Arithmetic 2

Click this to start.

6. Multiplication Basics

Multiplication basics:

Doing a multiplication, we multiply a number by another. Then, we get a number called the product.
And when doing it, we multiply every digit by every digit for each place value, and then, add together all those individual products.

Let’s now begin with the idea as follows.

Doing a multiplication, we multiply a number by another. Then, we get a number called the product.

So multiplying a number by another, we do a multiplication, and then, get a number called the product.

A multiplication is a math operation, and doing it, we use an *operator*, together with two numbers called the *operands*.

And as the operator, we use this: x, which is put between the numbers called the operands.

So for instance, multiplying 3 by 2, we do this: 3 x 2, read as 3 multiplied by 2, read as 3 times 2, or just read as 3 by 2 for short.

Then, we get 6 called the product.

And using math language, we can simply put all the ideas above this way: 3 x 2 = 6, which is read as 3 times 2 equals 6. And for short, we can just read it this way: 3 by 2 is 6.

So 3 x 2 = 6 is saying that multiplying 3 by 2, we get 6, or that 6 is the product of 3 and 2. What then about this: 2 x 3 x 4?

It’s made of two multiplications, and we can do it this way: 2 x 3 x 4 = 6 x 4 = 24
It’s because 2 x 3 = 6, so 2 x 3 x 4 = 6 x 4, and 6 x 4 = 24.

Then, 24 can be called the product of the three numbers 2, 3, and 4.

The numbers in multiplications as 4 x 5 are single digit numbers. And times table has all kinds of multiplications with single digit numbers and their products. So memorizing times table, we can get many products on the spot.

What then about multi-digit numbers? So, for instance, what if we multiply 23 by 45?

Then, we get to use math basics as follows.

Multiplication basics:

Multiply every digit by every digit for each place value, and then, add together all those individual products.

So doing a multiplication, we multiply every digit by every digit for each place value in the numbers, and add up all the products.
For instance, multiplying 23 by 45, we do this: 23 x 45, and can do it the way as follows.

We have these: 23 = 20 + 3 and 45 = 40 + 5

So multiply 20 by 40, 20 by 5, 3 by 40, and 3 by 5, and then, add together all the products.

So we can do 23 x 45 the way as follows.

23 x 45 = (20 + 3) x (40 + 5)

= 20 x 40 + 20 x 5 + 3 x 40 + 3 x 5

= 800 + 100 + 120 + 15

= 1000 + 35 = 1035

And if getting used to metal math, we can get the same this way:

23 x 45 = 800 + 100 + 120 + 15 = 1035

And let’s see now, why it is the case.

If taking the sum of 23 of 45s, that is, if adding together twenty three 45s, we do this:

23 x 45

So if taking the product of 23 and 45, we get the sum of twenty three 45s.
Thus, the product of 23 and 45 is the sum of 23 of 45s, which means, the sum of twenty three 45s.

And the sum of 23 of 45s equals the sum of 20 of 45s and 3 of 45s, too.

So using the language of math, we can put the idea this way: 23 x 45 = 20 x 45 + 3 x 45

Next, 45 is the sum of 40 and 5.
That is, 45 = 40 + 5

So we can say that 20 of 45s are equal to 20 of (40 + 5)s.

Thus, we have this:

20 x 45 = 20 x (40 + 5)……….(1)

Next, adding together 20 of 40s and 20 of 5s,

we get 20 of (40 + 5)s.

In other words, 20 of (40 + 5)s equal the sum

of 20 of 40s and 20 of 5s.

So we have this:

20 x (40 + 5) = 20 x 40 + 20 x 5……….(2)

So, we now have these two:

20 x 45 = 20 x (40 + 5)………………..(1)

20 x (40 + 5) = 20 x 40 + 20 x 5………(2)

Thus, we have this:

20 x 45 = 20 x 40 + 20 x 5……………(3)

Now again, 45 is the sum of 40 and 5.

That is, 45 = 40 + 5

So we can say that 3 of 45s are equal to 3 of (40 + 5)s.

Thus, we have this:

3 x 45 = 3 x (40 + 5)…………..(4)

Next, adding together 3 of 40s and 3 of 5s,

we get 3 of (40 + 5)s.

In other words, 3 of (40 + 5)s equal the sum

of 3 of 40s and 3 of 5s.

So we have this:

$3 \times (40 + 5) = 3 \times 40 + 3 \times 5$………….(5)

So, we now have these:

3 x 45 = 3 x (40 + 5)……………………(4)

3 x (40 + 5) = 3 x 40 + 3 x 5………….(5)

Thus, we have this:

3 x 45 = 3 x 40 + 3 x 5…………………(6)

So now, we have these:

$20 \times 45 = 20 \times 40 + 20 \times 5$…………..(3)

$3 \times 45 = 3 \times 40 + 3 \times 5$………………(6)

And we have this: $23 \times 45 = 20 \times 45 + 3 \times 45$

Thus, we get this:

$23 \times 45 = 20 \times 40 + 20 \times 5 + 3 \times 40 + 3 \times 5$

And we have these:

$23 = 20 + 3$ and $45 = 40 + 5$

So we get this:

23×45

$= (20 + 3) \times (40 + 5)$

$= 20 \times 40 + 20 \times 5 + 3 \times 40 + 3 \times 5$

Thus, multiplying 23 by 45, we multiply 20 by 40, 20 by 5, 3 by 40, and 3 by 5, and then, add together all the products.

So when taking the product of two numbers, how do we take it?

When taking the product of two numbers, we multiply every digit in one number by every digit in the other number, and then, get the sum of all those individual products.

And we have a math law which covers the multiplication process above.

The process is in fact, an example of one of the three basic laws, and that law is called the distributive law.

Distributive Law:

Multiplication is distributive over additions and subtractions.

