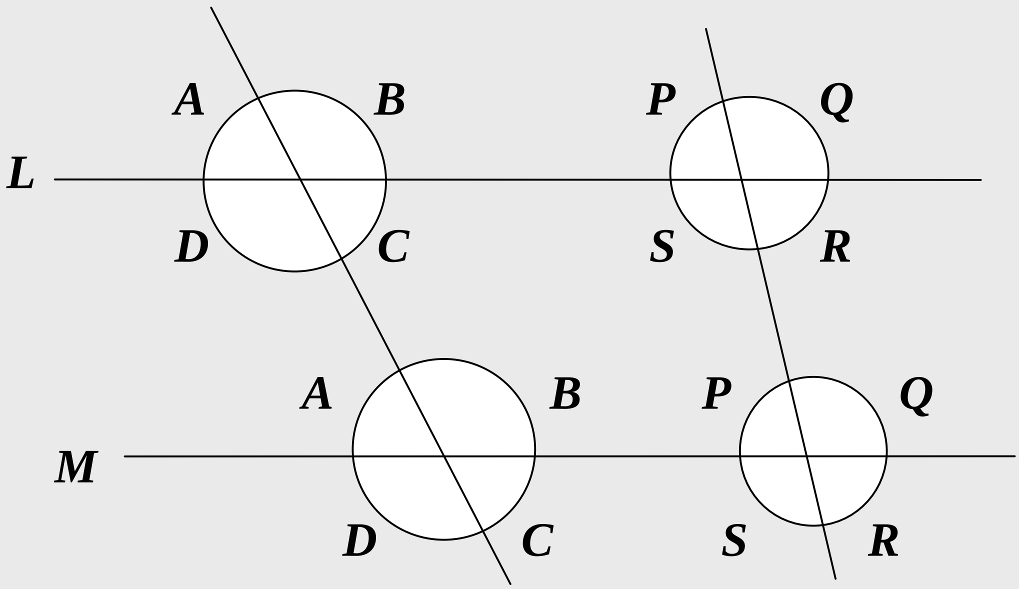
And that's not it.

We know, vertical angles are the same, two equal angles.

So we get these:

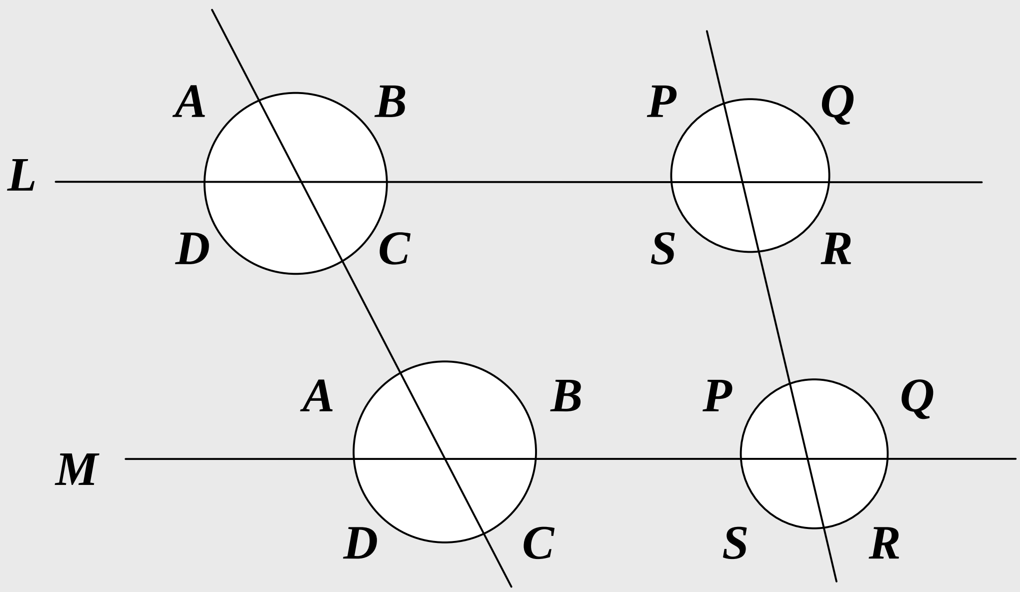
$\angle A = \angle C$, $\angle B = \angle D$, $\angle P = \angle R$, and $\angle Q = \angle S$.

Fig. 6 The line ***L*** is parallel to the line ***M***.



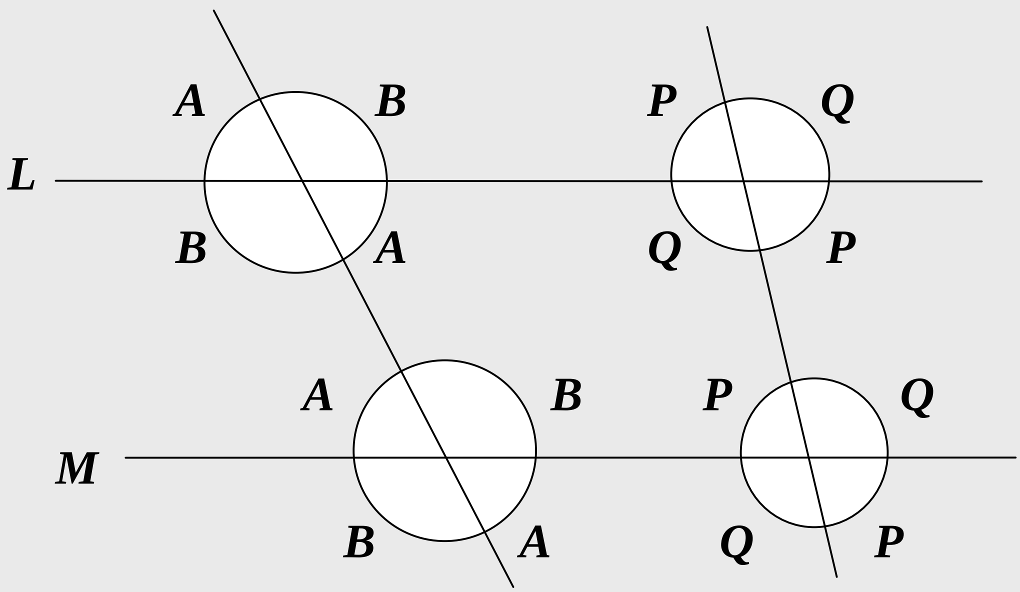
Therefore, along each of the parallel lines, we have now, the same group of four angles.

Fig. 6 The line ***L*** is parallel to the line ***M***.



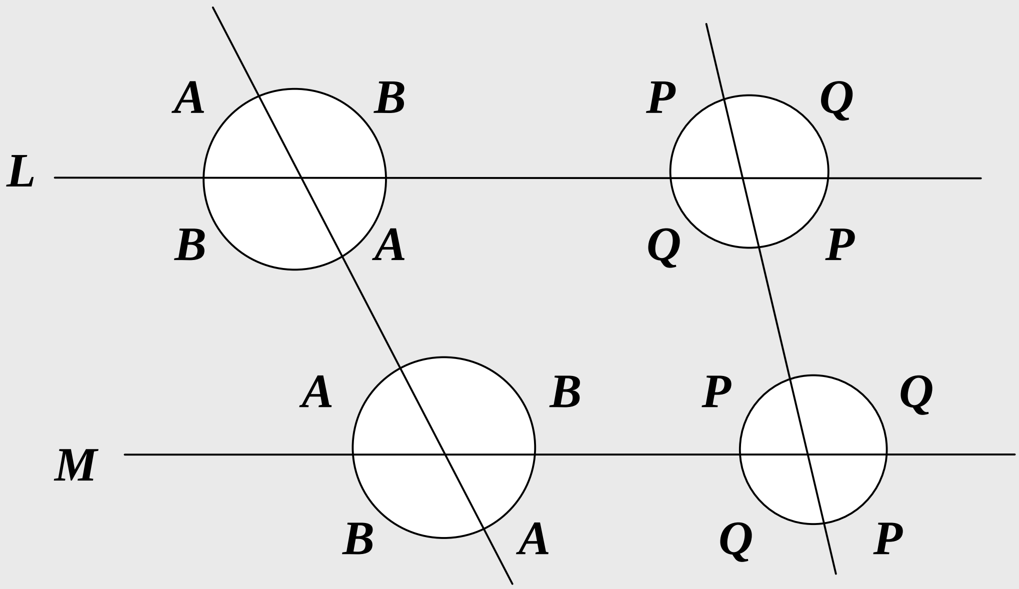
So we can now put the figure above the way as follows.

Fig. 7 The line ***L*** is parallel to the line ***M***.



And we know, alternate angles are in a pair, on the opposite sides of the transversal, and also, on the opposite sides of the two lines crossed by the transversal.

Fig. 7 The line ***L*** is parallel to the line ***M***.

