Arithmetic 3

Click this to start.

13. Arithmetic Basics 3

Doing arithmetic, we do operations called additions, subtractions, multiplications, and divisions. It can be said however, doing arithmetic, we do either of two operations, one is addition, and the other is multiplication. How?

Subtracting a number, we can add the negative of the number.

For instance, we can have 2 – 5 = 2 + (-5) = 2 + (-2) + (-3) = 0 + (-3) = -3.
And we can have -7 – 9 = -7 + (-9) = -16.
So we have 2 – 5 = 2 + (-5), and -7 – 9 = -7 + (-9).
Thus, in general, we have ***A* – *B* = *A* + (-*B*)**.

And we know that the negative of a negative is positive.

So for instance, we get 4 – (-2) = 4 + 2 = 6, and -9 – (-4) = -9 + 4 = -5.
Thus in general, we have ***A* – (-*B*) = *A* + *B***.

What then about divisions?

Dividing by a number, we can multiply by the reciprocal of the number.

For instance, dividing 6 by 3, we can multiply 6 by 1/3, since 1/3 is the reciprocal of 3. Then, we get 2, which is the quotient we get dividing 6 by 3. And we can take 2 as a product, too, because it is the product we get multiplying 6 by 1/3.

And dividing 8 by -2, we can multiply 8 by -1/2, since -1/2 is the reciprocal of -2. Then, we get -4, which is the quotient we get dividing 8 by -2. And we can take -4 as a product, too, because it is the product we get multiplying 8 by -1/2.

For another instance, dividing 6/5 by 2/5, we can multiply 6/5 by 5/2, since 5/2 is the reciprocal of 2/5. Then, we get 3, which is the quotient we get dividing 6/5 by 2/5. And we can take 3 as a product, too, because it is the product we get multiplying 6/5 by 5/2.

And dividing -9/5 by -3/5, we can multiply -9/5 by -5/3, because -5/3 is the reciprocal of -3/5. Then, we get 3, which is the quotient we get dividing -9/5 by -3/5. And we can take 3 as a product, too, because it is the product we get multiplying -9/5 by -5/3.

What then is a reciprocal?

Dividing 1 by a number nonzero, we get the reciprocal of the number.

So for instance, 1/3 is the reciprocal of 3, because dividing 1 by 3, we get 1/3.

And -1/5 is the reciprocal of -5, because dividing 1 by -5, we get -1/5.

A zero has no reciprocal, because we have no division by 0.

And multiplying a number by its reciprocal, we get 1. And if we get 1 multiplying two numbers by each other, the two are said to be reciprocal of each other.

For instance, the reciprocal of 2/3 is 3/2, because we get $\frac{2}{3} \cdot \frac{3}{2} = \frac{6}{6} = 1$.

And the reciprocal of -5/4 is -4/5, and we get -(5/4) · -(4/5) = 20/20 = 1.

Thus, if the product of two numbers is 1, the two are reciprocal of each other.

For instance, 7/5 is the reciprocal of 5/7, which is the reciprocal of 7/5, too, because we get: (7/5) · (5/7) = 1. And -4/3 is the reciprocal of -3/4, which is the reciprocal of -4/3, also, because we get -(4/3) · -(3/4) = 1.

So in general, if $\boldsymbol{AB = 1}$, $\boldsymbol{A}$ is the reciprocal of $\boldsymbol{B}$, and $\boldsymbol{B}$ is the reciprocal of $\boldsymbol{A}$, too, and we say that $\boldsymbol{A}$ and $\boldsymbol{B}$ are reciprocal of each other.

And if $\boldsymbol{A}$ and $\boldsymbol{B}$ both are not zero, $\boldsymbol{\frac{A}{B}}$ and $\boldsymbol{\frac{B}{A}}$ are reciprocal of each other.

So $\boldsymbol{\frac{A}{B}}$ is the reciprocal of $\boldsymbol{\frac{B}{A}}$, which is the reciprocal of $\boldsymbol{\frac{A}{B}}$, too.

Swapping thus, the numerator and the denominator, we get the reciprocal.
What then about the reciprocal of 5?

We know 5 = 5/1, so taking the reciprocal of 5, we get 1/5.
What then about the reciprocal of 1?

The reciprocal of 1 is 1, because 1 x 1 = 1, and if the product of two numbers is 1, the two are reciprocal of each other, that is, one is the reciprocal of the other, and vice versa.

And dividing by a number, we can multiply by the reciprocal of the number.

Dividing for instance, 6 by 3, we can multiply 6 by 1/3, and get 2.

And dividing 1 by 1/4, we can multiply 1 by 4, and get 4.
In fact, adding together four of (1/4)s, we get 1, that is, multiplying 4 by 1/4, we get 1.

Next, we know if in multiplication, the number of negatives is odd, the product is negative.
And if the number of negatives is even, the product is positive.

The same is true for divisions, too.

So in division, if the number of negatives is odd, the quotient is negative.
And if the number of negatives is even, the quotient is positive.

It's because a division by a number is a multiplication by its reciprocal, and taking a reciprocal of a number, we don't change the sign of the number.

So for instance, we get 8 ÷ (-2) = -4, that is, $\frac{8}{-2}$ = 8/(-2) = -4. So we get 8/(-2) = -8/2.

In fact, multiplying by the same number, the numerator and the denominator, we get the same fraction. So in the fraction 8/(-2), multiplying 8 and -2 both by –1, we get -8/2.

And for another instance, 24 ÷ (-4) ÷ (-2) = 3, that is, 24/(-4)/(-2) = 3.

That's because 24/(-4) = -6. So we get 24/(-4)/(-2) = -6/(-2) = 3.

And we can put it this way, too: 24/(-4)/(-2) = 24/8 = 3,

because 24/(-4)/(-2) = 24/{(-4)·(-2)} = 24/8 = 3.

That is to say that $\dfrac{\frac{24}{-4}}{-2} = \dfrac{24}{8} = 3$, because $\dfrac{\frac{24}{-4}}{-2} = \dfrac{24}{\{(-4)\cdot(-2)\}} = \dfrac{24}{8} = 3$.

And for another instance, we get -24 ÷ (-4) ÷ (-2) = -3, that is, -24/(-4)/(-2) = -3.

That's because -24/(-4) = 6.   So we get -24/(-4)/(-2) = 6/(-2) = -3.

And we can put it this way, too: -24/(-4)/(-2) = -24/8 = -3,

because -24/(-4)/(-2) = -24/{(-4)·(-2)} = -24/8 = -3.

And for another instance, we get $\dfrac{-2}{3}\cdot\dfrac{5}{-7}\cdot\dfrac{-8}{9}\cdot\dfrac{-12}{11}\cdot\dfrac{8}{-3} = -\dfrac{2}{3}\cdot\dfrac{5}{7}\cdot\dfrac{8}{9}\cdot\dfrac{12}{11}\cdot\dfrac{8}{3}$

That's because the entire operation has an odd number of negatives.

And for another instance, we can get

-37/(-7)/5/(-2) = -37/70, and -37/(-7)/(-5)/(-2) = 37/70.

So in general, doing divisions of numbers positive and negative mixed together, we can just take all the numbers as positive numbers, take the quotient, and then, add the negative sign to the quotient if odd is the number of all the negatives.

For instance, doing this: -37/(-7)/5/(-2), do this: -37/7/5/2, because the number of negatives is 3, which is odd.

For another instance, doing this: -37/(-7)/(-5)/(-2), just do this: 37/7/5/2, because the number of negatives is 4, which is even.

14. Rational Numbers 1

To begin with, what are rational numbers?

The word ‘rational’ in ‘rational number’ is the adjective of ratio, and has little to do with the dictionary meaning of reasonable. What ratio then is a rational number?

It can be called an *integer ratio*. More specifically, a rational number is a ratio of an integer to another as $\frac{1}{3}$, which can be taken as a ratio of 1 to 3.

So though it sounds obvious, 1 is a third of three, and 1 is three times a third, that is, $1 = 3 \times \frac{1}{3}$. And usually, we take $\frac{1}{3}$ as a third of 1 or one third or just a third.

Taking more instances of rational numbers, we have $\frac{2}{5}$, -$\frac{3}{4}$, $0.2 = \frac{2}{10} = \frac{1}{5}$, etc.

And we can take 2/5 as a ratio of 2 to 5, and 2 is two fifths of 5, and 2 is five times 2/5, that is, 2 = 5 x (2/5). Usually though, we take 2/5 as a number called two fifths.

So the nature of a rational number is an integer ratio, a ratio of an integer to another.

And rational numbers can be quickly just called rationals, and include integers, too.
So calling a number a rational, we mean it's a rational number.
Why is an integer a rational, too, though?

We can put every integer in an integer ratio, too, as $7 = \frac{7}{1}$, $-5 = \frac{-5}{1}$, $0 = \frac{0}{1}$, etc.
Thus, the set of all integers is a part of the set of all rational numbers.
So if expressing a number in terms of an integer ratio, we get a rational number.

Usually though, saying a rational number, we mean a fractional number as $\frac{2}{3}$, 1.3, etc.

So if expressing an amount that has a part of 1 as a half or a third, we use a rational number.
For instance, 1.3 = 1 + 0.3, which means 1 and three tenths.

What do we mean by though, three tenths?

Dividing 1 by 10, we don’t just divide 1 into 10 parts.
Dividing 1 by 10, we divide 1 into 10 equal parts, and then, take one of those ten parts, and we call the one part one tenth.

So dividing 1 by 10, we get one tenth. And we can put it this way: $1 \div 10 = \frac{1}{10} = 1/10$.