Putting the idea in math language, we can, for instance, put it the way as follows:

$$A \times (B + C) = A \times B + A \times C$$

$$(A + B) \times (C + D)$$

$$= A \times C + A \times D + B \times C + B \times D$$

So to begin with, what do we mean by this?

$$A \times (B + C) = A \times B + A \times C$$

For instance, since 26 = 20 + 6, doing 3 x 26,

we can do it this way:

3 x 26 = 3 x (20 + 6) = 3 x 20 + 3 x 6

= 60 + 18 = 78

Next, we can have this:

$$(\boldsymbol{A} + \boldsymbol{B}) \times (\boldsymbol{C} + \boldsymbol{D})$$

$$= \boldsymbol{A} \times \boldsymbol{C} + \boldsymbol{A} \times \boldsymbol{D} + \boldsymbol{B} \times \boldsymbol{C} + \boldsymbol{B} \times \boldsymbol{D}$$

For instance, we can have these:

23 = 20 + 3

45 = 40 + 5

So multiplying 23 by 45, we multiply 20 by 40, 20 by 5, 3 by 40, and 3 by 5, and then, add together all the products.

That is to say that we do 23 x 45 this way:

23 x 45 = 20 x 40 + 20 x 5 + 3 x 40 + 3 x 5.

The details of the distributive law are covered in a separate lesson. The law can be the reason that when we multiply a number by another, we *multiply every digit by every digit for each place value*, and then, add all the products together.

What then about this: 123 x 456?

We have these:

$123 = 100 + 20 + 3$ and $456 = 400 + 50 + 6$

So we do this: 123×456 this way:

$123 \times 456 = (100 + 20 + 3) \times (400 + 50 + 6)$

$= 100 \times 400 + 100 \times 50 + 100 \times 6$

$+ 20 \times 400 + 20 \times 50 + 20 \times 6$

$+ 3 \times 400 + 3 \times 50 + 3 \times 6$

So we can do multiplication to take the sum of many same numbers. That’s not the only reason, though, that we do multiplication. We can do it for some other reasons, too, and will cover those reasons and how we do those multiplications in separate sections, called Reasons for Multiplications.

7. Division Basics

Division basics:

Doing a division, we divide a dividend by a divisor.

When doing it, we do multiplication, though, to find a number so that the product of that number and the divisor equals the dividend.

In short, dividing, we find a number so that the product of that number and the divisor is the dividend.

For instance, if we divide 12 by 3, 12 is the dividend, 3 is the divisor, and the number we get to find is 4, because $3 \times 4 = 12$

The basics stated above may sound too theoretical. So it can be put this way, too:

Doing a division in math, we divide an amount into equal parts, and take one part.

So for instance, dividing 6 into 2 equal parts, and taking one part, we get 3, the result.

Thus, in math, dividing 6 into 2 equal parts, we do this: $6 \div 2$, and then, we get 3 as the result.

So in math, we put the idea above this way:

$6 \div 2 = 3$.

Let’s now take another example.

Dividing 12 by 3, we do this: 12 ÷ 3, read as 12 divided by 3.

And after doing this: 12 ÷ 3, we get 4 as the result of or the solution to it. So we have this:

$12 \div 3 = 4$

And we call the result 4 the quotient.

So if dividing 12 by 3, we do this 12 ÷ 3, and get 4, called the quotient, which is the result.

So doing a division, we find a number called the quotient so that the product of the divisor and that number we find equals the dividend.

What then is the dividend?

The dividend is the product of the divisor and the quotient.

So, **Divisor x Quotient = Dividend**, and thus, the dividend is a multiple of the divisor, and also, a multiple of the quotient.

We have 12 ÷ 3 = 4, since 4 x 3 = 12, so 12 is a multiple of 3, and also, is a multiple of 4.

In short, 12 is a multiple common to 3 and 4.

Memorizing times table, a.k.a. multiplication table, we can do many divisions instantly, and can do many others very quickly, too.

For instance, 4 x 3 = 12 is in times table, so memorizing the table, we can do 12 ÷ 3 and 12 ÷ 4 instantly.

What if now, the dividend is not a multiple of the divisor?

Doing such a division, too, we still find a quotient, but the product of it and the divisor is not the dividend but the divisor’s multiple *the closest to and less than* the dividend.

It sounds confusing, isn’t it.

So in short, the product of the divisor and the quotient is the divisor’s multiple the closest to and less than the dividend.

A bit better but still confusing. Clarity matters.

So for instance, if we divide 14 by 5, the quotient is 2, because 5 x 2 = 10, and 10 is the multiple the closest to and less than 14. And we have 14 – 10 = 4, which is called the remainder, which is less than the divisor.

So the difference between the dividend and the multiple is called the remainder, and the magnitude of the remainder is less than the magnitude of the divisor.

What then is the quotient?

The quotient is the maximum number of times the divisor can be removed from the dividend, and the remainder is what’s left.

So it is the maximum number of the divisors that can be subtracted from the dividend. Example is the best teacher, isn’t it?

So for instance, if we divide 19 by 5, the quotient is 3, because 3 is the maximum number of times 5 can be removed from 19.

19 – 5 – 5 – 5 = 4

And 4 is the remainder, because we have these: 3 x 5 = 15 and 19 – 15 = 4, which is less than the divisor 5.

In math, we can put the whole idea above this way: 19 = 3 x 5 + 4 and 19 ÷ 5 = 3…4, which is thus, saying that 3 is the quotient and 4 is the remainder.

What then about 18 ÷ 5?

18 ÷ 5 = 3…3, because 18 = 3 x 5 + 3, which is saying that 3 is the quotient and 3 is the remainder.

What then about 17 ÷ 5?

17 ÷ 5 = 3…2, because 17 = 3 x 5 + 2, which is saying that 3 is the quotient and 2 is the remainder.

What then about 16 ÷ 5?

16 ÷ 5 = 3…1, because 16 = 3 x 5 + 1, which saying that 3 is the quotient and 1 is the remainder.

What then about 15 ÷ 5?

$15 \div 5 = 3\ldots0$, which is just $15 \div 5 = 3$, because we have this: $15 = 3 \times 5 + 0 = 3 \times 5$, which is saying that 3 is the quotient and 0 is the remainder.