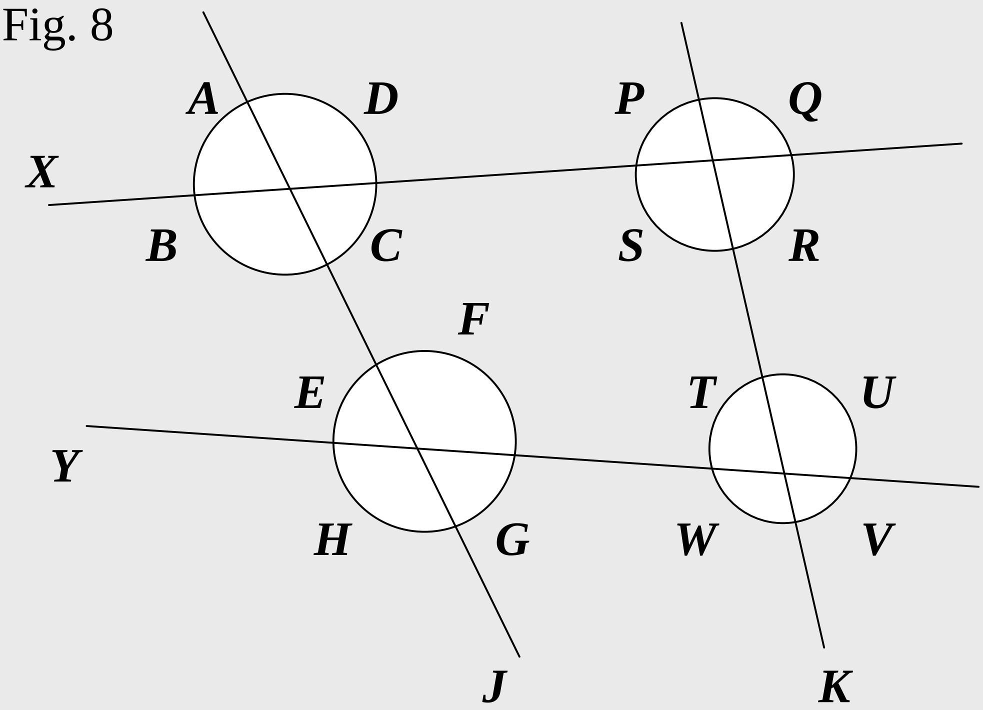


So now, in the figure above, we can see more clearly that if the two lines crossed by the transversal are parallel, alternate angles are equal, and of course, corresponding angles are equal, too, as well as vertical angles.

In short, angles with parallel lines are equal.

Note that when taking angle either corresponding or alternate, we consider ***three lines at a time***, and the two of the three are the ***two lines crossed*** by the other line called ***the transversal***. So for instance, in the figure below, $\angle A$ and $\angle T$ are not corresponding angles, and neither are these two angles: $\angle Q$ and $\angle F$.   Why not, though?

Fig. 8

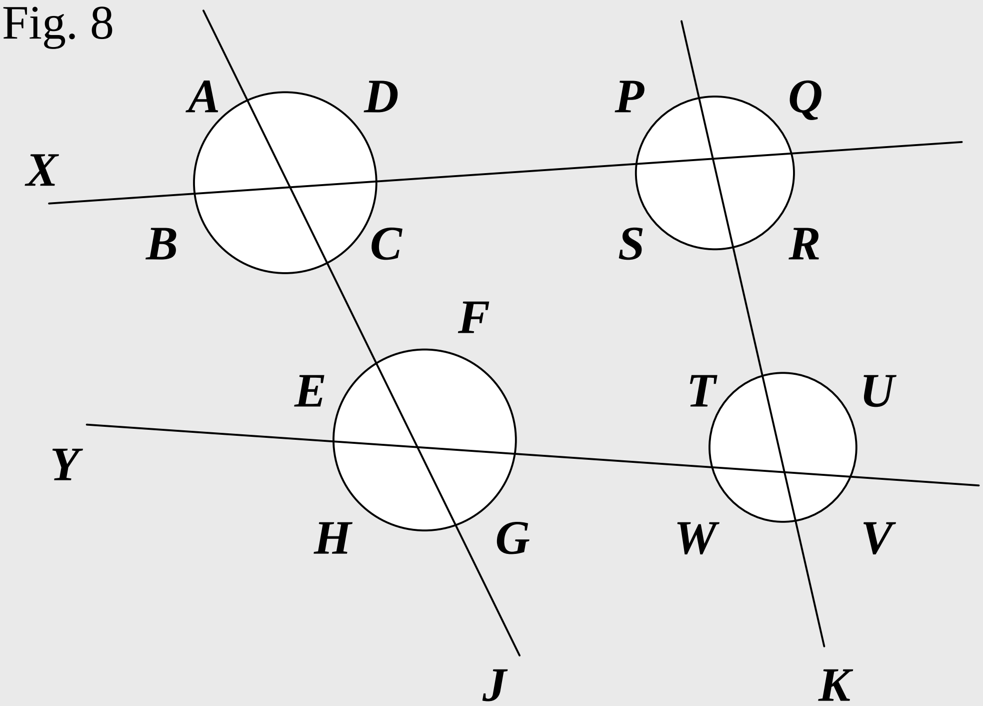


It's because with no transversal, no two angles can be considered to be corresponding or alternate. So we need to note that whether corresponding or alternate, the two angles in a pair need to share a line called the ***transversal***.

In the case of the figure below, we can consider either of the four sets of three lines as follows.

$\{X, Y, J\}$, $\{X, Y, K\}$, $\{L, M, J\}$, and $\{L, M, K\}$.

Fig. 8

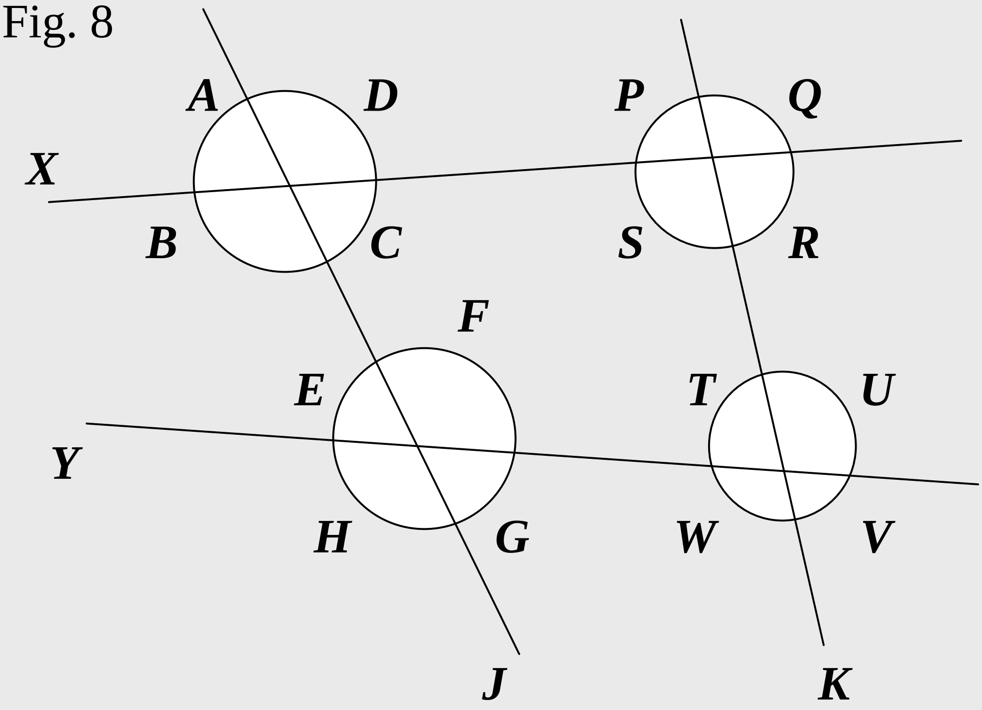


Again, the ***transversal*** matters. With no transversal, no two angles can be considered to be corresponding or alternate.

And we consider two lines at a time, along with the transversal.

So for instance, in the case of $\{X, Y, J\}$, where the line J is the transversal crossing the two lines X and Y, the angles that can be taken into consideration are these: $\angle A$, $\angle B$, $\angle C$, $\angle D$, $\angle E$, $\angle F$, $\angle G$, and $\angle H$.

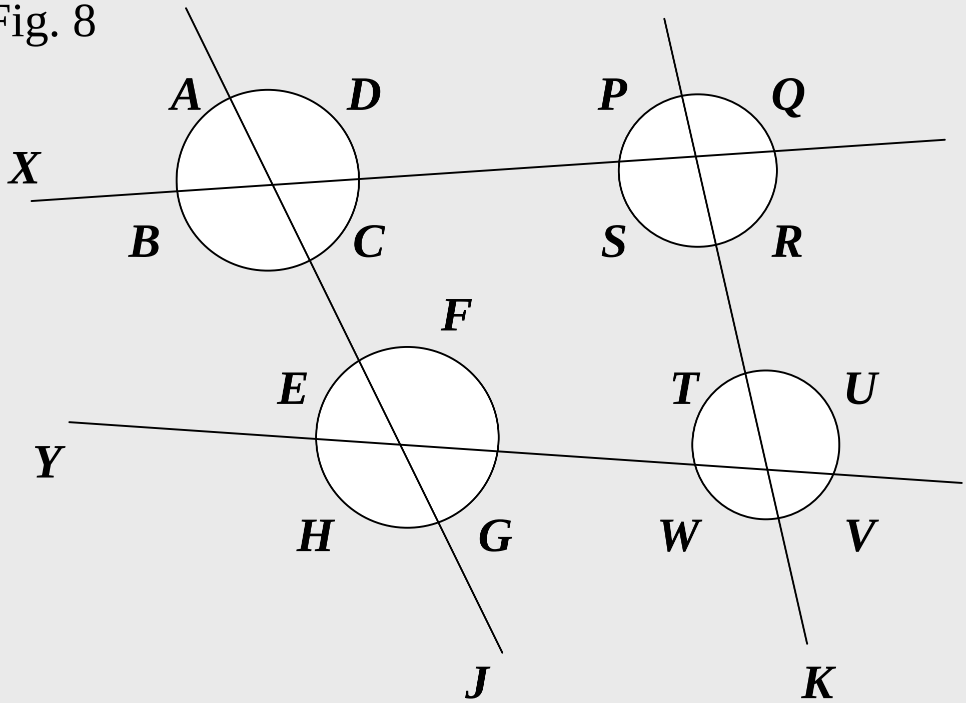
Fig. 8



Next, in the case of $\{X, Y, K\}$, where the line K is the transversal, the angles taken into consideration are these: $\angle P$, $\angle Q$, $\angle R$, $\angle S$, $\angle T$, $\angle U$, $\angle V$, and $\angle W$.