What then, about dividing 3 by 10?

Dividing 3 by 10, we divide 3 into ten equal parts, and then, take one of the ten parts, and we call the one part three tenths, which is 3/10. How?

We know 3 is three of 1s.
So we can do the same division 3 times, and the same division is the division of 1 by 10, that is $1 \div 10$. And from each of the same divisions, we get 1 tenth, which is $\frac{1}{10}$.

So dividing 3 by 10, we can do $(1 \div 10)$ three times, and then, put together the three of (1/10)s, that is, we get $\frac{1}{10}+\frac{1}{10}+\frac{1}{10}$.

So we get three tenths, and can put it this way: 3/10, which is $\frac{3}{10}$

So 3/10 is three of $\frac{1}{10}$s. That is to say that we get $\frac{3}{10} = 3 \text{ x } \frac{1}{10}$

And by the same token, dividing 7 by 10, we can do 7 times the division of 1 by 10, that is, we can do $(1 \div 10)$ seven times, and then, add together the seven of (1/10)s.

Then, we get seven tenths, and write it this way: $\frac{7}{10}$ or 7/10.

What then about using such a number as a ratio?

Assuming ***A*** is a third of ***B***, we can put it this way: ***A***/***B*** = 1/3, since 1 is a third of 3.
And in that case, we can say that ***B*** is three times ***A***.

So we get ***A*/*B* = 1/3** $\Rightarrow$ ***B*/*A* = 3/1 = 3** $\Rightarrow$ ***B* = 3** x ***A***, which means ***B*** is 3 times ***A***.

And since ***A*** is a third of ***B***, we can put it this way, too: ***A*/*B* = 2/6**, since 2 is a third of 6.

So we get ***A*/*B* = 2/6 = 1/3**.

And again, since ***A*** is a third of ***B***, we can put it this way, too: ***A*/*B* = 3/9**, since 3 is a third of 9. So we get ***A*/*B* = 3/9 = 2/6 = 1/3**.

And again, since ***A*** is a third of ***B***, we can put it this way, too: ***A*/*B* = 4/12**, since 4 is a third of 12. So we get ***A*/*B* = 1/3 = 2/6 = 3/9 = 4/12 = 5/15 …** And in fact, we have

$$\frac{2}{6}=\frac{1 \text{ x } 2}{3 \text{ x } 2}=\frac{1}{3},\ \frac{3}{9}=\frac{1 \text{ x } 3}{3 \text{ x } 3}=\frac{1}{3},\ \frac{4}{12}=\frac{1 \text{ x } 4}{3 \text{ x } 4}=\frac{1}{3},\ \frac{5}{15}=\frac{1 \text{ x } 5}{3 \text{ x } 5}=\frac{1}{3}, \text{ etc.}$$

Putting it in a slightly different form, we have

2/6 = (1 x 2)/(3 x 2) = 1/3, 3/9 = (1 x 3)/(3 x 3) = 1/3, 4/12 = (1 x 4)/(3 x 4) = 1/3, etc.

So we can see that multiplying by the same number both the numerator and denominator, we get the same ratio. (The same number cannot be 0, of course.)

<u>So the same ratios can look different.</u>

And we usually simplify a ratio so that the numerator and denominator do not share the same divisor other than 1. What same divisor though?

In the case of 2/6, 2 is a divisor of 2, and is a divisor of 6, too.

In the case of 3/9, 3 is a divisor of 3, and is a divisor of 9, too.

In the case of 4/12, 4 is a divisor of 4, and is a divisor of 12, too.

Simplifying a ratio, we divide by the same integer both the numerator and denominator so that we get smaller integers for both the numerator and denominator keeping the value of the ratio intact. (And of course, the same integer cannot be 0.)

For instance, 4/6, 6/9, and 8/12 are the same ratios, and all get simplified to 2/3, which is thus, the simplest.

And when a ratio is in its simplest form as 2/3, we say that the numerator and denominator are prime to each other, and do not share any divisor other than 1.

Now, let's move on to arithmetic on rationals.
By the way, doing a multiplication, we don't usually use the operator x, and instead, we just use a dot · or nothing if no ambiguity is expected. So for instance, meaning 3 x 4, we often put it this way: 3·4, and we usually put ***A*** x ***B*** this way: ***AB***.

So to begin with, we have

- $\frac{A}{B} = \frac{AC}{BC} = \frac{AE}{BE} = \frac{AXY}{BXY} = \ldots$, where B, C, D, E, X, and $Y \neq 0$.

And putting it in a slightly different form, we have

- $A/B = AC/BC = AD/BD = AE/BE = AXY/BXY = \ldots$ where B, C, D, E, X, and $Y \neq 0$.

So other than 0, multiplying by the same both the numerator and the denominator, we get the same ratio, that is, the value of the fraction (or the ratio) does not change.

And because of the fact above, adding fractions, we do the addition the way below:

- $\boldsymbol{\frac{A}{B}+\frac{C}{D}=\frac{AD}{BD}+\frac{BC}{BD}=\frac{AD+BC}{BD}}$, where ***B*** and $\boldsymbol{D \neq 0}$.

And putting it a bit differently, we have

$\boldsymbol{A/B + C/D = AD/BD + BC/BD = (AD + BC)/BD}$, where ***B*** and $\boldsymbol{D \neq 0}$, of course.

For instance, $\frac{2}{3}+\frac{5}{7}=\frac{2 \text{ x } 7}{3 \text{ x } 7}+\frac{5 \text{ x } 3}{7 \text{ x } 3}=\frac{14}{21}+\frac{15}{21}=\frac{14+15}{21}=\frac{29}{21}$.

And putting it in a slightly different form, we have

2/3 + 5/7 = (2 x 7)/(3 x 7) + (5 x 3)/(7 x 3) = 14/21 + 15/21 = (14 + 15)/21 = 29/21.

And next, moving on to subtractions, we can do a subtraction the way below.

- $\boldsymbol{\frac{A}{B} - \frac{C}{D} = \frac{AD}{BD} - \frac{BC}{BD} = \frac{AD - BC}{BD}}$, where ***B*** and ***D*** **≠ 0**.

And putting it a bit differently, we have

A*/*B* – *C*/*D* = *AD*/*BD* – *BC*/*BD* = (*AD* – *BC*)/*BD, where ***B*** and ***D*** **≠ 0**, of course.

For instance, $\frac{2}{3} - \frac{5}{7} = \frac{2 \text{ x } 7}{3 \text{ x } 7} - \frac{5 \text{ x } 3}{7 \text{ x } 3} = \frac{14}{21} - \frac{15}{21} = \frac{14 - 15}{21} = \frac{-1}{21} = -\frac{1}{21}$.

And putting it a bit differently, we have

2/3 – 5/7 = (2 x 7)/(3 x 7) – (5 x 3)/(7 x 3) = 14/21 – 15/21 = (14 – 15)/21 = -1/21.

So putting threads together, we have

$$\frac{A}{B}+\frac{C}{D}=\frac{AD}{BD}+\frac{BC}{BD}=\frac{AD+BC}{BD}\text{, where } B \text{ and } D\neq 0.$$

$$\frac{A}{B}-\frac{C}{D}=\frac{AD}{BD}-\frac{BC}{BD}=\frac{AD-BC}{BD}\text{, where } B \text{ and } D\neq 0.$$

And putting the two above into one expression, we get

$$\frac{A}{B}\pm\frac{C}{D}=\frac{AD}{BD}\pm\frac{BC}{BD}=\frac{AD\pm BC}{BD}\text{, where } B \text{ and } D\neq 0.$$

15. Rational Numbers 2

To begin with, going over briefly the ideas covered in the previous section, we have

- $\frac{A}{B} = \frac{AC}{BC} = \frac{AE}{BE} = \frac{AXY}{BXY} = \ldots$, where B, C, D, E, X, and $Y \neq 0$.

So other than 0, multiplying by the same both the numerator and the denominator, we get the same ratio, that is, the value of the fraction (or the ratio) does not change.

And because of the fact above, adding fractions, we can do additions the way below.

- $\frac{A}{B}+\frac{C}{D}=\frac{AD}{BD}+\frac{BC}{BD}=\frac{AD+BC}{BD}$, where B and $D \neq 0$.

For instance, $\frac{2}{3}+\frac{5}{7}=\frac{2 \text{ x } 7}{3 \text{ x } 7}+\frac{5 \text{ x } 3}{7 \text{ x } 3}=\frac{14}{21}+\frac{15}{21}=\frac{14+15}{21}=\frac{29}{21}$.

And next, moving on to subtractions, we can do a subtraction the way below.

- $\frac{A}{B}-\frac{C}{D}=\frac{AD}{BD}-\frac{BC}{BD}=\frac{AD-BC}{BD}$, where B and $D \neq 0$.

For instance, $\frac{2}{3}-\frac{5}{7}=\frac{2 \text{ x } 7}{3 \text{ x } 7}-\frac{5 \text{ x } 3}{7 \text{ x } 3}=\frac{14}{21}-\frac{15}{21}=\frac{14-15}{21}=\frac{-1}{21}=-\frac{1}{21}$.

So putting threads together, we have

$$\frac{A}{B}+\frac{C}{D}=\frac{AD}{BD}+\frac{BC}{BD}=\frac{AD+BC}{BD}, \text{ where } B \text{ and } D \neq 0.$$

$$\frac{A}{B}-\frac{C}{D}=\frac{AD}{BD}-\frac{BC}{BD}=\frac{AD-BC}{BD}, \text{ where } B \text{ and } D \neq 0.$$

And putting the two above into one expression, we get

$$\frac{A}{B}\pm\frac{C}{D}=\frac{AD}{BD}\pm\frac{BC}{BD}=\frac{AD\pm BC}{BD}, \text{ where } B \text{ and } D \neq 0.$$

Now, moving next, on to multiplications, and multiplying, for instance, 1 by 1/3, what do we get?