And in this case, we say that 5 divides 15, and 5 is a divisor of 15.

So if the remainder is 0 after we divide a number by a particular number, we call the particular number a divisor. And we say that *the particular number **divides** the number*.

For instance, 3 is a divisor of 6, and 2 divides 4, since the remainder is 0.
So in math, the word, divisor is overused.

Whether the remainder is 0 or not, any number by which we divide a number is called a divisor, too.

Dividing, we divide a dividend by a divisor. So in this: 7 ÷ 5, 7 is a dividend and 5 is a divisor, but the remainder is not 0.

So there is another word we can use if no remainder is made, that is, the remainder is 0.

And we call it a *factor*. For instance, 2 is a factor of 4, and 3 is a factor of 12, because 2 divides 4, and 3 divides 12, since in each case, the remainder is 0.

What then about this: $126 \div 5$?

126 = 100 + 25 + 1 = 20 x 5 + 5 x 5 + 1

= (20 + 5) x 5 + 1 = 25 x 5 + 1

So we get this: 126 ÷ 5 = 25…1, where 25 is the quotient and 1 is the remainder. And 5 doesn’t divide 126.

What then about 126 ÷ 3?

126 = 120 + 6 = 40 x 3 + 2 x 3 = (40 + 2) x 3

So we get this: 126 ÷ 3 = 42, where 42 is the quotient and 0 is the remainder.

So 3 divides 126, and is a factor of 126, of course. And, of course, 42 divides 126, also. So since we have 42 x 3 = 126, we have these: 126 ÷ 42 = 3 and 126 ÷ 3 = 42

And 126 is not only a multiple of 42 but a multiple of 3, too, so 126 is said to be a common multiple of 3 and 42.

What then about 126 ÷ 15?

$126 = 100 + 26$

$= 90 + 10 + 20 + 6$

$= 90 + 30 + 6$

$= 6 \times 15 + 2 \times 15 + 6$

$= (6 + 2) \times 15 + 6$

$= 8 \times 15 + 6$

So we get this: $126 \div 15 = 8 \ldots 6$

Oftentimes though, we just put a division the way as follows.

$126 \div 15 = 126/15$, which equals $\frac{126}{15}$, which is said to be in a fractional expression or fractional form.

So we often put a division as $3 \div 2$ in a fractional form as 3/2.

Thus, for instance, we can have this:

$3 \div 2 = 3/2$

What then do we mean by this: 3/2?

It means 3 divided by 2.

So dividing 3 by 2, we get 3/2, three halves.

Thus, 3/2 means three of halves, that is, 3 of (1/2)s, so we have this: 3/2 = 3 x (1/2)

And we can put it the way as follows.

First, dividing 1 by 2, we get 1/2, a half.

That is, we get this: 1 ÷ 2 = 1/2.

So if we do that division three times, we get these: 1/2, 1/2, and 1/2.

Thus, we get three halves, that is. 3 of (1/2)s,

and in other words, we get this: 3 x 1/2

So we get this: 1/2 + 1/2 + 1/2 = 3/2

And we have this: 1/2 + 1/2 = 2/2 = 1

So we get these:

1/2 + 1/2 + 1/2 = 3/2 = 1 + 1/2

3/2 = (2 + 1)/2 = 2/2 + 1/2 = 1 + 1/2

$14 \div 5 = 14/5$

$14 \div 5 = 2 + 4/5$

$14 \div 5 = 14/5 = 2 + 4/5 = 10/5 + 4/5$

$= (10 + 4)/5 = 14/5$

And in division, though it sounds pretty obvious, we need to keep in mind these:

No division by 0 is allowed.

So we cannot divide any number by 0.

No denominator can be 0.

I make a lot of mistakes not keeping the fact above. And we can solve many problems in math using the fact above.

8. Reciprocals

Doing math, we often do arithmetic. And doing it, we do *arithmetic operations*.

Doing the operations, we do *additions and multiplications*. And of course, we do these, too: subtractions and divisions.

We can replace, though, a subtraction with an addition, and a multiplication can replace a division.-

So doing a subtraction, too, we can still add a number, and even doing a division, we can still multiply by a number.
And the numbers are called the *inverses*.

So subtracting a number, we can add the inverse, and dividing by a number, we can multiply by the inverse.

In arithmetic, we have two kinds in inverses: *additive* and *multiplicative*.
So we have *additive inverses* and *multiplicative inverses*.

If additive, it’s said to be *equal and opposite*, and if multiplicative, it’s said to be *reciprocal*.

In each kind, the inverses are in a pair. And in each pair, one is the inverse of the other.

So a pair of numbers can be the inverse of each other.

We have two kinds in inverses.
One is additive and the other is multiplicative.

If multiplicative, the two numbers in a pair are reciprocal to each other.

The reciprocal of a number is one, 1 over the number, that is, one divided by the number.

So the reciprocal of 2 is 1 over 2, which is 1 divided by 2, which is 1/2 or 0.5.

And we have this: 2 x 1/2 = 2 x 0.5 = 1

If two numbers are reciprocal to each other, the product of the two is 1.

So 5 and 1/5 are reciprocal to each other.

And so are -3/4 and -4/3.

It’s because 5 x 1/5 = 1 and -3/4 x (-4/3) = 1

So we can use math basics as follows.

Math basics:

Taking the product of two numbers reciprocal to each other, we get 1.

Now, stated earlier, if multiplicative inverses, the two numbers are reciprocal to each other. And one is the inverse of the other.

For instance, the multiplicative inverse of 3 is 1/3. And that of 1/3 is 3.

It’s because 3 and 1/3 are reciprocal to each other, since we have this: 3 x 1/3 = 1.

What then do we get taking the product of a number and its multiplicative inverse?

We get 1.
Multiplying a number by its multiplicative inverse, we get 1.
So 3/4 and 4/3 are the multiplicative inverse of each other. And so are 2/9 and 9/2.