Next, in the case of {***J*, *K*, *X***}, where the line ***X*** is the transversal, the angles taken into consideration are these: ∠***A***, ∠***B***, ∠***C***, ∠***D***, ∠***P***, ∠***Q***, ∠***R***, and ∠***S***.

Fig. 8

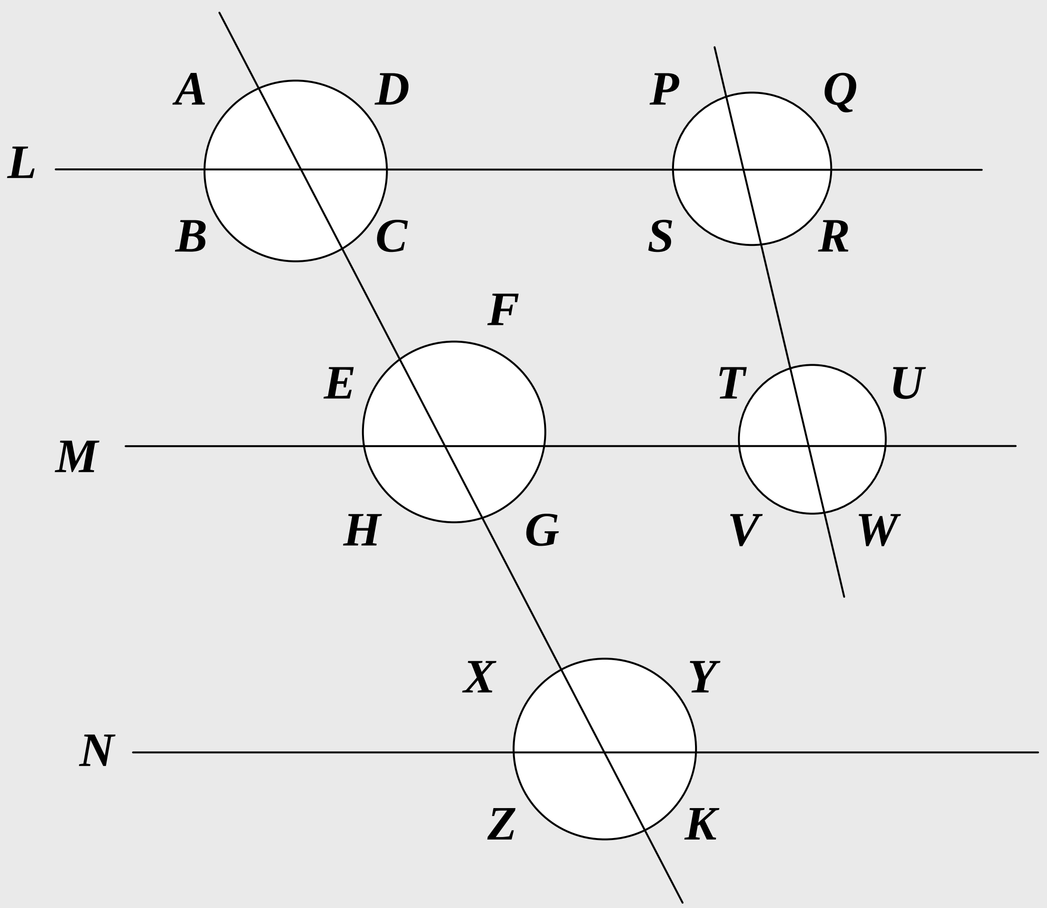


And in the case of {***J*, *K*, *Y***}, where the line ***Y*** is the transversal, the angles taken into consideration are these: ∠***E***, ∠***F***, ∠***G***, ∠***H***, ∠***T***, ∠***U***, ∠***V***, and ∠***W***.

So ∠***A*** and ∠***T*** cannot be considered at the same time.

What if now, there are more than two lines, and they are all parallel to each other?

Fig. 9

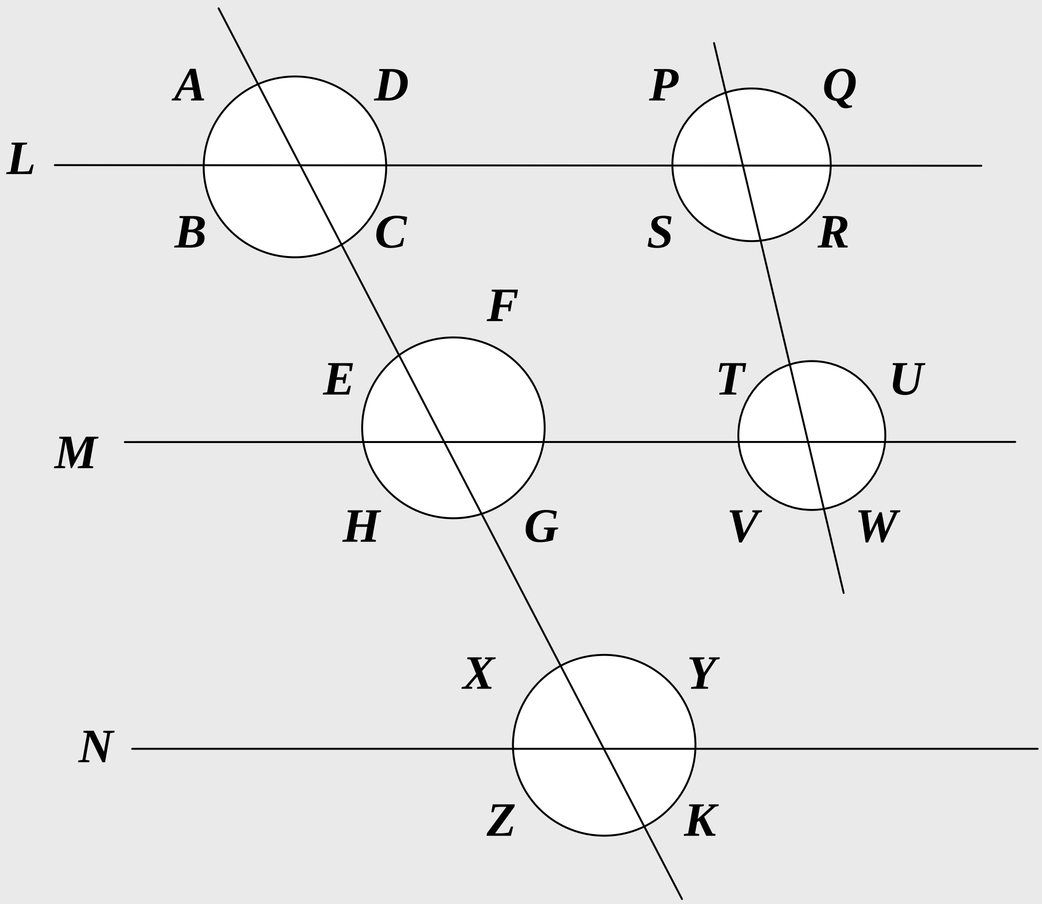


Corresponding or alternate, we consider two angles at a time, since the angles are in a pair. We don’t need to, however, if all the lines crossed by the transversal are parallel. That is, we can cut corners and get a short cut.

If the number of parallel lines crossed by the transversal are more than two, we can consider at a time not just two but as many angles as the number of the parallel lines.

Suppose now, in the figure below, we have ***L*** // ***M*** // ***N***.