We get 1 x 1/3 = 1/3. And we get 2 x 1/ 3 = 2/3, 3 x 1/3 = 3/3 = 1, 4 x 1/3 = 4/3, etc.

What do we mean by though, for instance, 1 x 1/3 = 1/3?

Taking 1 of (1/3)s, we can multiply 1 by 1/3, and get 1/3.

Taking 2 of (1/3)s, we can multiply 2 by 1/3, and get 2/3.

Taking 3 of (1/3)s, we can multiply 3 by 1/3, and get 3/3 = 1.

And taking 4 of (1/3)s, we can multiply 4 by 1/3, and get 4/3.

And we know 1 x $\frac{1}{3} = \frac{1}{3}$ x 1, 2 x $\frac{1}{3} = \frac{1}{3}$ x 2, 3 x $\frac{1}{3} = \frac{1}{3}$ x 3, and 4 x $\frac{1}{3} = \frac{1}{3}$ x 4.

What then do we mean by 1 x $\frac{1}{3} = \frac{1}{3}$, for instance?

Taking a third of 1, we can multiply 1/3 by 1, and get 1/3.
Taking a third of 2, we can multiply 1/3 by 2, and get 2/3.

Taking a third of 3, we can multiply 1/3 by 3, and get 3/3 = 1.

And taking a third of 4, we can multiply 1/3 by 4, and get 4/3.

What then do we mean by 2/3 x 5/7, that is, $\frac{2}{3} \times \frac{5}{7}$, for instance?

We know 2/3 is 2 of (1/3)s.
And we know taking $\frac{1}{3} \times \frac{5}{7}$, we can say that we take a third of 5/7.

So taking $\frac{2}{3} \times \frac{5}{7}$, we take $2 \times \frac{1}{3} \times \frac{5}{7}$, so we can say that we take twice a third of 5/7.

What then is a third of 5/7?

We know 5/7 = 15/21, and a third of 15/21 is 5/21, since 15/21 is 15 of (1/21)s, and a third of 15 is 5.

So a third of 5/7, that is, $\frac{1}{3} \times \frac{5}{7}$ is 5/21.

And we know multiplying 2/3 by 5/7, that is, taking $\frac{2}{3} \times \frac{5}{7}$, we take twice a third of 5/7.

So we get $\frac{2}{3} \times \frac{5}{7} = 2 \times (\frac{1}{3} \times \frac{5}{7}) = 2 \times \frac{5}{21}$, which is 10/21.

What then do we mean by $\frac{5}{7} \times \frac{2}{3}$?

We know 5/7 is 5 of (1/7)s.

And we know taking $\frac{1}{7} \times \frac{2}{3}$, we can say that we take a seventh of 2/3.

So taking $\frac{5}{7}$ x $\frac{2}{3}$, we take 5 x $\frac{1}{7}$ x $\frac{2}{3}$, so we can say that we take 5 times a seventh of 2/3.

And we know 2/3 = 14/21, and a seventh of 14/21 is 2/21, since 14/21 is 14 of (1/21)s, and a seventh of 14 is 2. So a seventh of 2/3, that is, $\frac{1}{7}$ x $\frac{2}{3}$ is 2/21.

And we know taking $\frac{5}{7}$ x $\frac{2}{3}$, we take 5 times a seventh of 2/3.
So we get 5 x $\frac{1}{7}$ x $\frac{2}{3} = 5$ x $\frac{2}{21}$, which is 10/21, which is the same as $\frac{5}{7}$ x $\frac{2}{3} = \frac{3}{2}$ x $\frac{5}{7}$.

Now, how then do we get 10 and 21 in the fraction 10/21 above?

We get 10 multiplying 5 by 2, and get 21 multiplying 7 by 3. Specifically,

• Taking the product of the numerators of 5/7 and 2/3, we get 10, which is the numerator of 10/21, which is the result of $\frac{5}{7}$ x $\frac{2}{3}$.

• Taking the product of the denominators of 5/7 and 2/3, we get 21, which is the denominator of 10/21.

So we can notice that taking a product of fractions, we get a fraction where the numerator is the product of the numerators, and the denominator is the product of the denominators of the fractions multiplied.

In short, we get ***A*/*B*** x ***C*/*D*** =***AC*/*BD***, that is, we get $\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}$.

And we get $\frac{A}{B} \times \frac{C}{D} \times \frac{E}{F} = \frac{ACE}{BDF}$, because $\frac{AC}{BD} \times \frac{E}{F} = \frac{ACE}{BDF}$. And so forth

In the next section, we will move on to the next operations, divisions with rational numbers, after doing some more examples on additions, shown in the next pages.

16. Rational Numbers 3

First, going over some ideas in the previous sections, we can begin with

- $\frac{A}{B} = \frac{AC}{BC} = \frac{AE}{BE} = \frac{AXY}{BXY} = \ldots$, where ***B***, ***C***, ***D***, ***E***, ***X***, and $Y \neq 0$.

So other than 0, multiplying by the same both the numerator and the denominator, we get the same ratio, that is, the value of the fraction (or the ratio) does not change.

And because of the fact above, adding fractions, we do the addition the way below.

- $\frac{A}{B}+\frac{C}{D}=\frac{AD}{BD}+\frac{BC}{BD}=\frac{AD+BC}{BD}$, where B and $D \neq 0$.

For instance, $\frac{2}{3}+\frac{5}{7}=\frac{2 \text{ x } 7}{3 \text{ x } 7}+\frac{5 \text{ x } 3}{7 \text{ x } 3}=\frac{14}{21}+\frac{15}{21}=\frac{14+15}{21}=\frac{29}{21}$.

And next, moving on to subtractions, we can do a subtraction the way below.

- $\frac{A}{B}-\frac{C}{D}=\frac{AD}{BD}-\frac{BC}{BD}=\frac{AD-BC}{BD}$, where B and $D \neq 0$.

For instance, $\frac{2}{3}-\frac{5}{7}=\frac{2 \text{ x } 7}{3 \text{ x } 7}-\frac{5 \text{ x } 3}{7 \text{ x } 3}=\frac{14}{21}-\frac{15}{21}=\frac{14-15}{21}=\frac{-1}{21}=-\frac{1}{21}$.

So in sum, we have $\frac{A}{B} \pm \frac{C}{D} = \frac{AD}{BD} \pm \frac{BC}{BD} = \frac{AD \pm BC}{BD}$, where ***B*** and $D \neq 0$.

And next, moving on to multiplications, we can say that taking a product of fractions, we get a fraction where the numerator is the product of the numerators, and the denominator is the product of the denominators of the fractions multiplied.

In short, we get ***A*/*B*** x ***C*/*D*** =***AC*/*BD***, that is, we get $\frac{A}{B} \text{ x } \frac{C}{D} = \frac{AC}{BD}$.

Now, moving on to divisions, and dividing, for instance, 1 by 1/3, what do we get?

We get 3, because 1 is three times 1/3. That is, three of (1/3)s make 1.

So we get $1 \div 1 \text{ x } 3 = 3$. That is, we get $1 \div \frac{1}{3} = 3$. And we have $\frac{1}{3} \text{ x } 3 = 3 \text{ x } \frac{1}{3} = 1$.

What then, do we get if we divide 2 by 1/3?

We know 2 is two of 1s.

So we can divide 2 by 1/3 doing twice the division of 1 by 1/3, and then, adding together the two results, which are two of 3s. Then, we get 6.

And putting the idea above in expressions, we can put it the way below.

$$(1 \div \frac{1}{3}) + (1 \div \frac{1}{3}) = 2 \text{ x } (1 \div \frac{1}{3}) = 2 \text{ x } 3 = 6.$$

And we can put the same the way below, too:

Putting one pizza on top of another same pizza, we get a stack of two same pizzas.
Then, dividing the stack by 3, what do we get?

We get six equal pieces of pizza, and every piece is 1/3 of one pizza.

So we get $2 \div \frac{1}{3} = 6$.

And we have this, too: $(1 \div \frac{1}{3}) + (1 \div \frac{1}{3}) = 2 \text{ x } (1 \div \frac{1}{3}) = 2 \text{ x } 3 = 6$.

And we know $2 \div \frac{1}{3} = (1 \div \frac{1}{3}) + (1 \div \frac{1}{3}) = 2 \text{ x } 3$.

So we get $2 \div \frac{1}{3} = 2 \text{ x } 3 = 6$. (Notice that dividing by 1/3, we can multiply by 3.)

And of course, we can put the result above the way below, too:

Since 1 is three times 1/3, and 2 is twice 1, we can say that 2 is six times 1/3.

Moving next, on to another example, and dividing 4/5 by 2/3, what do we get?

We have 4/5 = 12/15, and 2/3 = 10/15.

So dividing 4/5 by 2/3, we can divide 12/15 by 10/15.

And we know 12/15 is 12 of (1/15)s, and 10/15 is 10 of (1/15)s.

And thus, dividing 4/5 by 2/3, we in fact, divide 12 by 10, that is, we get 12/10 = 6/5.

So we get $\frac{4}{5} \div \frac{2}{3} = \frac{12}{15} \div \frac{10}{15} = 12 \div 10 = \frac{12}{10} = \frac{6}{5}$.

Now, how then do we get 12 and 10 in the calculation above?