It’s because 3/4 x 4/3 = 1, and 2/9 x 9/2 = 1

1/5 is the multiplicative inverse of 5, which is the multiplicative inverse of 1/5. And one is reciprocal to the other.

So taking the product of a number and its multiplicative inverse, we get 1.

What then can we do with the fact above?

Taking the product of a number and its multiplicative inverse, we get 1.

So multiplying by the multiplicative inverse of a number, we divide by the number.
For instance, multiplying 3 by 1/3, we get 1, so we have this: $3 \times 1/3 = 3 \div 3$

And 1/3 is the multiplicative inverse of 3.

So?

So, dividing by a number, we can multiply by the multiplicative inverse of the number.

For instance, if dividing by 4, we can multiply by 1/4, which is the multiplicative inverse of 4, since 1/4 is the reciprocal of 4.

So even doing a division, we can still multiply
by a number, called the multiplicative inverse.

And the multiplicative inverse is reciprocal to
the number by which we divide a number.
So if dividing by 7, we can multiply by the
number reciprocal to 7.
The number reciprocal to 7 is 1/7.

Dividing thus, by 7, we can multiply by 1/7.

So we have 14 ÷ 7 = 14 x 1/7, which is 14/7.

Dividing thus, by a number, we can multiply by the multiplicative inverse of the number, which is the reciprocal to the number, often just called the reciprocal for short.

Long story short, dividing, we can multiply by the reciprocal.

$9 \div 3 = 9 \times 1/3 = 9/3$

$3 \div (-3) = 3 \times (-1/3) = -3/3$

$-9 \div 1/3 = -9 \times 3 = -27$

By the way, multiplying or dividing a number by –1, we change its sign.

3 x (-1) = -3, and 3 ÷ (-1) = 3/(-1) = -3

And there are only two numbers that are their own reciprocals. The two are 1 and –1.

1/1 = 1, and 1/(-1) = -1.

9. Reasons for Multiplications 1

This section covers some reasons for multiplications, so you'll get some answers to this question: Why multiplications?

To begin with, getting the sum of a few same numbers, we can get it doing a few additions.

5 + 5 = 10

7 + 7 + 7 = 14 + 7 = 21

What if now, we want to get a sum of many same numbers as fifteen nines, 15 of 9s? Though we can do many additions if adding together many of the same numbers, we may want to do something else.

Taking a sum like this: 7 + 7 + 7 + 7 + 7 + 7 or this: 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5, what are we adding up?

Multiple same numbers. And their sum?

It’s a multiple. So why bother doing many additions? We just get a multiple, a multiple of 7 as 42 or a multiple of 5 as 40.
So it’s a multiplication, and doing it, we can get a sum of many same numbers at once.

Multiplying, for instance, some 5s together, we get a multiple of 5 as 25 or 40.

So doing a multiplication, we can get a sum of many same numbers in a snap.

Multiplying one of the same numbers by the number of the same numbers, we get a multiple of the same number, and it's the sum of all those same numbers.

$$7 + 7 + 7 + 7 + 7 + 7 = 6 \times 7$$

$$5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 = 8 \times 5$$

Doing thus, a multiplication, we can get such a sum at once, that is, we can get a sum of many same numbers in a snap.

It’s not the only purpose of multiplication, though. Doing it, we can get a portion, too, which will be covered in the next section.

Let’s now take some examples, and check some basics on multiplications.

Instead of adding together five same numbers, we can multiply a number by 5. So getting the sum of five 7s, instead of doing many additions, we can multiply 7 by 5. Multiplying 7 by 5, we do this: 7 x 5. And doing it, we get 35, called the product. Then, the product is the sum, that is, the product 35 is the sum of five 7s.

Multiplying numbers together, we get a number called the product.

Memorizing multiplication table a.k.a. times table, we can get many products fast at once, so we can get the sum of five 7s in no time.

It’s not easy to memorize it, but it’s worth it. And it depends on the help you get. Doing right examples, you’ll get it soon enough. Multiplying together 5 and 7, we can multiply either 5 by 7 or 7 by 5, and in both cases, get 35, called the product of 5 and 7.

Also, we can call it a multiple of 5 and a multiple of 7 at the same time. So we can call 35 a common multiple of 5 and 7.

35 is a multiple common to 5 and 7.
And we have math basics saying this:
Multiplication commutes.

It’s a math law, called the ***commutative law***.

So the calculation can commute, that is, it can go forward and backward.

So we can have this: 5 x 7 = 7 x 5, which means this:

5 + 5 + 5 + 5 + 5 + 5 + 5 = 7 + 7 + 7 + 7 + 7

Thus, we have this: 5 x 7 = 7 x 5 = 35, and multiplication commutes.

Let's now take another example.

Taking the sum of nineteen of 9s or nine of 19s, we can multiply 19 by 9 or 9 by 19 instead of doing many additions.

So doing 19 x 9 or 9 x 19, we can get the sum of those many at once, and we have this: 19 x 9 = 9 x 19 due to the ***commutative law***.

Memorizing multiplication table and practicing some mental math, we can get many products fast, so we can get the sum of nineteen 9s or the sum of nine 19s quickly at once, and it’s 171.

$19 \times 9 = 9 \times 19 = 90 + 81 = 171$ How?

We have another math basics saying this:

Multiplication is distributive.

So multiplication can be distributed over addition and subtraction:

$$A \times (B + C) = A \times B + A \times C$$

$$A \times (B - C) = A \times B - A \times C$$

In fact, it can be distributed over any number of additions and subtractions, and also, over any combinations of both.