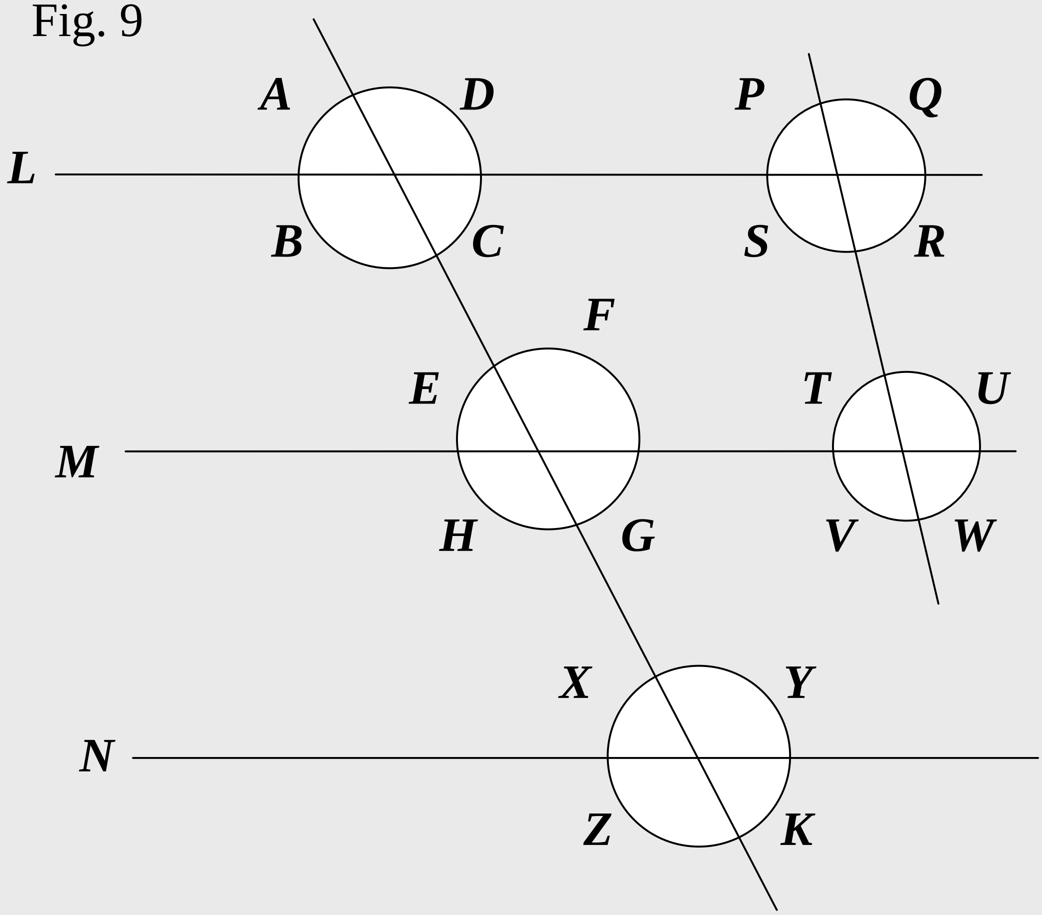
Fig. 9



Then, at once, we can get this: $\angle A = \angle E = \angle X$. How?

We have this: $\angle A = \angle E$, because $L // M$, and we have this, too: $\angle E = \angle X$, because $M // N$. And thus, $\angle A = \angle E = \angle X$.

Fig. 9



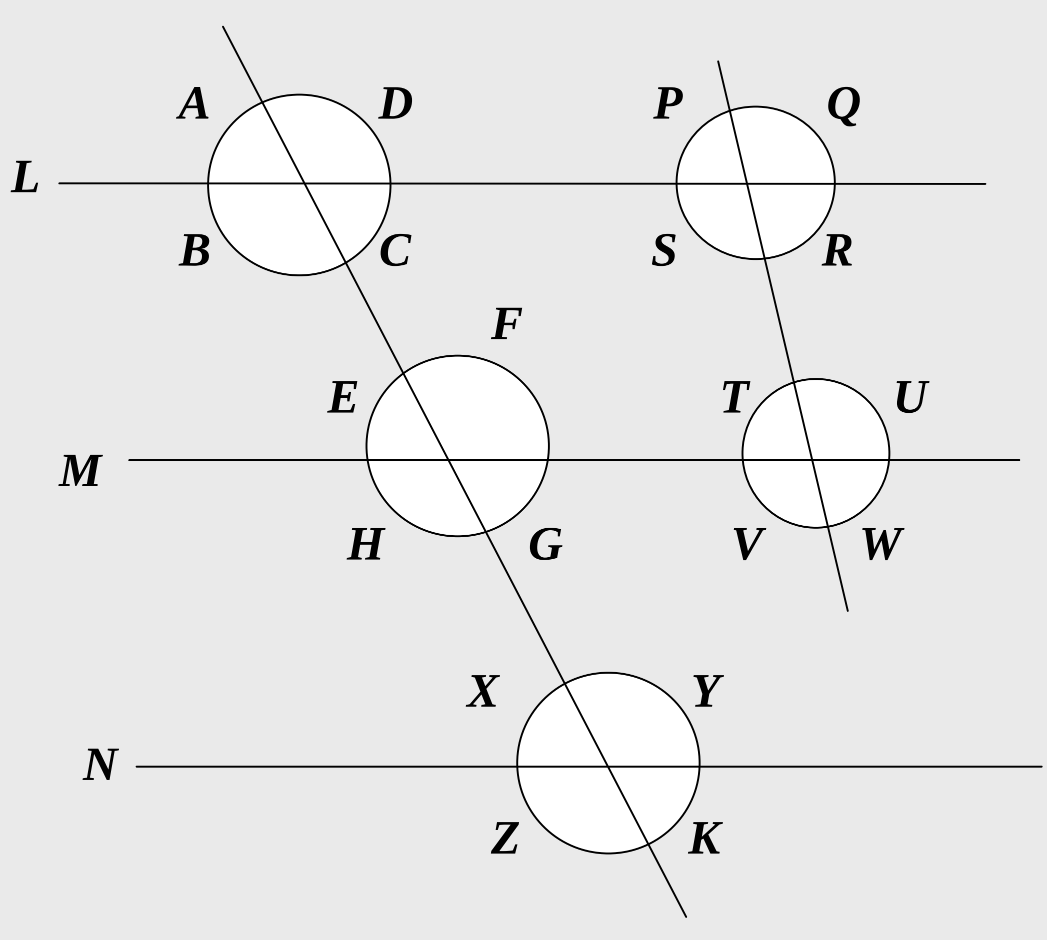
By the same token, we can get this: $\angle B = \angle H = \angle Z$, and of course, we can also, get these, too: $\angle C = \angle G = \angle K$, and $\angle D = \angle F = \angle Y$.

What then about alternate angles?

The same is true of alternate angles, too.

Suppose now again, in the figure below, ***L*** // ***M*** // ***N***, which means that the lines ***L***, ***M***, and ***N*** are parallel to each other.

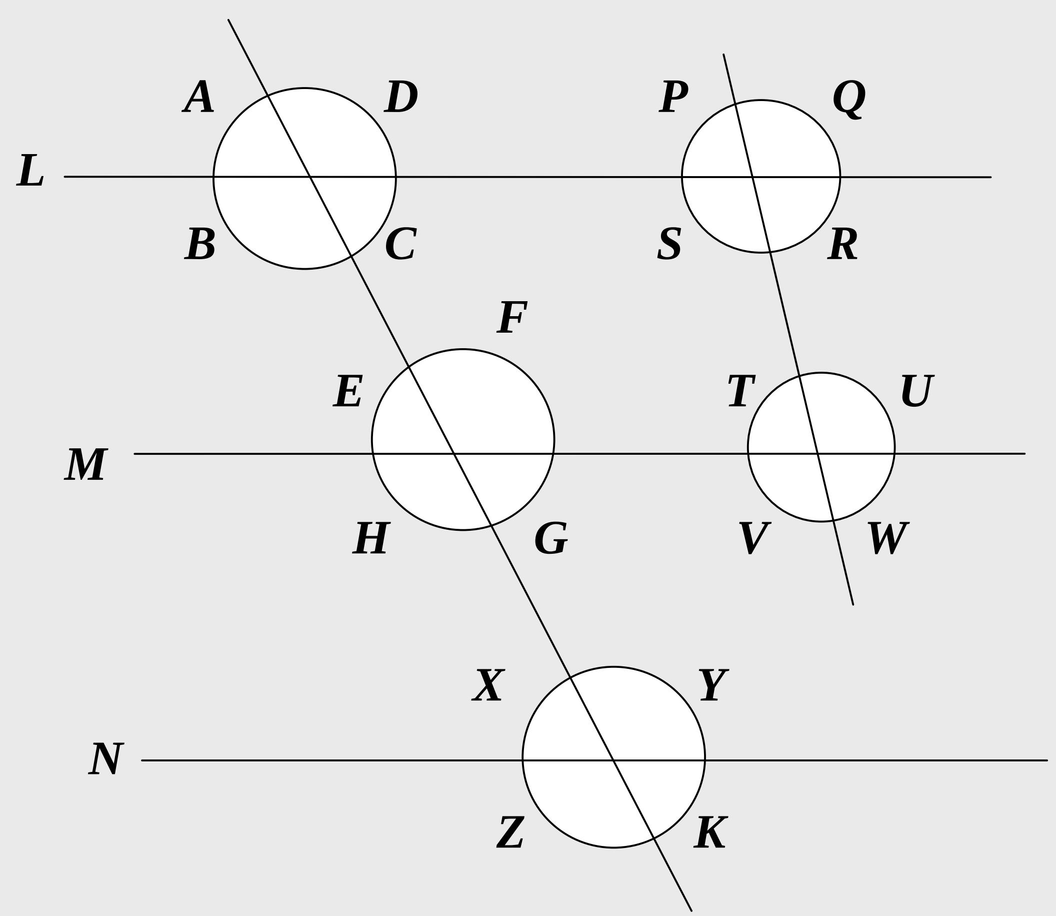
Fig. 9



Then, we can get this, at once: ∠***B*** = ∠***F*** = ∠***Z***.   How?