We get 12 multiplying 4 by 3, and get 10 multiplying 5 by 2. Specifically,

- Multiplying the numerator of 4/5 by the denominator of 2/3, we get 12, which is the numerator of 12/10, which is the result.

- Multiplying the denominator of 4/5 by the numerator of 2/3, we get 10, which is the denominator of 12/10.

So we can notice that dividing a fraction by another, we can multiply the fraction by the reciprocal of the other.   What reciprocal though?

For instance, the reciprocal of 2/3 is 3/2, and the reciprocal of 3/2 is 2/3.

So the product of two numbers reciprocal to each other is 1.

Thus, we can put the division above this way: $\frac{4}{5} \div \frac{2}{3} = \frac{4}{5} \times \frac{3}{2} = \frac{12}{10} = \frac{6}{5}$.

In short, we get $\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \text{ x } \frac{D}{C} = \frac{AD}{BC}$, where B, C, and $D \neq 0$, of course.

And now, putting threads altogether, we have

$$\frac{A}{B} \pm \frac{C}{D} = \frac{AD}{BD} \pm \frac{BC}{BD} = \frac{AD \pm BC}{BD}, \text{ where } B \text{ and } D \neq 0.$$

$$\frac{A}{B} \text{ x } \frac{C}{D} = \frac{A}{B} \cdot \frac{C}{D} = \frac{A}{B}\frac{C}{D} = \frac{AC}{BD}, \text{ where } B \text{ and } D \neq 0.$$

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \text{ x } \frac{D}{C} = \frac{AD}{BC}, \text{ where } B, C, \text{ and } D \neq 0.$$

And in fact, we can use any real number in the operations above if it is appropriate, that is, if it does not make any denominator 0.
For instance, we can have these:

$$\frac{0.31}{0.2}+\frac{1.7}{2.5}=\frac{0.31 \text{ x } 2.5}{0.2 \text{ x } 2.5}+\frac{0.2 \text{ x } 1.7}{0.2 \text{ x } 2.5}=\frac{(0.31 \text{ x } 2.5)+(0.2 \text{ x } 1.7)}{0.2 \text{ x } 2.5}=\frac{0.775+0.34}{0.2 \text{ x } 2.5}=\frac{1.115}{0.5}.$$

$$\frac{0.31}{0.2}-\frac{1.7}{2.5}=\frac{0.31 \text{ x } 2.5}{0.2 \text{ x } 2.5}-\frac{0.2 \text{ x } 1.7}{0.2 \text{ x } 2.5}=\frac{(0.31 \text{ x } 2.5)-(0.2 \text{ x } 1.7)}{0.2 \text{ x } 2.5}=\frac{0.775-0.34}{0.2 \text{ x } 2.5}=\frac{0.435}{0.5}.$$

$$\frac{0.31}{0.2} \text{ x } \frac{1.7}{2.5}=\frac{0.31}{0.2}\cdot\frac{1.7}{2.5}=\frac{0.31 \text{ x } 1.7}{0.2 \text{ x } 2.5}=\frac{0.527}{0.5}=\frac{5.27}{5}=\frac{527}{500}.$$

$$\frac{0.31}{0.2} \div \frac{1.7}{2.5} = \frac{0.31}{0.2} \text{ x } \frac{2.5}{1.7} = \frac{0.31 \text{ x } 2.5}{0.2 \text{ x } 1.7} = \frac{0.775}{0.34} = \frac{775}{340} = \frac{155}{68}.$$

$$\frac{\frac{2}{3}}{\frac{5}{7}} + \frac{\frac{9}{5}}{\frac{3}{8}} = \frac{\frac{2}{3} \text{ x } \frac{3}{8}}{\frac{5}{7} \text{ x } \frac{3}{8}} + \frac{\frac{5}{7} \text{ x } \frac{9}{5}}{\frac{5}{7} \text{ x } \frac{3}{8}} = \frac{(\frac{2}{3} \text{ x } \frac{3}{8}) + (\frac{5}{7} \text{ x } \frac{9}{5})}{\frac{5}{7} \text{ x } \frac{3}{8}} = \frac{\frac{6}{24} + \frac{45}{35}}{\frac{15}{56}} = \frac{\frac{6 \text{ x } 35 + 24 \text{ x } 45}{24 \text{ x } 35}}{\frac{15}{56}} = \frac{\frac{210 + 1080}{840}}{\frac{15}{56}} = \frac{\frac{1290}{840}}{\frac{15}{56}}$$

$$= \frac{\frac{129}{84}}{\frac{15}{56}} = \frac{\frac{129}{84} \text{ x } 56}{\frac{15}{56} \text{ x } 56} = \frac{\frac{7224}{84}}{15} = \frac{\frac{7224}{84} \text{ x } 84}{15 \text{ x } 84} = \frac{7224}{1260} = \frac{1806 \text{ x } 4}{315 \text{ x } 4} = \frac{1086}{315} = \frac{602 \text{ x } 3}{105 \text{ x } 3} = \frac{602}{105} = \frac{86}{15}.$$

So we can calculate $\frac{\frac{2}{3}}{\frac{5}{7}} + \frac{\frac{9}{5}}{\frac{3}{8}}$ the way below.

First, we can have.

$$\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3} \text{ x } 7}{\frac{5}{7} \text{ x } 7} = \frac{\frac{14}{3}}{5} = \frac{\frac{14}{3} \text{ x } 3}{5 \text{ x } 3} = \frac{14}{15}, \text{ and } \frac{\frac{9}{5}}{\frac{3}{8}} = \frac{\frac{9}{5} \text{ x } 8}{\frac{3}{8} \text{ x } 8} = \frac{\frac{72}{5}}{3} = \frac{\frac{72}{5} \text{ x } 5}{3 \text{ x } 5} = \frac{72}{15}.$$

So next, we get $\frac{\frac{2}{3}}{\frac{5}{7}} + \frac{\frac{9}{5}}{\frac{3}{8}} = \frac{14}{15} + \frac{72}{15} = \frac{86}{15}$.

And we can get the sum faster the way below.

To begin with, we can have $\frac{\frac{A}{B}}{\frac{C}{D}} = \frac{\frac{A}{B}D}{\frac{C}{D}D} = \frac{\frac{AD}{B}}{C} = \frac{\frac{AD}{B}\frac{1}{C}}{C\frac{1}{C}} = \frac{\frac{AD}{BC}}{1} = \frac{AD}{BC}$.

And we can put it the way below, too.

We know we can get $\frac{1}{\frac{C}{D}} = \frac{1 \cdot D}{\frac{C}{D}D} = \frac{D}{C}$. So we can get $\frac{\frac{A}{B}}{\frac{C}{D}} = \frac{A}{B} \cdot \frac{1}{\frac{C}{D}} = \frac{A}{B}\frac{D}{C} = \frac{AD}{BC}$.

So either way, we get $\dfrac{\frac{A}{B}}{\frac{C}{D}} = \dfrac{AD}{BC}$.

Thus, calculating $\dfrac{\frac{2}{3}}{\frac{5}{7}} + \dfrac{\frac{9}{5}}{\frac{3}{8}}$, we can get the sum faster the way below.

$$\frac{\frac{2}{3}}{\frac{5}{7}} + \frac{\frac{9}{5}}{\frac{3}{8}} = \frac{2\cdot 7}{3\cdot 5} + \frac{9\cdot 8}{5\cdot 3} = \frac{14+72}{15} = \frac{86}{15}.$$

And by the same token, we can calculate $\dfrac{\frac{2}{3}}{\frac{5}{7}} - \dfrac{\frac{9}{5}}{\frac{3}{8}}$ the way below.

$$\frac{\frac{2}{3}}{\frac{5}{7}} + \frac{\frac{9}{5}}{\frac{3}{8}} = \frac{2\cdot 7}{3\cdot 5} - \frac{9\cdot 8}{5\cdot 3} = \frac{14-72}{15} = -\frac{58}{15}.$$

Taking some more instances, we can have

$$\frac{-\frac{2}{5}}{\frac{7}{9}} - \frac{\frac{3}{4}}{-\frac{5}{3}} = -\frac{\frac{2}{5}}{\frac{7}{9}} + \frac{\frac{3}{4}}{\frac{5}{3}} = -\frac{2\cdot 9}{5\cdot 7} + \frac{3\cdot 3}{4\cdot 5} = \frac{-2\cdot 9\cdot 4 + 3\cdot 3\cdot 7}{4\cdot 5\cdot 7} = \frac{-72+63}{140} = -\frac{9}{140}.$$

$$\frac{\frac{12}{25}}{\frac{8}{15}} = \frac{\frac{3}{25}}{\frac{2}{15}} = \frac{\frac{3}{5}}{\frac{2}{3}} = \frac{9}{10}. \quad \frac{\frac{12}{5}}{\frac{9}{25}} = \frac{\frac{4}{5}}{\frac{3}{25}} = \frac{\frac{4}{1}}{\frac{3}{5}} = \frac{20}{3}. \quad \frac{\frac{12}{15}}{\frac{9}{15}} = \frac{\frac{12}{1}}{\frac{9}{1}} = \frac{12}{9} = \frac{4}{3}.$$

$$\frac{\frac{12}{15}}{\frac{9}{15}} = \frac{\frac{12}{1}}{\frac{9}{1}} = \frac{\frac{4}{1}}{\frac{3}{1}} = \frac{4}{3}. \quad \frac{\frac{4}{3}}{\frac{8}{9}} = \frac{\frac{1}{3}}{\frac{2}{9}} = \frac{\frac{1}{1}}{\frac{2}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}. \quad \frac{\frac{16}{15}}{\frac{8}{24}} = \frac{\frac{16}{5}}{\frac{8}{8}} = \frac{\frac{16}{5}}{1} = \frac{16}{5}.$$

$$\frac{\frac{a+b}{c}}{\frac{p-q}{r}} = \frac{(a+b)r}{(p-q)c}.$$

18. Irrational Numbers 1

To begin with, what are irrational numbers?