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C} + \boldsymbol{A} \times \boldsymbol{D}$$

$$\boldsymbol{A} \times (\boldsymbol{B} - \boldsymbol{C} - \boldsymbol{D}) = \boldsymbol{A} \times \boldsymbol{B} - \boldsymbol{A} \times \boldsymbol{C} - \boldsymbol{A} \times \boldsymbol{D}$$

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C} - \boldsymbol{D}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C} - \boldsymbol{A} \times \boldsymbol{D}$$

Reasons for Multiplications 1

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C} - \boldsymbol{D} + \boldsymbol{E} \ldots)$$

$$= \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C} - \boldsymbol{A} \times \boldsymbol{D} + \boldsymbol{A} \times \boldsymbol{E} \ldots$$

$$\boldsymbol{A} \times (\boldsymbol{B} - \boldsymbol{C} + \boldsymbol{D} - \boldsymbol{E} \ldots)$$

$$= \boldsymbol{A} \times \boldsymbol{B} - \boldsymbol{A} \times \boldsymbol{C} + \boldsymbol{A} \times \boldsymbol{D} - \boldsymbol{A} \times \boldsymbol{E} \ldots$$

$$9 \times 19 = 9 \times (10 + 9) = 9 \times 10 + 9 \times 9$$

$$= 90 + 81 = 171$$

$$9 \times 19 = 9 \times (5 + 6 + 8)$$

$$= 9 \times 5 + 9 \times 6 + 9 \times 8 = 45 + 54 + 72 = 171$$

$$9 \times 19 = 9 \times (11 + 10 - 2) = 99 + 90 - 18$$

$$= 189 - 18 = 171$$

$$9 \times 19 = 9 \times (30 - 11) = 9 \times 30 - 9 \times 11$$

$$= 270 - 99 = 170 + 100 - 99 = 171$$

9×19

$= 9 \times (10 + 9)$

$= 9 \times (5 + 6 + 8)$

$= 9 \times (5 - 3 + 9 + 8)$

$= 9 \times (2 + 3 + 5 + 4 + 5)$

$= 9 \times (4 + 2 + 7 - 3 + 5 + 4) \quad = 171$

9 x 19

= 9 x (30 – 3 – 8) = 9 x 30 – 9 x 3 – 9 x 8

= 270 – 27 – 72 = 243 – 72

= 143 + 100 – 72 = 143 + 28 = 171

We can expand the calculation this way, too:

9 x 19 = (3 + 6) x (10 + 9)

Then, multiplications happen in a league.

So we get this:

9 x 19 = (3 + 6) x (10 + 9)

= 3 x 10 + 6 x 10 + 3 x 9 + 6 x 9

= 30 + 60 + 27 + 54 = 90 + 81 = 171

$9 \times 19 = (3 + 2 + 4) \times (1 + 7 + 6 + 5)$

$= 3 \times 1 + 2 \times 1 + 4 \times 1 + 3 \times 7 + 2 \times 7 + 4 \times 7$

$+ 3 \times 6 + 2 \times 6 + 4 \times 6 + 3 \times 5 + 2 \times 5 + 4 \times 5$

$= 3 + 2 + 4 + 21 + 14 + 28$

$+ 18 + 12 + 24 + 15 + 10 + 20$

$= 72 + 99 = 171$

In short, $19 \times 9 = 9 \times 19 = 90 + 81 = 171$

So multiplying 9 by 19, we get 171, the sum of nineteen 9s, and it’s a multiple of 9. And of course, it’s a multiple of 19, too, since 171 is the sum of nine 19s, also. So it can be called a multiple common to 9 and 19.

And in the next section, we’ll cover why and how we multiply by a ratio.

10. Reasons for Multiplications 2

Doing a multiplication, we can get a sum of many same numbers at once. And it’s a multiple. So doing multiplications, we can get multiples fast. And that’s a reason for multiplication. One reason.

So another is covered in this section. And you’ll get some answers to this question:

Why multiply by a ratio as 1/2 or 3/5?

To begin with, what do we mean by a ratio?

A ratio is a number indicating a relative amount.
So for instance, the ratio of 3 to 3 is 1.
The amount of 3 relative to 3 is 1.

The ratio of 3 to 9 is a third, 1/3.
The amount of 3 relative to 9 is a third, because 3 is a third of 9.

The ratio of 8 to 4 is 2.

The amount of 8 relative to 4 is 2, because 8 is two times 4.

So in general, indicating a portion, a multiple, a portion and a whole, and a portion and a multiple, we use a ratio.

Multiplying by such a ratio, we can get a portion. Then, the next question is this: How do we do the multiplication to get a portion?

So you'll get some answers to these: What's the purpose of doing the multiplications as follows and how do we do them?

1/3 x 12,   2/3 x 12,   0.5 x 12

So, what if multiplying a number by a half?

Normally, a number indicates or represents an amount or a quantity. So if saying, for instance, *half the number*, we mean *half the amount* or *half the quantity*.

And saying that we multiply a number by 2,
we mean that we multiply an amount by 2.

So what if now, we multiply a number by 1/2
or 0.5?

We get half the number. So we divide the number by 2. Thus, we get a portion. What portion?

A portion of the number. Dividing a number by two, we get one of two same portions. And each portion is half the number.

So dividing a number by 2, we can multiply the number by a half, 1/2, the reciprocal of 2.

And we have math basics as follows.

Math basics:

Dividing, we can multiply by the reciprocal.

2 is the reciprocal of 1/2, which is the reciprocal of 2, so 2 and 1/2 are reciprocal to each other. And taking the product of two numbers reciprocal to each other, we get 1.

Now, 1/2 is a fraction, and is a portion of 1.
And multiplying a number by a fraction as 1/2,
we can get a portion of the number.

Multiplying a number by a third, we divide the
number by 3, and get a third of the number.
So we get a portion of the number.

Multiplying 6 by a third, that is, 1/3, we divide 6 by 3, and get 2, which is a third of 6, so we get a third of 6, which is a portion of 6.

And in math language, we can put the idea this way: $6 \times 1/3 = 6/3 = 6 \div 3 = 2$, which is therefore, a third of six, a portion of 6.

And as we can see in the sequence of expressions here: 6 x 1/3 = 6/3 = 6 ÷ 3 = 2, when doing a multiplication to get a portion, we do a division, too.