We have this: $\angle B = \angle F$, because L // M, and we have this, too: $\angle F = \angle Z$, because M // N. And thus, $\angle B = \angle F = \angle Z$.

Fig. 9



By the same token, we can get this: $\angle A = \angle G = \angle X$, and of course, we can also, get these, too: $\angle D = \angle H = \angle Y$, and $\angle C = \angle E = \angle K$.

9.12. Angles and Lines 12

Now, summing up, we have a math property as follows.

> “Corresponding angles with parallel lines are equal, and so are alternate angles with parallel lines.”

We can put the math property above the way as follows, too.
First off:

> **Parallel lines ⇔ The Same Corresponding Angles**

That is to say that if two lines crossed by the transversal are parallel, corresponding angles are equal, and also, if corresponding angles are equal, the two lines are parallel.

Next:

> **Parallel lines ⇔ The Same Alternate Angles**

That is to say that if two lines crossed by the transversal are parallel, alternate angles are equal, and also, if alternate angles are equal, the two lines are parallel.

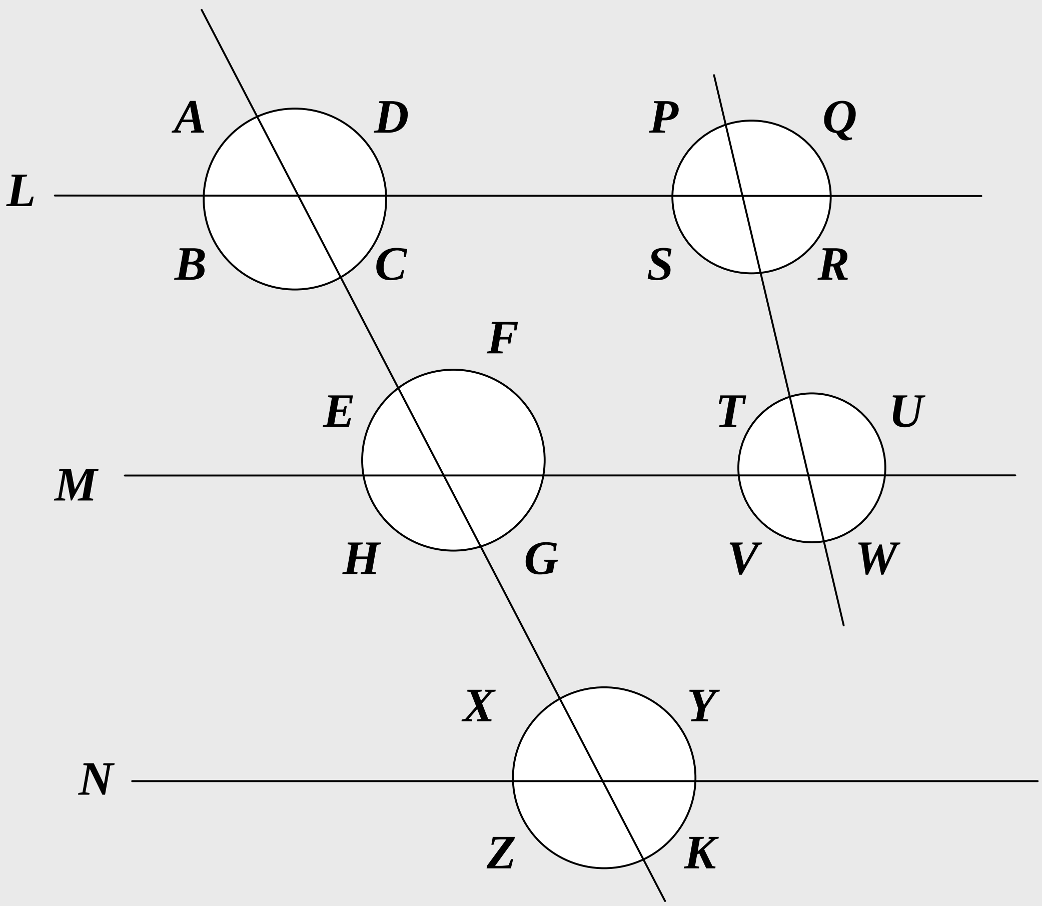
And putting the two together, we get this:

Parallel lines ⇔ The Same Corresponding Angles

Parallel lines ⇔ The Same Alternate Angles

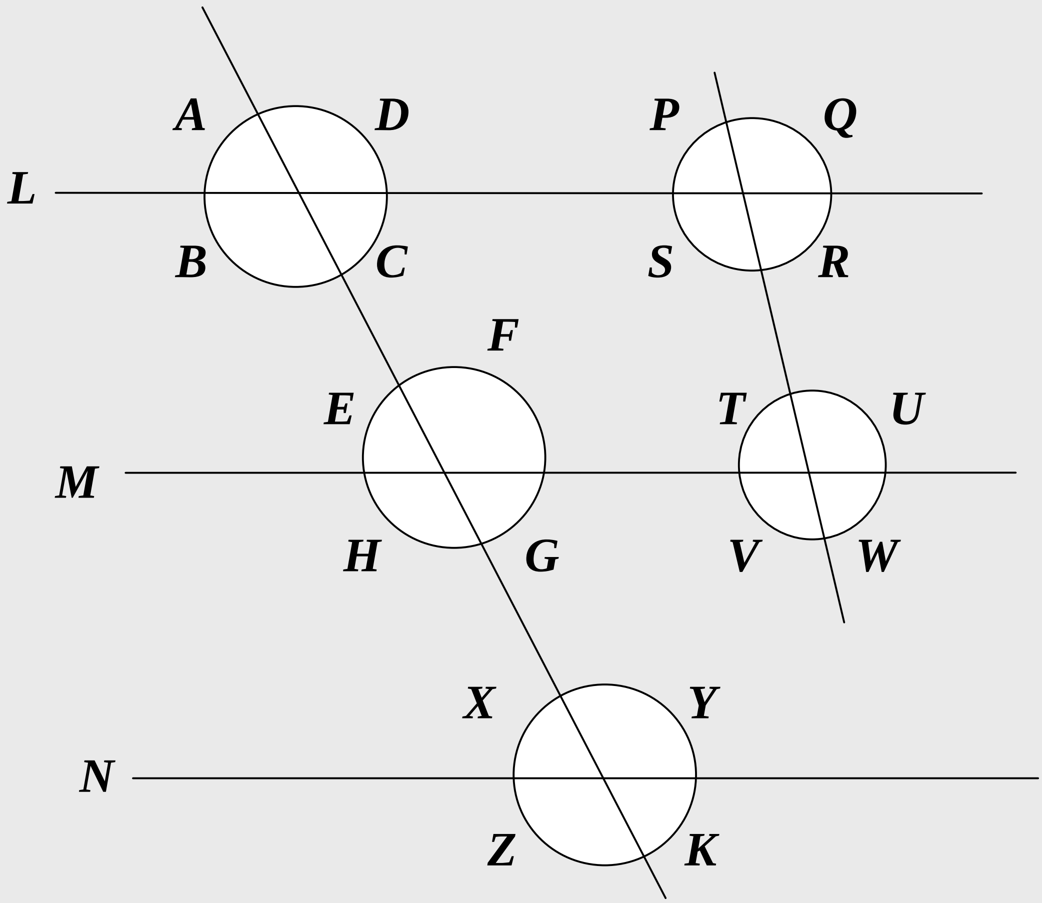
Now, for instance, if we have this: ***L*** // ***M*** // ***N*** in the figure below, what are all the angles that are equal to the angle ***A***?

Fig. 1



If the lines crossed by the transversal are parallel, not only vertical angles but corresponding and alternate angles are equal, too.