The word ‘irrational’ in ‘irrational number’ is the antonym of the adjective of ratio, and has little to do with the dictionary meaning of unreasonable.
What then, is an irrational number?

We know a rational number can be called an integer ratio. More specifically, a rational number is a ratio between two integers as 2/3, 3/5, -3/7, 0.12 = 12/100 = 6/50 = 3/25, etc.

So if a number is rational, it is an integer ratio, and can be expressed by a ratio between two integers. If however, we cannot put a number in an integer ratio, it is not rational, and said to be irrational. So an irrational number cannot be expressed by a ratio between two integers, and thus, is not like 1/2, 0.3, or 3/7.

And irrational numbers can be quickly called *irrationals*.
So calling a number an irrational, we mean it's an irrational number.
And a typical example of an irrational is: $\sqrt{\mathbf{2}}$, called a square root of 2.
What then is $\sqrt{\mathbf{2}}$?

If we get 2 multiplying 1 by a positive number twice, the number is $\sqrt{\mathbf{2}}$.
So multiplying 1 by $\sqrt{\mathbf{2}}$ twice, we get 2. That is, we have $\mathbf{1} \times \sqrt{\mathbf{2}} \times \sqrt{\mathbf{2}} = \mathbf{2}$.

Why does the number multiplied twice have to be a positive number though?

That's because $\sqrt{\mathbf{2}}$ is positive.

And of course, we can get 2 if we multiply 1 by $-\sqrt{\mathbf{2}}$ twice, too.

So in short, multiplying 1 by $\pm\sqrt{\mathbf{2}}$ twice, we get 2.

That is to say that we get $(\pm\sqrt{\mathbf{2}})^{2} = \mathbf{2}$. And we read $\pm\sqrt{\mathbf{2}}$ as plus or minus $\sqrt{\mathbf{2}}$.

And we want to note that any number inside the sign $\sqrt{\ }$ is positive or 0, and <u>cannot be negative</u>. And we usually call the sign a square root sign.

So in short, what's inside the square root sign is $\geq \mathbf{0}$, and can <u>never</u> be $< \mathbf{0}$. Why not?

That's because we get the number inside the square root sign multiplying 1 by a number twice, and if 1 is multiplied by a number twice, the product is $\geq \mathbf{0}$, and cannot be $< \mathbf{0}$.

So *what's inside the square root sign* is $\geq \mathbf{0}$, and can *never* be *negative*.

How then is $\sqrt{\mathbf{2}}$ an irrational?

That's because it cannot be put in an integer ratio, that is, a ratio between two integers. We will get to see why it is the case doing one of the examples on irrationals.
Showing though, it is an irrational, we can use the fact below.

• If a number is rational, it can be put in a ratio between two integers prime to each other.

That's because if we simplify such a ratio to its simplest, the ratio simplest has to be between two integers that are prime to each other. If no divisor other than 1 is common to integers, the integers are said to be prime to each other. So for instance, 12 and 25 are prime to each other, because 1 is the only divisor common to the two integers. That is, other than 1, there is no integer that can divide 12 and 25 both.

We have 12/25 = 48/100 = 0.48. So 0.48 is a rational number, and can be put in a ratio between two integers prime to each other.

And we have many kinds in irrationals. Among those, we have the circular ratio denoted by π, which is read as pi, and is 3.141592…, which is usually approximated to be 3.14.

And another example irrational can be the Euler's number, which is denoted by $\boldsymbol{e}$, and is 2.718181828459045…, which is usually approximated to be 2.718.

And such a number as $\sqrt{\mathbf{2}}$ is called a radical, too, specifically called a radical of degree two or a second degree radical.

And we call a *radical* a *root*, too. So $\sqrt{\mathbf{2}}$ is called a second root or a square root.
And in fact, a radical has its degree.

Expressing a radical in general, we can put it this way: $\sqrt[n]{\boldsymbol{a}}$, where $\boldsymbol{n}$ is an integer $\geq \mathbf{2}$, and indicates the degree of the radical. So for instance, $\sqrt[3]{\boldsymbol{a}}$ is a radical of degree 3, and the degree of $\sqrt{\mathbf{5}}$ is 2, because in fact, $\sqrt{\mathbf{5}} = \sqrt[2]{\mathbf{5}}$. What number then is $\sqrt[n]{\boldsymbol{a}}$?

If we get $\boldsymbol{a}$ if multiplying 1 by a positive number $\boldsymbol{n}$ times, that positive number is $\sqrt[n]{a}$, which is called the n^{th} root of $\boldsymbol{a}$ or the n^{th} radical of $\boldsymbol{a}$.

- And the statement above is the definition of $\sqrt[n]{a}$.

So for instance, multiplying 1 by $\sqrt[3]{2}$ three times, we get 2. And we call it the third root of 2 or the third radical of 2.

Usually though, we call $\sqrt[3]{2}$ a *cube* root of 2, and thus, call $\sqrt[3]{\ }$ a cube root sign.

Unlike the case of the square root sign, we can put any real number inside the cube root sign.

So what's inside the cube root sign can be negative, too.

That's because we get the number inside the cube root sign multiplying 1 by a number three times, and if 1 is multiplied by a number three times, the product can be negative as well as positive or 0. For instance, we can have $\sqrt[3]{\mathbf{-2}}$ as well as $\sqrt[3]{\mathbf{2}}$.

So <u>what's inside the cube root sign</u> can be <u>any real number</u>.

And covered in this book are the radicals of degrees 2 and 3. For more details on radicals and all the degrees, refer to the book called Powers and Logarithms.

So let's now move on to the arithmetic on irrationals in those two kinds.

First, the following three basic laws on arithmetic apply.

- Commutative Law

$1 + 2 = 2 + 1$, and in general, $a + b = b + a$.

$2 \times 3 = 3 \times 2$, and in general, $a \times b = b \times a$.

- Associative Law

$1 + 2 + 3 = (1 + 2) + 3 = 1 + (2 + 3)$, and in general, $a + b + c = a + b + c = a + b + c$.

$2 \times 3 \times 4 = (2 \times 3) \times 4 = 2 \times (3 \times 4)$, and in general, $a \times b \times c = (a \times b) \times c = a \times (b \times c)$.

- Distributive Law

$2(3 + 4) = 2 \times 3 + 2 \times 4$, and in general, $a(b + c) = ab + ac$

By the way, doing a multiplication, we don't usually use the operator x, and instead, we just use a dot, ·, as **2** x **3 = 2·3**, or use nothing as ***a*** x ***b* = *ab***, if no ambiguity is expected.

So to begin with, adding together 1 and $\sqrt{\mathbf{2}}$, we just get $\mathbf{1}+\sqrt{\mathbf{2}}=\sqrt{\mathbf{2}}+\mathbf{1}$.

And in general, adding together ***a*** and $\sqrt{\boldsymbol{b}}$, we just get $\boldsymbol{a}+\sqrt{\boldsymbol{b}}=\sqrt{\boldsymbol{b}}+\boldsymbol{a}$.

Next, adding together $\sqrt{\mathbf{2}}$ and $\sqrt{\mathbf{3}}$, we just get $\sqrt{\mathbf{2}}+\sqrt{\mathbf{3}}$.

And in general, adding together $\sqrt{\boldsymbol{a}}$ and $\sqrt{\boldsymbol{b}}$, we just get $\sqrt{\boldsymbol{a}}+\sqrt{\boldsymbol{b}}$.

Next, adding together $\sqrt{\mathbf{2}}$ and $\sqrt{\mathbf{2}}$, we get $\mathbf{2}\sqrt{\mathbf{2}}$. That is, we get $\sqrt{\mathbf{2}}+\sqrt{\mathbf{2}}=\mathbf{2}\sqrt{\mathbf{2}}$.

That's because two of $\sqrt{\mathbf{2}}$ s is: 2 times $\sqrt{\mathbf{2}}$, which is $\mathbf{2}\sqrt{\mathbf{2}}$.

So in general, adding together ***n*** of $\sqrt{\boldsymbol{a}}$ s, we get $\boldsymbol{n}\sqrt{\boldsymbol{a}}$.

Thus next, adding together $\sqrt{\mathbf{2}}$ and $\mathbf{3}\sqrt{\mathbf{2}}$, we get $\sqrt{\mathbf{2}}+\mathbf{3}\sqrt{\mathbf{2}}=\mathbf{4}\sqrt{\mathbf{2}}$.

And also, adding together $\mathbf{2}\sqrt{\mathbf{3}}$ and $\mathbf{5}\sqrt{\mathbf{3}}$, we get $\mathbf{2}\sqrt{\mathbf{3}}+\mathbf{5}\sqrt{\mathbf{3}}=\mathbf{7}\sqrt{\mathbf{3}}$.

So in general, adding together $\boldsymbol{n}\sqrt{\boldsymbol{a}}$ and $\boldsymbol{m}\sqrt{\boldsymbol{a}}$, we just get $\boldsymbol{n}\sqrt{\boldsymbol{a}}+\boldsymbol{m}\sqrt{\boldsymbol{a}}=(\boldsymbol{m}+\boldsymbol{n})\sqrt{\boldsymbol{a}}$.