So doing a multiplication, along with a division, we can get a portion.

By the way, if knowing some divisions, we know some multiplications, too.

For instance, 6 ÷ 3 = 2, because 2 x 3 = 6 and 3 x 2 = 6, since 2 multiplied by 3 equals 6, which equals also, 3 times 2.

So without knowing some multiplications as those in times table, we can’t do quite a few divisions as this: 15 ÷ 3 or this: 15/3.

Without times table in your memory, you can’t do many divisions with no calculator.
Times table matters.
And it does do a lot.

So now, getting a portion that is a third of 6, for instance, we do a multiplication, along with a division. A third of an amount is one of three equal parts adding up to the amount.

So dividing 1 into three equal parts and taking one part, we get a third, 1/3.
What then about two thirds?

Two thirds of a number?

Twice a third of a number?

Yeah, it’s twice a third of an amount.

Taking the product of two thirds and 9, we can get a portion of 9, and it’s two thirds of 9.

And getting it, we multiply two thirds by 9, that is, we do this: 2/3 x 9, and then, we get 18/3, which is 6, and is two thirds of 9. In math language, we can put the idea this way:

$$2/3 \times 9 = (2 \times 9)/3 = 18/3 = 18 \div 3 = 6$$

So multiplying two thirds by 9, we get 6.

And we have this: 2 x 1/3 = 1/3 + 1/3 = 2/3

So two thirds equals twice a third.

And 6 is twice a third of 9.

So in math language, we can put 2/3 of 9 this way, too: 2 x 1/3 x 9 = 2/3 x 9, which says, “Twice a third of 9 equals two thirds of 9.”

And we can put the same the way as follows.
Dividing 9 into 3 equal parts, and taking two parts, we get two thirds of 9.

A third of 9 is 1/3 x 9, which is 9/3, which is 3.
So taking two of 3s, we get two thirds of 9.
Then, we get 6, which is thus, two thirds of 9.

If taking two thirds of 9, we take twice a third of 9. And we can get it doing any of these:

2 x 1/3 x 9

2 x 9/3

2/3 x 9

It’s because they all produce the same.

And they can be related any of the ways as follows:

Reasons for Multiplications 2

2 x 1/3 x 9 = 2 x 9/3 = 2 x 3 = 6

2 x 1/3 x 9 = 2 x 9/3 = 18/3 = 6

2 x 1/3 x 9 = 2/3 x 9 = 18/3

So we have 2 x 1/3 x 9 = 2/3 x 9 = 2 x 9/3

And we have math basics as follows.

Math basics: *Multiplications associate, but divisions don't.*

It’s a math law, called ***associative law***, so multiplications are associative, but ***divisions are not***.

$$(8 \times 4) \times 2 = 8 \times (4 \times 2)$$

$$(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$$

In general, we have these:

$$(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = \boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$$

$$(\boldsymbol{A} \div \boldsymbol{B}) \div \boldsymbol{C} \neq \boldsymbol{A} \div (\boldsymbol{B} \div \boldsymbol{C}) \text{ if } \boldsymbol{C} \neq \mathbf{1}$$

It’s because multiplications are associative, but divisions are not.

And in multiplications, we can group them anyway we want, and can do any group first.

$$(\boldsymbol{A} \text{ x } \boldsymbol{B}) \text{ x } \boldsymbol{C} \text{ x } \boldsymbol{D} \text{ x } \boldsymbol{E} \text{ x } \boldsymbol{F} \text{ x } \boldsymbol{G}$$

$$= \boldsymbol{A} \text{ x } (\boldsymbol{B} \text{ x } \boldsymbol{C}) \text{ x } \boldsymbol{D} \text{ x } \boldsymbol{E} \text{ x } \boldsymbol{F} \text{ x } \boldsymbol{G}$$

$$= \boldsymbol{A} \text{ x } \boldsymbol{B} \text{ x } (\boldsymbol{C} \text{ x } \boldsymbol{D}) \text{ x } (\boldsymbol{E} \text{ x } \boldsymbol{F}) \text{ x } \boldsymbol{G}$$

$$= \boldsymbol{A} \text{ x } \boldsymbol{B} \text{ x } (\boldsymbol{C} \text{ x } \boldsymbol{D}) \text{ x } \boldsymbol{E} \text{ x } (\boldsymbol{F} \text{ x } \boldsymbol{G})$$

2 x 3 x 4 x 5 x 6 = (2 x 3) x 4 x 5 x 6

= 6 x 4 x 5 x 6 = 6 x (4 x 5) x 6 = 6 x 20 x 6 =

6 x (20 x 6) = 6 x 120 = 720

$$2 \times 3 \times 4 \times 5 \times 6 = 2 \times (3 \times 4) \times (5 \times 6)$$

$$= 2 \times 12 \times 30 = 2 \times (12 \times 30) = 2 \times 360$$

$$= (2 \times 12) \times 30 = 24 \times 30 = 720$$

So now, taking a portion of an amount, we multiply the amount by the ratio that explains the portion.

Taking, for instance, a portion that is two thirds of 9, we take the product of two thirds and 9, and then, we get the portion.

So getting it, we multiply two thirds by 9, that is, we do this: 2/3 x 9, and then, we get 18/3, which is 6, and is two thirds of 9.

In math language, we can put the idea this way: $2/3 \times 9 = (2 \times 9)/3 = 18/3 = 18 \div 3 = 6$

So we do a division, too, doing this: $2/3 \times 9$. Thus, doing a multiplication, along with a division, we can get a portion, a part of an amount based on a ratio.

And doing a multiplication, we can get a
multiple fast, that is, we can get a sum of two
or more of the same amounts at once.
What then about a combination?
A combination of a multiple and a portion?
As twice and a third, three times and a
quarter, or seventeen times and three fifths?
We’ll cover the answer in the next section.

11. Reasons for Multiplications 3

Doing a multiplication, we can get a multiple. And doing a multiplication, along with a division, we can get a portion.