Fig. 1

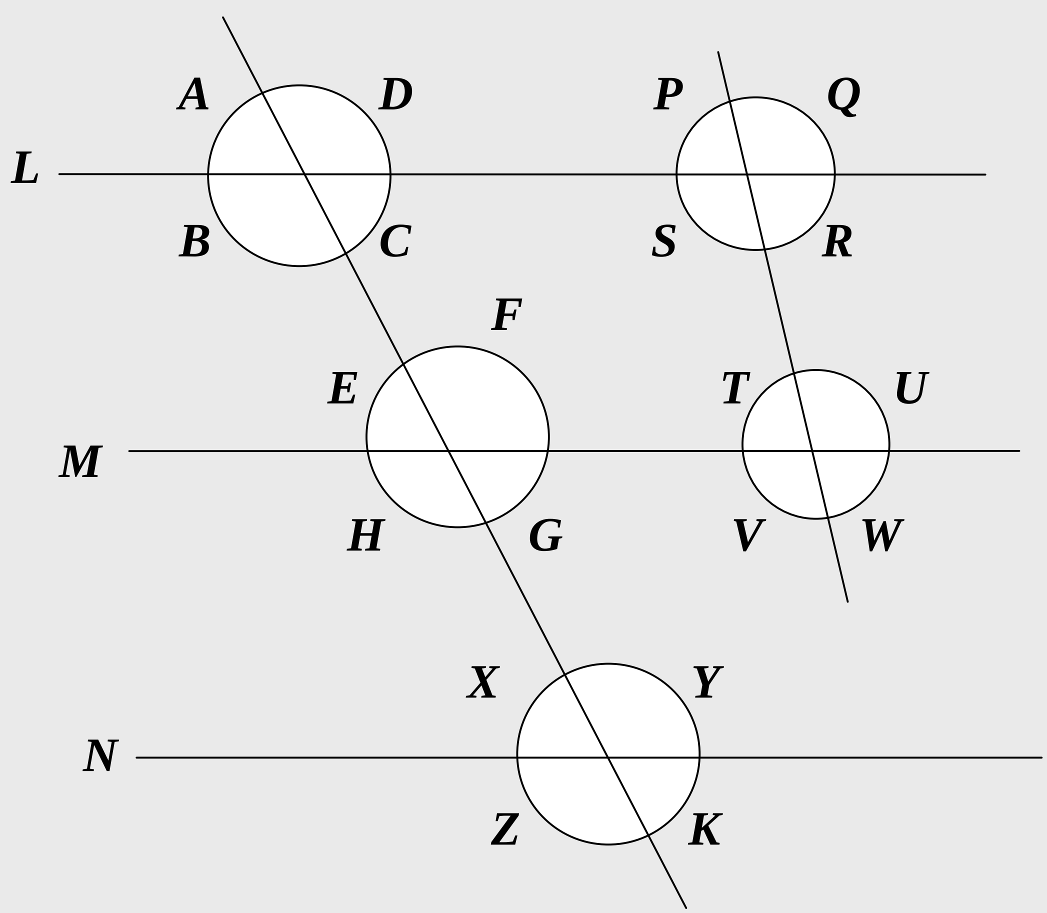


So the answer is as follows.

$\angle A = \angle C = \angle E = \angle G = \angle X = \angle K$

What then, about ∠***D***, ∠***P***, and ∠***Q*** ?

Fig. 2



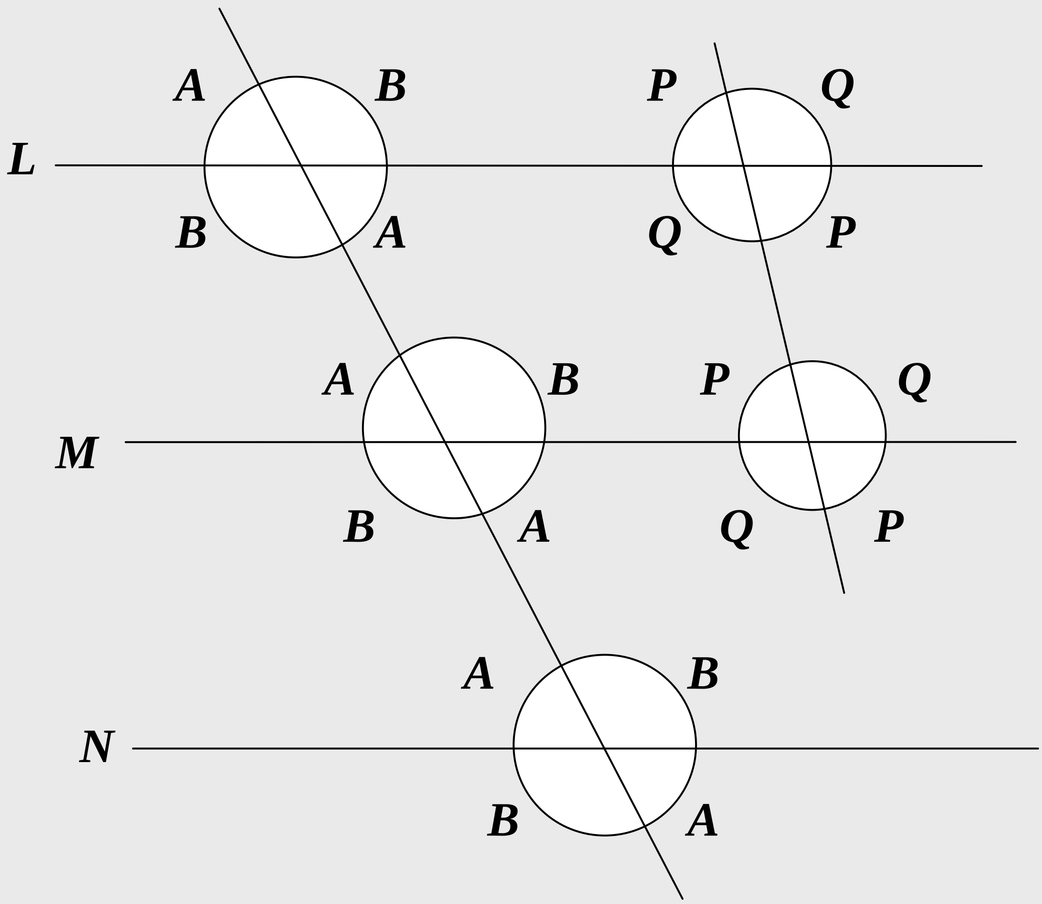
Since we have ***L*** // ***M*** // ***N***, we can have these:

∠***D*** = ∠***B*** = ∠***F*** = ∠***H*** = ∠***Y*** = ∠***Z***,

∠***P*** = ∠***R*** = ∠***T*** = ∠***W***, and ∠***Q*** = ∠***S*** = ∠***U*** = ∠***V***.

So we can put the figure above the way as follows.

Fig. 3



So the bottom line is as follows.

First, vertical angles are two same angles.

If two lines meet at a point, we get two pairs of vertical angles, which are two same angles.

Next, angles with parallel lines are the same.

The angles made by parallel lines and the transversal are the same.

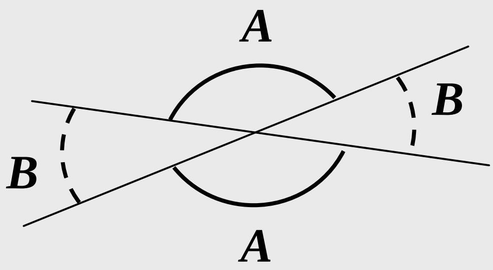
And next, with parallel lines, corresponding angles are equal, and so are alternate angles.

If the lines crossed by the transversal are parallel, corresponding angles are equal, and so are alternate angles.

And we can put the math properties above in figures the way as follows.

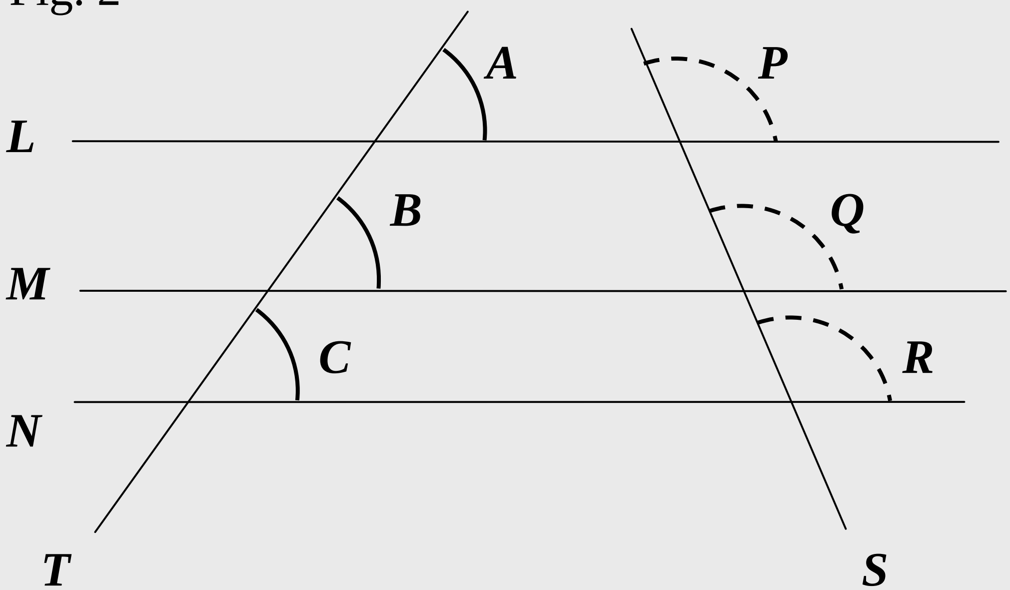
First off, vertical angles are two same angles, and are made by two lines crossing at a point as shown below.