Thus, for more instance, we get $\sqrt{\mathbf{5}}+\mathbf{2}\sqrt{\mathbf{5}}=\mathbf{3}\sqrt{\mathbf{5}}$, and $\mathbf{2}\sqrt{\mathbf{7}}+\mathbf{5}\sqrt{\mathbf{7}}=\mathbf{7}\sqrt{\mathbf{7}}$.

Next, adding together $\sqrt{\mathbf{2}}$ and $\mathbf{2}\sqrt{\mathbf{3}}$, we just get $\sqrt{\mathbf{2}}+\mathbf{2}\sqrt{\mathbf{3}}$.
And also, adding together $\mathbf{2}\sqrt{\mathbf{2}}$ and $\mathbf{5}\sqrt{\mathbf{3}}$, we just get $\mathbf{2}\sqrt{\mathbf{2}}+\mathbf{5}\sqrt{\mathbf{3}}$.

And in general, adding together $\boldsymbol{c}\sqrt{\boldsymbol{a}}$ and $\boldsymbol{d}\sqrt{\boldsymbol{b}}$, we just get $\boldsymbol{c}\sqrt{\boldsymbol{a}}+\boldsymbol{d}\sqrt{\boldsymbol{b}}$.

Next, moving on to subtractions, and subtracting 1 from $\sqrt{\mathbf{3}}$, we just get $\sqrt{\mathbf{3}}-\mathbf{1}$.

And in general, subtracting $\boldsymbol{a}$ from $\sqrt{\boldsymbol{b}}$, we just get $\sqrt{\boldsymbol{b}}-\boldsymbol{a}$.

Next, subtracting $\sqrt{\mathbf{3}}$ from 1, we just get: $\mathbf{1}-\sqrt{\mathbf{3}}$.

And in general, subtracting $\sqrt{\boldsymbol{a}}$ from $\boldsymbol{b}$, we just get: $\boldsymbol{b}-\sqrt{\boldsymbol{a}}$.

Next, subtracting $\sqrt{\mathbf{2}}$ from $\sqrt{\mathbf{3}}$, we just get $\sqrt{\mathbf{3}}-\sqrt{\mathbf{2}}$.

And in general, subtracting $\sqrt{\boldsymbol{a}}$ from $\sqrt{\boldsymbol{b}}$, we just get $\sqrt{\boldsymbol{b}}-\sqrt{\boldsymbol{a}}$.

Next, subtracting $\mathbf{3}\sqrt{\mathbf{2}}$ from $\mathbf{2}\sqrt{\mathbf{3}}$, we just get $\mathbf{2}\sqrt{\mathbf{3}}-\mathbf{3}\sqrt{\mathbf{2}}$.

And in general, subtracting $\boldsymbol{n}\sqrt{\boldsymbol{a}}$ from $\boldsymbol{m}\sqrt{\boldsymbol{b}}$, we just get $\boldsymbol{m}\sqrt{\boldsymbol{b}}-\boldsymbol{n}\sqrt{\boldsymbol{a}}$.

Next, subtracting $\mathbf{3\sqrt{2}}$ from $\mathbf{5\sqrt{2}}$, we get $\mathbf{5\sqrt{2} - 3\sqrt{2} = 2\sqrt{2}}$.

Also, subtracting $\mathbf{5\sqrt{2}}$ from $\mathbf{2\sqrt{2}}$, we get $\mathbf{2\sqrt{2} - 5\sqrt{2} = -3\sqrt{2}}$.

And in general, subtracting $\boldsymbol{n\sqrt{a}}$ from $\boldsymbol{m\sqrt{a}}$, we just get $\boldsymbol{m\sqrt{a} - n\sqrt{a} = (m - n)\sqrt{a}}$.

Next, moving on to multiplications, and multiplying 2 by $\mathbf{\sqrt{3}}$, we get $\mathbf{2} \text{ x } \mathbf{\sqrt{3} = 2\sqrt{3}}$.

And in general, multiplying $\boldsymbol{a}$ by $\boldsymbol{\sqrt{b}}$, we get: $\boldsymbol{a\sqrt{b}}$.

Next, multiplying $\mathbf{\sqrt{2}}$ by $\mathbf{\sqrt{3}}$, we get $\mathbf{\sqrt{2}\sqrt{3} = \sqrt{2 \cdot 3} = \sqrt{6}}$.

And in general, taking the product of $\boldsymbol{\sqrt{a}}$ and $\boldsymbol{\sqrt{b}}$, we get $\boldsymbol{\sqrt{a}\sqrt{b} = \sqrt{ab}}$.

- How can we get $\boldsymbol{\sqrt{a}\sqrt{b} = \sqrt{ab}}$, though?

By definition, we have $\mathbf{1}\cdot\sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{a}}=\sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{a}}=\boldsymbol{a}$, and $\mathbf{1}\cdot\sqrt{\boldsymbol{b}}\sqrt{\boldsymbol{b}}=\sqrt{\boldsymbol{b}}\sqrt{\boldsymbol{b}}=\boldsymbol{b}$.

So by definition, we get $\mathbf{1}\cdot\sqrt{\boldsymbol{ab}}\sqrt{\boldsymbol{ab}}=\sqrt{\boldsymbol{ab}}\sqrt{\boldsymbol{ab}}=\boldsymbol{ab}$.

And by definition again, we get $\mathbf{1}\cdot\sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{b}}\cdot\sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{b}}=\sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{b}}\sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{b}}=\sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{b}}\sqrt{\boldsymbol{b}}=\boldsymbol{ab}$.

Thus, we get $\sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{b}}=\sqrt{\boldsymbol{ab}}$.

So next, multiplying $\mathbf{2}\sqrt{\mathbf{2}}$ by $\mathbf{5}\sqrt{\mathbf{3}}$, we get $\mathbf{2}\sqrt{\mathbf{2}}\cdot\mathbf{5}\sqrt{\mathbf{3}}=\mathbf{2}\cdot\mathbf{5}\sqrt{\mathbf{2}}\sqrt{\mathbf{3}}=\mathbf{10}\sqrt{\mathbf{6}}$.

And in general, multiplying $\boldsymbol{c}\sqrt{\boldsymbol{a}}$ by $\boldsymbol{d}\sqrt{\boldsymbol{b}}$, we get $\boldsymbol{c}\sqrt{\boldsymbol{a}}\cdot\boldsymbol{d}\sqrt{\boldsymbol{b}}=\boldsymbol{cd}\sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{b}}=\boldsymbol{cd}\sqrt{\boldsymbol{ab}}$.

Then, what if we multiply $\mathbf{2}+\mathbf{3}\sqrt{\mathbf{2}}$ by $\mathbf{3}-\mathbf{4}\sqrt{\mathbf{5}}$?

We can use the distributive law. Then, we get

$$(2+3\sqrt{2})(3-4\sqrt{5}) = 2\cdot 3 - 2\cdot 4\sqrt{5} + 3\cdot 3\sqrt{2} - 3\sqrt{2}\cdot 4\sqrt{5} = 6 - 8\sqrt{5} + 9\sqrt{2} - 12\sqrt{10}.$$

And by the same token, we can get

$$(\sqrt{3}+3\sqrt{2})(2\sqrt{7}-4\sqrt{5}) = 2\sqrt{3}\sqrt{7} - 4\sqrt{3}\sqrt{5} + 6\sqrt{2}\sqrt{7} - 12\sqrt{2}\sqrt{5} = 2\sqrt{21} - 4\sqrt{15} + 6\sqrt{14}.$$

And in general, we have

$$(a\sqrt{x}+b\sqrt{y})(s\sqrt{u}+t\sqrt{v}) = as\sqrt{ux} + at\sqrt{vx} + bs\sqrt{uy} + bt\sqrt{vy}.$$

18. Irrational Numbers 2

Let’s first, go over some of the material in the previous section.

To begin with, what are irrational numbers?

If a number is rational, it is an integer ratio, and can be expressed by a ratio between two integers. If however, we cannot put a number in an integer ratio, it is not rational, and said to be irrational. So an irrational number cannot be expressed by a ratio between two integers, and thus, is not like 1/2, 0.3, or 3/7.

And irrational numbers can be quickly called *irrationals*.
So calling a number an irrational, we mean it’s an irrational number.

And a typical example of an irrational is: $\sqrt{\mathbf{2}}$, called a square root of 2.

And we want to note that any number inside the sign $\sqrt{\ }$ is positive or 0, and cannot be negative. And we usually call the sign a square root sign.
So in short, what's inside the square root sign is ≥ **0**, and can never be < **0**.

That's because we get the number inside the square root sign multiplying 1 by a number twice, and if 1 is multiplied by a number twice, the product is ≥ **0**, and cannot be < **0**.

So *what's inside the square root sign* is ≥ **0**, and can *never* be *negative*.

And such a number as $\sqrt{\mathbf{2}}$ is called a radical, too, specifically called a radical of degree two or a second degree radical.
And we call a *radical* a *root*, too. So $\sqrt{\mathbf{2}}$ is called a second root or a square root.

And in fact, a radical has its degree.

Expressing a radical in general, we can put it this way: $\sqrt[n]{\boldsymbol{a}}$, where $\boldsymbol{n}$ is an integer $\geq \mathbf{2}$, and indicates the degree of the radical. So for instance, $\sqrt[3]{\boldsymbol{a}}$ is a radical of degree 3, and the degree of $\sqrt{\mathbf{5}}$ is 2, because in fact, $\sqrt{\mathbf{5}} = \sqrt[2]{\mathbf{5}}$. What number then is $\sqrt[n]{\boldsymbol{a}}$?