A multiple is a sum of two or more of the same amounts as twice an amount. And a portion is a part of an amount as half an amount.

If taking the sum of three nines, we get this:

3 x 9 = 27, which is thus, a multiple of 9.

Also, if taking the sum of nine 3s, we get 27, too, because we get this: 9 x 3 = 27. So 27 is a multiple of 3, as well as a multiple of 9, so can be called a common multiple of 3 and 9.

And next, if a portion is two thirds of an amount 9, we can get the portion this way:

2/3 x 9 = 18/3 = 18 ÷ 3 = 6.

So 6 is the portion that amounts to 2/3 of 9.

What then about seven thirds?

Seven thirds of a number as 9?

So seven times a third of a number 9?

Yes, it’s seven times a third of an amount 9.
And we can get it doing a multiplication,
along with a division, too.

So taking, for instance, the product of seven
thirds and 9, do we get seven thirds of 9?

Yes, we do. What we get is not, though, just a portion. It’s a sum of a multiple of 9 and a portion of 9. And the sum equals seven thirds of 9.

So it’s a combination of a multiple and a portion. And we can get it doing a multiplication, along with a division, too.

Taking thus, the product of seven thirds and 9, we get seven thirds of 9, which equals the sum of a multiple of 9 and a portion of 9.

And getting it, we multiply seven thirds by 9, that is, we do this: 7/3 x 9, and then, we get 63/3, which is 21, and is seven thirds of 9.

And in the language of math, we can put the idea this way:

$7/3 \times 9 = (7 \times 9)/3 = 63/3 = 63 \div 3 = 21$

So multiplying seven thirds by 9, we get 21.

And we have this: $7 \times 1/3 = 7/3$

So seven thirds equals seven times a third.

And 21 is seven times a third of 9.

So in math language, we can put seven thirds of 9 this way, too: 7 x 1/3 x 9 = 7/3 x 9, which says in plane language, “Seven times a third of 9 equals seven thirds of 9.”

And we can put the same the way as follows.

Of several 9s, dividing each into 3 equal parts, and taking seven parts, we get seven thirds of 9.

What is each of the seven parts?

It’s a third of 9. A third of 9 is 1/3 x 9, which is 9/3, which is 3.

So taking seven of 3s, we take seven thirds of 9. Then, we get 21, which is thus, seven thirds of 9.

If taking seven thirds of 9, we take seven times a third of 9. And we can get it doing any of these: 7 x 1/3 x 9,   7 x 9/3,   7/3 x 9

And they can be related any of the ways as follows:

7 x 1/3 x 9 = 7 x 9/3 = 7 x 3 = 21

7 x 1/3 x 9 = 7 x 9/3 = 63/3 = 21

7 x 1/3 x 9 = 7/3 x 9 = 63/3

How then, is 21 a combination of a multiple of 9 and a portion of 9?

We can put 7/3 the way as follows.

7/3 = (6 + 1)/3 = 6/3 + 1/3 = 2 + 1/3

So two and a third equals seven thirds.

Two and a third of 9 is the sum of twice 9 and a third of 9. And we can get it at once.

Multiplying two and a third by 9, we get the sum of twice 9 and a third of 9, that is, we get the sum of 2 x 9 and 1/3 x 9.

And we have math basics as follows.

Multiplication commutes and is distributive.

In math language, it’s put this way:

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{B} + \boldsymbol{C}) \times \boldsymbol{A} = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C}$$

Since multiplication commutes, that is, it can go forward and backward, we can flip the operation, so we have this:

$$A \times (B + C) = (B + C) \times A$$

And it's distributive over addition and subtraction, so we have this:

$$A \times (B + C) = (B + C) \times A = A \times B + A \times C$$

So we can get two and a third of 9 this way:

(2 + 1/3) x 9

= 2 x 9 + 1/3 x 9 = 18 + 9/3 = 18 + 3 = 21

So two and a third of 9 is 21, which is the sum of twice 9 and a third of 9, that is, the sum of 2 x 9 and 1/3 x 9.

Let’s now take another example.

Two and a half of 6 is the sum of twice 6 and a half of 6. And we can get the sum at once.

How then can we get it?

Multiplying two and a half by 6, we get the sum of twice 6 and a half of 6, that is, we get the sum of 2 x 6 and 1/2 x 6.

And we have math basics as follows.

Multiplication commutes and is distributive.

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{B} + \boldsymbol{C}) \times \boldsymbol{A} = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C}$$

So we can get the sum of twice 6 and half a 6,

that is, the sum of 2 x 6 and 1/2 x 6 this way:

$(2 + 1/2) \times 6$

$= 2 \times 6 + 1/2 \times 6 = 12 + 6/2 = 12 + 3 = 15$

So two and a half of 6 is 15.

And we have this: $2 + 1/2 = 4/2 + 1/2 = 5/2$

That is, two and a half equals five halves.

So we can get the same faster this way:

$(2 + 1/2) \times 6 = 5/2 \times 6 = 30/2 = 15$

Doing thus, a multiplication, or doing a multiplication with a division, we can get a multiple, a portion, or a combination of both.

A multiple, a portion, or their combination can be called a proportional. And a ratio as 2/3, 2, or 5/2 explains a proportional.

So multiplying an amount by a ratio as 1/2, 2/3, 2, or 5/2, we get a proportional of the amount as half, two thirds of, twice, three times, or seven halves the amount.

So when taking a proportional of an amount, we multiply the amount by the ratio that describes the proportional.

For instance, 1 and 1/3 of 6 is (1 + 1/3) of 6, which is a proportional, and we can get it doing this: (1 + 1/3) x 6

First, 1 + 1/3 = 3/3 + 1/3 = (3 + 1)/3 = 4/3

So the ratio of the proportional to 6 is 4/3.
And taking (1 + 1/3) of 6, that is, 4/3 of 6, we can do 4/3 x 6, and can do it this way:
4/3 x 6 = (4 x 6)/3 = 24/3 = 8, which is the proportional.