Fig. 1



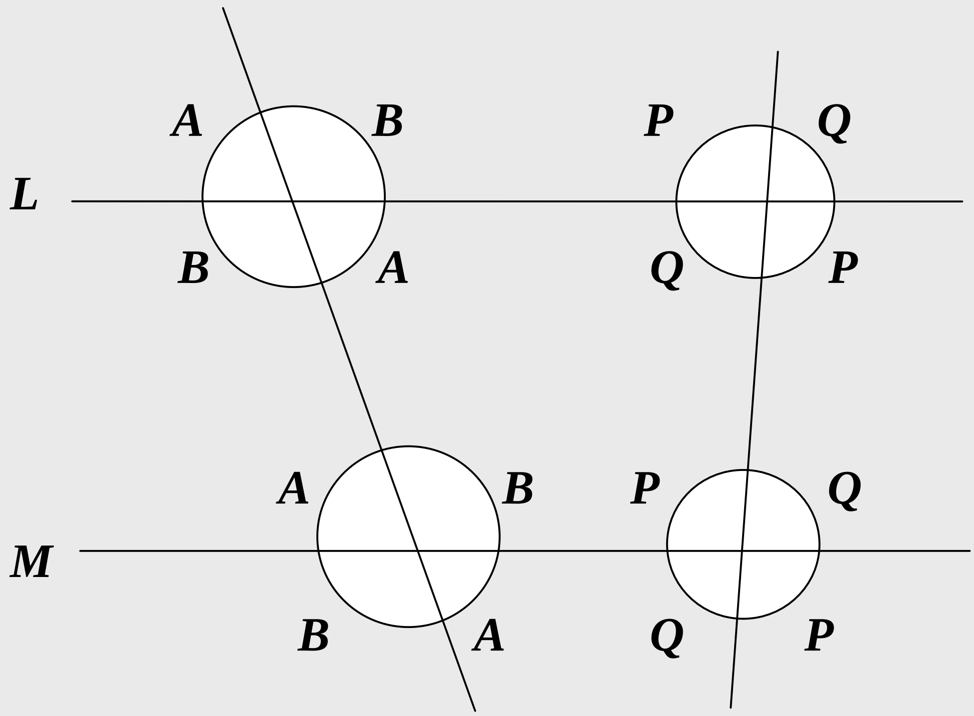
Next, in the figure below, if we have this: ***L*** // ***M*** // ***N***, we get these: $\angle$***A*** = $\angle$***B*** = $\angle$***C***, and $\angle$***P*** = $\angle$***Q*** = $\angle$***R***.

Fig. 2



And assuming ***L*** // ***M***, we have this:

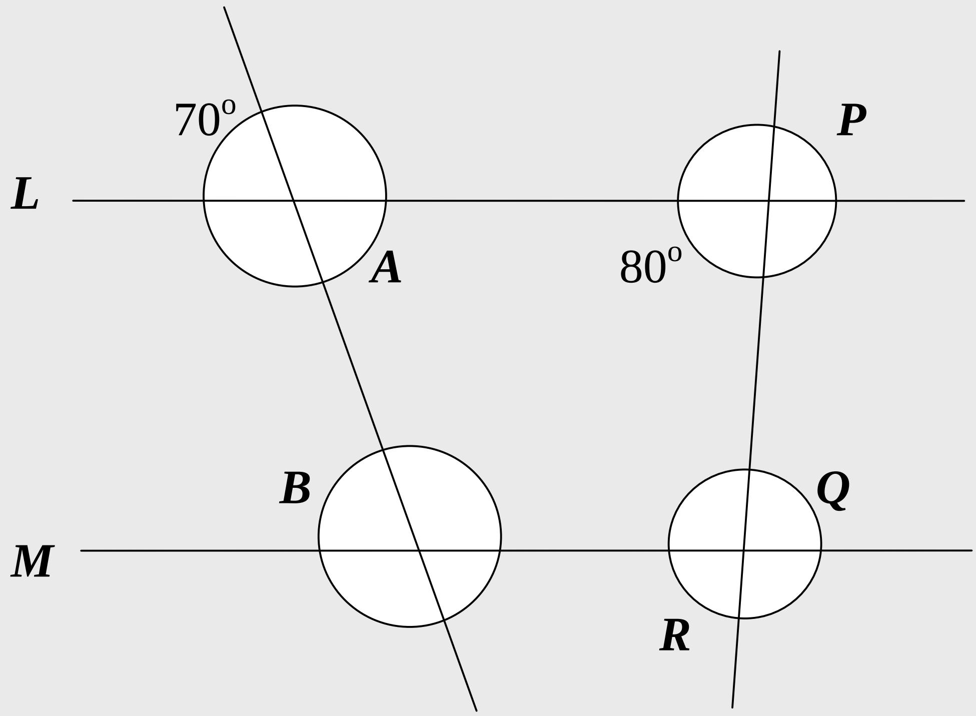
Fig. 3



So if the lines crossed by the transversal are parallel, not only vertical angles but corresponding and alternate angles are the same, too.

So for instance, what are the two angles, $\angle \boldsymbol{A}$ and $\angle \boldsymbol{B}$ in the figure below?

Fig. 4

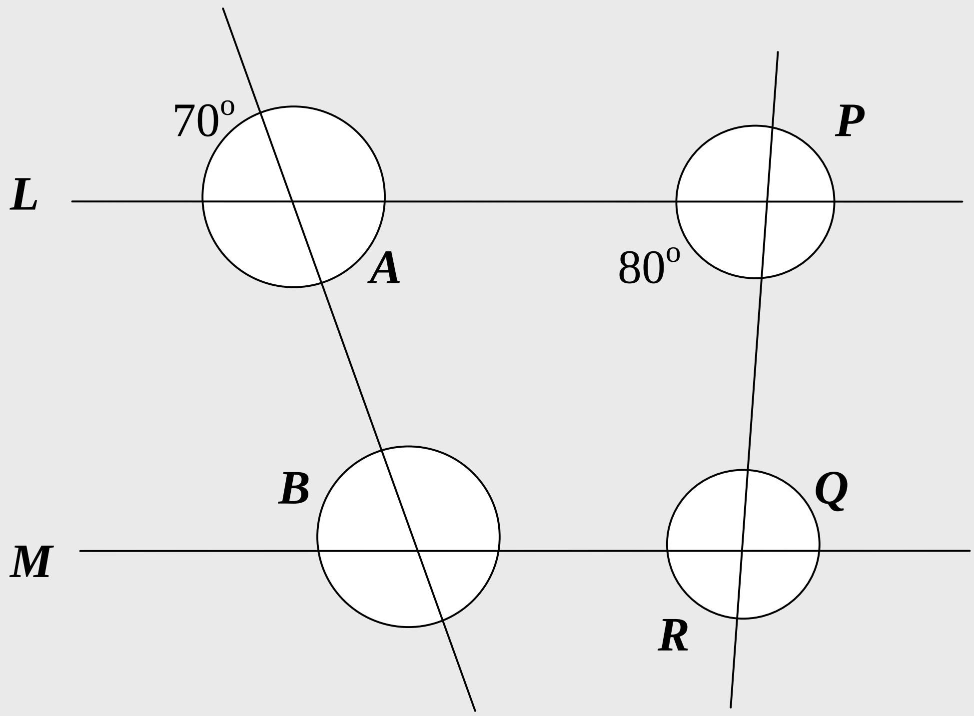


$\angle \boldsymbol{A} = \mathbf{70^o}$, because $\angle \boldsymbol{A}$ and 70^o are vertical angles, two same angles.

$\angle \boldsymbol{B} = \mathbf{70^o}$, because $\boldsymbol{L}$ // $\boldsymbol{M}$, so $\angle \boldsymbol{A}$ and $\angle \boldsymbol{B}$ are the same alternate angles, or because $\angle \boldsymbol{B}$ and 70^o are the same corresponding angles.

Next, what are $\angle P$, $\angle Q$, and $\angle R$?

Fig. 4



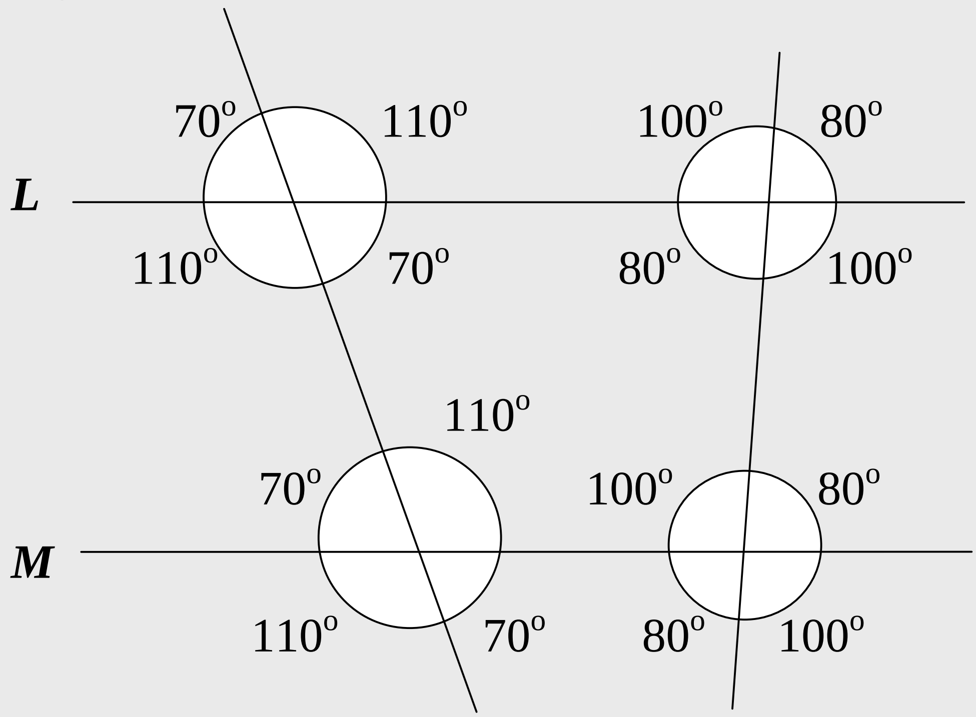
$\angle P$ **= 80°**, because $\angle P$ and 80° are vertical angles.