If we get $\boldsymbol{a}$ multiplying 1 by a positive number $\boldsymbol{n}$ times, the positive number is $\sqrt[n]{\boldsymbol{a}}$, which is called the $\boldsymbol{n}^{\mathbf{th}}$ root of $\boldsymbol{a}$ or the $\boldsymbol{n}^{\mathbf{th}}$ radical of $\boldsymbol{a}$.

• And the statement above is the definition of $\sqrt[n]{\boldsymbol{a}}$.

So for instance, multiplying 1 by $\sqrt[3]{\mathbf{2}}$ three times, we get 2. And we call it the third root of 2 or the third radical of 2.

Usually though, we call $\sqrt[3]{\mathbf{2}}$ a *cube* root of 2, and thus, call $\sqrt[3]{}$ a cube root sign.

Unlike the case of the square root sign, we can put any real number inside the cube root sign.
So what's inside the cube root sign can be negative, too.

That's because we get the number inside the cube root sign multiplying 1 by a number three times, and if 1 is multiplied by a number three times, the product can be negative as well as positive or 0. For instance, we can have $\sqrt[3]{\mathbf{-2}}$ as well as $\sqrt[3]{\mathbf{2}}$.

So <u>what's inside the cube root sign</u> can be <u>any real number</u>.

And covered in this book are the radicals of degrees 2 and 3.

So let's now move on to the arithmetic on irrationals in those two kinds.
First, the following three basic laws on arithmetic apply.

- Commutative Law

$1 + 2 = 2 + 1$, and in general, $a + b = b + a$.

$2 \times 3 = 3 \times 2$, and in general, $a \times b = b \times a$.

- Associative Law

$1 + 2 + 3 = (1 + 2) + 3 = 1 + (2 + 3)$, and in general, $a + b + c = a + b + c = a + b + c$.

$2 \times 3 \times 4 = (2 \times 3) \times 4 = 2 \times (3 \times 4)$, and in general, $a \times b \times c = (a \times b) \times c = a \times (b \times c)$.

- Distributive Law

$2(3 + 4) = 2 \times 3 + 2 \times 4$, and in general, $a(b + c) = ab + ac$

By the way, doing a multiplication, we don't usually use the operator x, and instead, we just use a dot, ·, as **2 x 3 = 2·3**, or use nothing as ***a*** x ***b*** **=** ***ab***, if no ambiguity is expected.

So to begin with, adding together $\sqrt{2}$ and $3\sqrt{2}$, we get $\sqrt{2}+3\sqrt{2}=4\sqrt{2}$.

Also, adding together $2\sqrt{3}$ and $5\sqrt{3}$, we get $2\sqrt{3}+5\sqrt{3}=7\sqrt{3}$.

So in general, adding together $n\sqrt{a}$ and $m\sqrt{a}$, we just get $n\sqrt{a}+m\sqrt{a}=(m+n)\sqrt{a}$.

Next, adding together $\sqrt{2}$ and $2\sqrt{3}$, we just get $\sqrt{2}+2\sqrt{3}$.

Also, adding together $2\sqrt{2}$ and $5\sqrt{3}$, we just get $2\sqrt{2}+5\sqrt{3}$.

And in general, adding together $c\sqrt{a}$ and $d\sqrt{b}$, we just get $c\sqrt{a}+d\sqrt{b}$.

Next, subtracting $3\sqrt{2}$ from $2\sqrt{3}$, we just get $2\sqrt{3} - 3\sqrt{2}$.

And in general, subtracting $n\sqrt{a}$ from $m\sqrt{b}$, we just get $m\sqrt{b} - n\sqrt{a}$.

Next, subtracting $3\sqrt{2}$ from $5\sqrt{2}$, we get $5\sqrt{2} - 3\sqrt{2} = 2\sqrt{2}$.

And also, subtracting $5\sqrt{2}$ from $2\sqrt{2}$, we get $2\sqrt{2} - 5\sqrt{2} = -3\sqrt{2}$.

And in general, subtracting $n\sqrt{a}$ from $m\sqrt{a}$, we just get: $m\sqrt{a} - n\sqrt{a} = (m - n)\sqrt{a}$.

Next, moving on to multiplications, and multiplying 2 by $\sqrt{3}$, we get $2 \text{ x } \sqrt{3} = 2\sqrt{3}$.

And in general, multiplying a by $\sqrt{b}$, we get $a\sqrt{b}$.

Next, multiplying $\mathbf{3\sqrt{5}}$ by $\mathbf{2\sqrt{3}}$, we get $\mathbf{3\sqrt{5}\cdot 2\sqrt{3} = 3\cdot 2\sqrt{5\cdot 3} = 6\sqrt{15}}$.

And in general, multiplying $\boldsymbol{c\sqrt{a}}$ by $\boldsymbol{d\sqrt{b}}$, we get $\boldsymbol{c\sqrt{a}\cdot d\sqrt{b} = cd\sqrt{a}\sqrt{b} = cd\sqrt{ab}}$.

Then, what if we multiply $\mathbf{2+3\sqrt{2}}$ by $\mathbf{3-4\sqrt{5}}$?

We can use the distributive law. Then, we get

$$\mathbf{(2+3\sqrt{2})(3-4\sqrt{5}) = 2\cdot 3 - 2\cdot 4\sqrt{5} + 3\cdot 3\sqrt{2} - 3\sqrt{2}\cdot 4\sqrt{5} = 6 - 8\sqrt{5} + 9\sqrt{2} - 12\sqrt{10}}.$$

And by the same token, we can get

$$\mathbf{(\sqrt{3}+3\sqrt{2})(2\sqrt{7}-4\sqrt{5}) = 2\sqrt{3}\sqrt{7} - 4\sqrt{3}\sqrt{5} + 6\sqrt{2}\sqrt{7} - 12\sqrt{2}\sqrt{5} = 2\sqrt{21} - 4\sqrt{15} + 6\sqrt{14}}.$$

And in general, we have

$$(a\sqrt{x}+b\sqrt{y})(s\sqrt{u}+t\sqrt{v})=as\sqrt{ux}+at\sqrt{vx}+bs\sqrt{uy}+bt\sqrt{vy}.$$

Next, moving on to divisions, and dividing $\sqrt{2}$ by $\sqrt{3}$, we get $\sqrt{2}\div\sqrt{3}=\frac{\sqrt{2}}{\sqrt{3}}=\sqrt{\frac{2}{3}}$.

And in general, dividing $\sqrt{a}$ by $\sqrt{b}$, we get $\sqrt{a}\div\sqrt{b}=\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$.

How do we get $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$ though?

By definition, we have $\mathbf{1}\cdot\sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{a}} = \sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{a}} = \boldsymbol{a}$, and $\mathbf{1}\cdot\sqrt{\boldsymbol{b}}\sqrt{\boldsymbol{b}} = \sqrt{\boldsymbol{b}}\sqrt{\boldsymbol{b}} = \boldsymbol{b}$.

So by definition, we get $\mathbf{1}\cdot\sqrt{\frac{\boldsymbol{a}}{\boldsymbol{b}}}\cdot\sqrt{\frac{\boldsymbol{a}}{\boldsymbol{b}}} = \sqrt{\frac{\boldsymbol{a}}{\boldsymbol{b}}}\sqrt{\frac{\boldsymbol{a}}{\boldsymbol{b}}} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$.

And by definition again, we get $\mathbf{1}\cdot\frac{\sqrt{\boldsymbol{a}}}{\sqrt{\boldsymbol{b}}}\cdot\frac{\sqrt{\boldsymbol{a}}}{\sqrt{\boldsymbol{b}}} = \frac{\sqrt{\boldsymbol{a}}\sqrt{\boldsymbol{a}}}{\sqrt{\boldsymbol{b}}\sqrt{\boldsymbol{b}}} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$.

Thus, we get $\frac{\sqrt{\boldsymbol{a}}}{\sqrt{\boldsymbol{b}}} = \sqrt{\frac{\boldsymbol{a}}{\boldsymbol{b}}}$.

So next, dividing $2\sqrt{2}$ by $5\sqrt{3}$, we get $2\sqrt{2} \div 5\sqrt{3} = \dfrac{2\sqrt{2}}{5\sqrt{3}} = \dfrac{2}{5} \cdot \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{2}{5}\sqrt{\dfrac{2}{3}}$.

And in general, dividing $c\sqrt{a}$ by $d\sqrt{b}$, we get $c\sqrt{a} \div d\sqrt{b} = \dfrac{c\sqrt{a}}{d\sqrt{b}} = \dfrac{c}{d} \cdot \dfrac{\sqrt{a}}{\sqrt{b}} = \dfrac{c}{d}\sqrt{\dfrac{a}{b}}$.

Next, dividing $2+3\sqrt{2}$ by $3-4\sqrt{5}$, we just get $(2+3\sqrt{2}) \div (3-4\sqrt{5}) = \dfrac{2+3\sqrt{2}}{3-4\sqrt{5}}$.

And by the same token, we just get $(\sqrt{3}+3\sqrt{2}) \div (2\sqrt{7}-4\sqrt{5}) = \dfrac{\sqrt{3}+3\sqrt{2}}{2\sqrt{7}-4\sqrt{5}}$.

And in general, we have $(a\sqrt{x}+b\sqrt{y}) \div (s\sqrt{u}+t\sqrt{v}) = \dfrac{a\sqrt{x}+b\sqrt{y}}{s\sqrt{u}+t\sqrt{v}}$.

However, in all the cases above, it is *not* the case where we can use all real numbers as the letters used in all the operations above. Why not though?

Though it sounds quite natural, we need to keep in mind that *what's inside the square root sign* is positive or 0, simply because no number can be squared to be negative, and also, keep in mind that a denominator cannot be 0, since *no division by* 0 is allowed.