So the ratio of 8 to 6 is 4/3. And of course, we can do it the way as follows, too.

$$(1 + 1/3) \times 6 = 1 \times 6 + 1/3 \times 6 = 6 + 6/3 = 8$$

Multiplying an amount by the ratio of the proportional to the amount, we get the proportional.

Thus, multiplying an amount by a ratio as 1/2, 2, or 5/2, we get a proportional as half the amount, twice the amount, or five halves the amount.

If an amount is 10, we can get half the amount this way: $1/2 \times 10 = 10/2 = 5$, where 1/2 is the ratio, and 5 is the proportional.

If an amount is 7.5, we can get a proportional which is twice the amount this way:

2 x 7.5 = 2 x (7 + 0.5) = 14 + 1 = 15, where 2 is the ratio, and 15 is the proportional.

And if an amount is 4, and a proportional is five halves the amount, we can get it this way: 5/2 x 4 = (5 x 4)/2 = 20/2 = 10, where 5/2 is the ratio, and 10 is the proportional.

So getting a proportional from an amount, we can multiply the amount by a ratio.
That is to say that taking the product of an amount and a ratio, we get a proportional.

In other words, a proportional from an amount is the product of the amount and a ratio.

For instance, if a proportional from 9 is two thirds of 9, we can get the proportional doing this: 2/3 x 9.

And we can do it this way: 2/3 x 9 = 18/3 = 6, which is the proportional.

Thus, 6 is two thirds of 9.

And the ratio of 6 to 9 is 2/3.

What is a ratio?

A ratio is a relative value.

It’s a value relative to a base value.

For instance, if a base value is 9, the value of 6 relative to the value of 9 is 2/3.

In short, 6 relative to 9 is 2/3.

In other words, 6 is 2/3 of 9.

And we say that the ratio of 6 to 9 is 2/3.

And there is an important ratio we need to know very well, because we often use it. We call it percent, and use this symbol: %.

12. Precedence

Math basics: *Arithmetic operations proceed from the left to the right.*

So for instance, we have these:

$9 - 3 + 2 = 6 + 2 = 8$

$9 - 3 + 2 \neq 9 - 5$, so we can’t do $3 + 2$ first.

$9 - 3 - 2 = 6 - 2 = 4$

$9 - 3 - 2 \neq 9 - 1$, so we can’t do $3 - 2$ first.

Math basics:

Do what’s inside brackets first.

Though arithmetic operations proceed from the left to the right, we can assign *precedence* to operations using brackets. For instance, we can have this:

$$9 - (4 - 3) + 2 = 9 - 1 + 2 = 8 + 2 = 10$$

So we do 4 – 3 first, and then, proceed from the left to the right.

Using brackets, we can use any pair of these:
round brackets or parentheses ( ),
curly brackets or braces { },
or square brackets [ ].

Normally though, we don't use angled brackets for that purpose, so we don't use brackets like these: < >.

We use angled brackets for relational expressions as this: $\boldsymbol{A} > \boldsymbol{B}$, which means that $\boldsymbol{A}$ is bigger than $\boldsymbol{B}$.

And brackets can be nested.

If nested, the innermost is first.

$9 - \{(3 + 4) - 2\} = 9 - (7 - 2) = 9 - 5$

$7 - \{(2 + 4) - 3\} + 5 = 7 - \{6 - 3\} + 5$

$= 7 - 3 + 5 = 4 + 5$

$$8 - [\{(2 + 3) - 4\} + 6]$$
$$= 8 - [\{5 - 4\} + 6]$$
$$= 8 - [1 + 6]$$
$$= 8 - 7$$

$$12 - [\{(2 + 3) - 4\} + 7 + (5 - 2)]$$
$$= 12 - [\{5 - 4\} + 7 + 3]$$
$$= 12 - [1 + 7 + 3]$$
$$= 12 - 11$$

Use of different brackets could make the expression more readable, but it doesn't matter.

So we can use the same brackets, too.

$$12 - (((2 + 3) - 4) + 7 + (5 - 2))$$

$$= 12 - ((5 - 4) + 7 + 3)$$

$$= 12 - (1 + 7 + 3)$$

$$= 12 - 11$$

Math basics: *Precedence matters*.

Multiplication and division have precedence over addition and subtraction.

Neither between multiplication and division and nor between addition and subtraction.

$$9 - 3 \times 2 + 5 = 9 - 6 + 5 = 3 + 5$$

$$7 - 6 \div 2 + 4 = 7 - 3 + 4 = 4 + 4$$

$$7 - 3 \times 2 + 8 \div 4 + 5 = 7 - 6 + 2 + 5$$

$$9 - 6 \div 3 \times 4 + 5 = 9 - 2 \times 4 + 5 = 9 - 8 + 5$$

Neither between multiplication and division and nor between addition and subtraction.

So no precedence between multiplication and division, and no precedence between addition and subtraction.
Operations proceed from the left to the right.

$6 \div 3 \times 2 = 2 \times 2 = 4$, but

$6 \div 3 \times 2 \neq 6 \div 6 = 1$, so we do $6 \div 3$ first

$6 \times 8 \div 4 = 48 \div 4 = 12$, but

$6 \times 8 \div 4 \neq 6 \div 2 = 3$, so we do 6×8 first

Operations proceed from the left to the right.

$$9 - 6 \div 3 \times 2 + 6 \times 8 \div 4$$

$$= 9 - 2 \times 2 + 48 \div 4$$

$$= 9 - 4 + 12$$

$$9 + 6 \div 3 \times 2 - 6 \times 8 \div 4$$

$$= 9 + 2 \times 2 - 48 \div 4$$

$$= 9 + 4 - 12$$

By the way, instead of ÷, we frequently use a slash, /, as a division operator

And of course, we often use a horizontal bar, —, as a division operator, too.

So for instance, we have these: $9 \div 3 = 9/3$ and $(8 + 2) \div (4 + 1) = (8 + 2)/(4 + 1)$