$\angle Q$ **= 80°**, since $\angle P$ and $\angle Q$ are the same corresponding angles, or $\angle Q$ and 80° are the same alternate angles.

$\angle R$ **= 80°**, because $\angle R$ and $\angle Q$ are vertical angles, or because $\angle R$ and 80° are the same corresponding angles, or because $\angle R$ and $\angle P$ are the same alternate angles.

And showing all the angles, we have this:

Fig. 5



So first, vertical angles are two same angles.

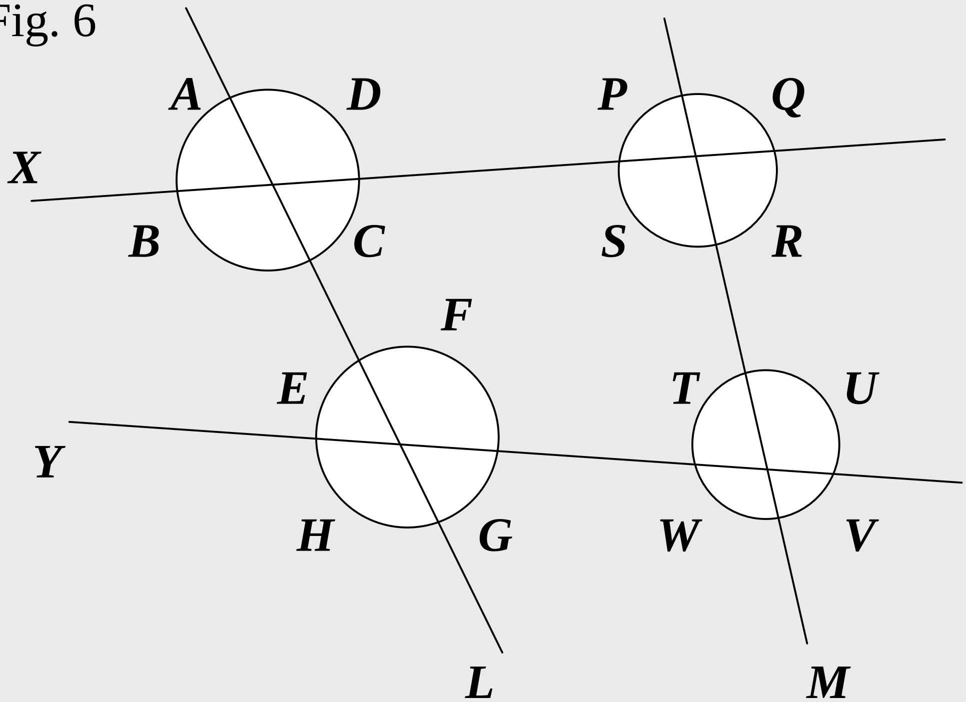
Next, angles with parallel lines are equal. And of those angles, some are the same corresponding angles, and some are the same alternate angles.

And thus, corresponding angles with parallel lines are the same, and so are alternate angles with parallel lines.

Next, in the case of the figure below, we can consider any of the four sets of three lines as follows.

$\{X, Y, L\}$, $\{X, Y, M\}$, $\{L, M, X\}$, and $\{L, M, Y\}$.

Fig. 6

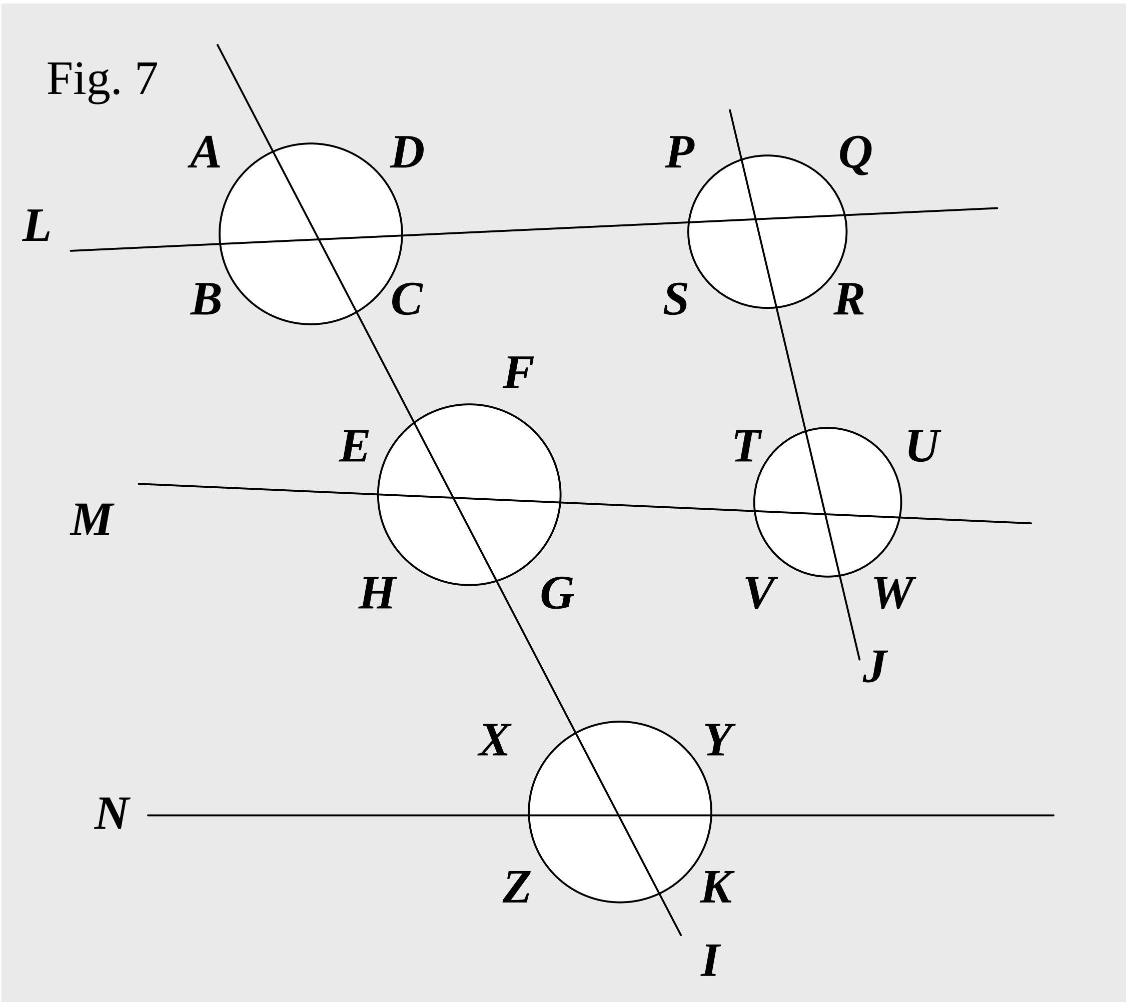


And in the case of $\{L, M, Y\}$, where the line Y is the transversal, the angles taken into consideration are these:

$\angle E$, $\angle F$, $\angle G$, $\angle H$, $\angle T$, $\angle U$, $\angle V$, and $\angle W$.

So $\angle A$ and $\angle T$ cannot be considered at the same time, and the two angles cannot be corresponding angles.

Finally, corresponding or alternate, we consider two angles at a time. And parallel or not, we consider three lines at a time, and the two of the three are the two lines crossed by the other line called the transversal.

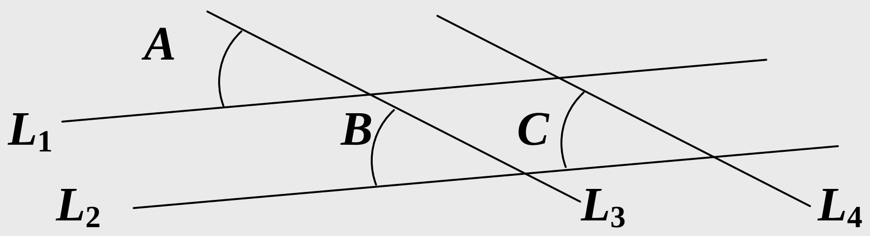


So in the case of the figure above, we can consider at a time one set of three lines as follows.

$\{L, M, I\}$, $\{L, N, I\}$, $\{I, J, M\}$, $\{L, M, J\}$, etc.

And in the figure below, if we have this: $\angle A = \angle B$, we get this: $L_1 // L_2$, because $\angle A$ and $\angle B$ are corresponding angles, since L_3 is the transversal.

Fig. 8



And if we have this: $L_1 // L_2$, and also, have this: $L_3 // L_4$, then we get this: $\angle A = \angle B = \angle C$.

It's because we have this: $\angle A = \angle B$, since $\angle A$ and $\angle B$ are corresponding angles with parallel lines, and also, we have this: $\angle B = \angle C$, since $\angle B$ and $\angle C$ are corresponding angles with parallel lines, too.