And if in a fraction, the denominator is an irrational number, we often convert the fraction to another fraction where the denominator is a rational, because it makes more sense. And if doing so, we say that we rationalize the denominator.

So assuming now, for instance $\boldsymbol{k} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$, and rationalizing the denominator of $\boldsymbol{k}$, we get

$\boldsymbol{k} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} = \frac{\mathbf{1} \cdot \sqrt{\mathbf{2}}}{\sqrt{\mathbf{2}}\sqrt{\mathbf{2}}} = \frac{\sqrt{\mathbf{2}}}{\mathbf{2}}$, which has now, a rational number as its denominator.

And rationalizing the denominator of $\frac{a}{\sqrt{b}}$, we can get $\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b}\sqrt{b}} = \frac{a\sqrt{b}}{b} = \frac{a}{b}\sqrt{b}$.

What then, about this: $\frac{\sqrt{3}}{\sqrt{2}+3}$?

Rationalizing the denominator of a fraction as the one above, we can use tools called *factorization identities*. And in this case, we can use this identity: $x^2 - y^2 = (x - y)(x + y)$.

Using thus, the identity above, we can get

$$\frac{\sqrt{3}}{\sqrt{2}+3} = \frac{\sqrt{3}(\sqrt{2}-3)}{(\sqrt{2}+3)(\sqrt{2}-3)} = \frac{\sqrt{6}-3\sqrt{3}}{2-9} = \frac{\sqrt{6}-3\sqrt{3}}{-7} = \frac{3\sqrt{3}-\sqrt{6}}{7}.$$

And rationalizing the denominator of $\frac{a\sqrt{b}+c\sqrt{d}}{x\sqrt{y}+u\sqrt{v}}$, we can get

$$\frac{a\sqrt{b}+c\sqrt{d}}{x\sqrt{y}+u\sqrt{v}}=\frac{(a\sqrt{b}+c\sqrt{d})(x\sqrt{y}-u\sqrt{v})}{(x\sqrt{y}+u\sqrt{v})(x\sqrt{y}-u\sqrt{v})}=\frac{(a\sqrt{b}+c\sqrt{d})(x\sqrt{y}-u\sqrt{v})}{x^2y-u^2v}.$$

What then about this: $\frac{\sqrt{5}}{\sqrt[3]{3}+2}$?

Using the identities $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, we can rationalize some denominators the way below.

$$\frac{1}{\sqrt[3]{a}+\sqrt[3]{b}}=\frac{\{(\sqrt[3]{a})^2-\sqrt[3]{a}\sqrt[3]{b}+(\sqrt[3]{b})^2\}}{(\sqrt[3]{a}+\sqrt[3]{b})\{(\sqrt[3]{a})^2-\sqrt[3]{a}\sqrt[3]{b}+(\sqrt[3]{b})^2\}}=\frac{\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}}{(\sqrt[3]{a}+\sqrt[3]{b})(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2})}$$

$$=\frac{\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}}{a+b}.$$

$$\frac{1}{\sqrt[3]{a}-\sqrt[3]{b}}=\frac{\{(\sqrt[3]{a})^2+\sqrt[3]{a}\sqrt[3]{b}+(\sqrt[3]{b})^2\}}{(\sqrt[3]{a}+\sqrt[3]{b})\{(\sqrt[3]{a})^2+\sqrt[3]{a}\sqrt[3]{b}+(\sqrt[3]{b})^2\}}=\frac{\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}}{(\sqrt[3]{a}-\sqrt[3]{b})(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2})}$$

$$=\frac{\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}}{a-b}.$$

So in sum, we can get

$$\frac{1}{\sqrt[3]{a}+\sqrt[3]{b}} = \frac{\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}}{a+b}, \text{ and } \frac{1}{\sqrt[3]{a}-\sqrt[3]{b}} = \frac{\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}}{a-b}.$$

And thus, getting back to $\frac{\sqrt{5}}{\sqrt[3]{3}+2}$, and rationalizing the denominator, we can get

$$\frac{\sqrt{5}}{\sqrt[3]{3}+2} = \frac{\sqrt{5}\{(\sqrt[3]{3})^2-2\sqrt[3]{3}+2^2\}}{(\sqrt[3]{3}+2)\{(\sqrt[3]{3})^2-2\sqrt[3]{3}+2^2\}} = \frac{\sqrt{5}(\sqrt[3]{9}-2\sqrt[3]{3}+4)}{(\sqrt[3]{3}+2)(\sqrt[3]{9}-2\sqrt[3]{3}+4)}$$

$$= \frac{\sqrt{5}(\sqrt[3]{9}-2\sqrt[3]{3}+4)}{(\sqrt[3]{3})^3+2^3} = \frac{\sqrt{5}(\sqrt[3]{9}-2\sqrt[3]{3}+4)}{3+8} = \frac{\sqrt{5}(\sqrt[3]{9}-2\sqrt[3]{3}+4)}{11}.$$

What then, about this: $\dfrac{\sqrt{5}}{2\sqrt[3]{3}+2}$?

$$\frac{\sqrt{5}}{2\sqrt[3]{3}+2}=\frac{\sqrt{5}}{2(\sqrt[3]{3}+1)}=\frac{\sqrt{5}\{(\sqrt[3]{3})^2-1\sqrt[3]{3}+1^2\}}{2(\sqrt[3]{3}+1)\{(\sqrt[3]{3})^2-1\sqrt[3]{3}+1^2\}}=\frac{\sqrt{5}(\sqrt[3]{9}-\sqrt[3]{3}+1)}{2\{(\sqrt[3]{3})^3+1^3\}}$$

$$=\frac{\sqrt{5}(\sqrt[3]{9}-\sqrt[3]{3}+1)}{2(3+1)}=\frac{\sqrt{5}(\sqrt[3]{9}-\sqrt[3]{3}+1)}{8}.$$

And for another instance, rationalizing the denominator of $\dfrac{\sqrt{5}}{2\sqrt[3]{3}+\sqrt[3]{2}}$, we can get

$$\frac{\sqrt{5}}{2\sqrt[3]{3}+\sqrt[3]{2}}=\frac{\sqrt{5}}{\sqrt[3]{8}\sqrt[3]{3}+\sqrt[3]{2}}=\frac{\sqrt{5}}{\sqrt[3]{8\cdot 3}+\sqrt[3]{2}}=\frac{\sqrt{5}}{\sqrt[3]{24}+\sqrt[3]{2}}$$

$$=\frac{\sqrt{5}(\sqrt[3]{24^2}-\sqrt[3]{24\cdot 2}+\sqrt[3]{2^2})}{(\sqrt[3]{24}+\sqrt[3]{2})(\sqrt[3]{24^2}-\sqrt[3]{24\cdot 2}+\sqrt[3]{2^2})}=\frac{\sqrt{5}(\sqrt[3]{24^2}-\sqrt[3]{48}+\sqrt[3]{2^2})}{24+2}$$

Meanwhile, we can get

$$\sqrt[3]{24^2}-\sqrt[3]{48}+\sqrt[3]{2^2}=\sqrt[3]{(8\cdot 3)^2}-\sqrt[3]{6\cdot 8}+\sqrt[3]{4}=\sqrt[3]{8^2}\sqrt[3]{3^2}-\sqrt[3]{8}\sqrt[3]{6}+\sqrt[3]{4}$$

$$=(\sqrt[3]{8})^2\cdot\sqrt[3]{9}-2\sqrt[3]{6}+\sqrt[3]{4}=4\sqrt[3]{9}-2\sqrt[3]{6}+\sqrt[3]{4}.$$

So we get $\dfrac{\sqrt{5}}{2\sqrt[3]{3}+\sqrt[3]{2}} = \dfrac{\sqrt{5}(\sqrt[3]{24^2}-\sqrt[3]{48}+\sqrt[3]{2^2})}{24+2} = \dfrac{\sqrt{5}(4\sqrt[3]{9}-2\sqrt[3]{6}+\sqrt[3]{4})}{26}$.

Also, using the identity $(x+y)^2 = x^2 + 2xy + y^2$ or $(x-y)^2 = x^2 - 2xy + y^2$, we can simplify some irrational numbers the way below.

$\sqrt{a+b+2\sqrt{ab}} = \sqrt{a}+\sqrt{b}$, because we have $(\sqrt{a}+\sqrt{b})^2 = a+2\sqrt{ab}+b$.

And by the same token, we can have $\sqrt{a+b-2\sqrt{ab}} = \sqrt{a}-\sqrt{b}$ if $a \geq b$.

If however, $a < b$, we get $\sqrt{a+b-2\sqrt{ab}} = \sqrt{b}-\sqrt{a}$, simply because $\sqrt{a+b-2\sqrt{ab}} > 0$.

For instance, we get $\sqrt{5-2\sqrt{6}} = \sqrt{2+3-2\sqrt{6}} = \sqrt{3}-\sqrt{2}$. That's because we can get

$$(\sqrt{3}-\sqrt{2})^2 = 3-2\sqrt{3}\sqrt{2}+2 = 3+2-2\sqrt{3\cdot 2} = 5-2\sqrt{6} \Rightarrow (\sqrt{3}-\sqrt{2})^2 = 5-2\sqrt{6}$$

$$\Rightarrow \sqrt{5-2\sqrt{6}} = \sqrt{(\sqrt{3}-\sqrt{2})^2} = \sqrt{3}-\sqrt{2} \Rightarrow \sqrt{5-2\sqrt{6}} = \sqrt{3}-\sqrt{2}.$$