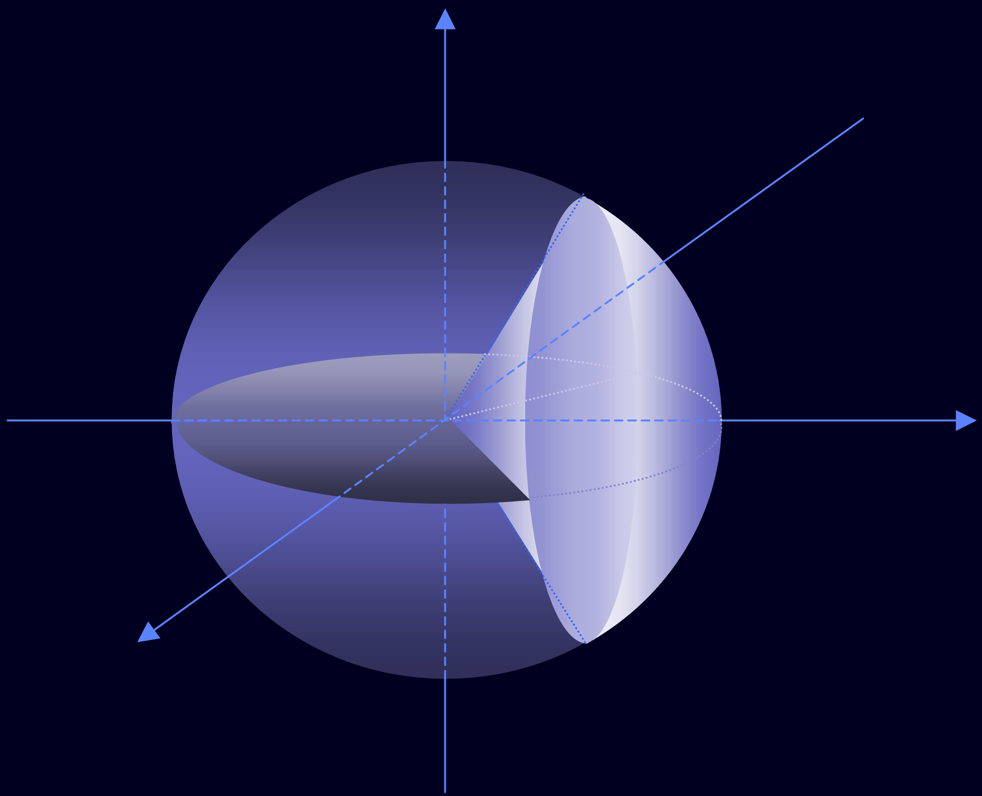
기본 함수

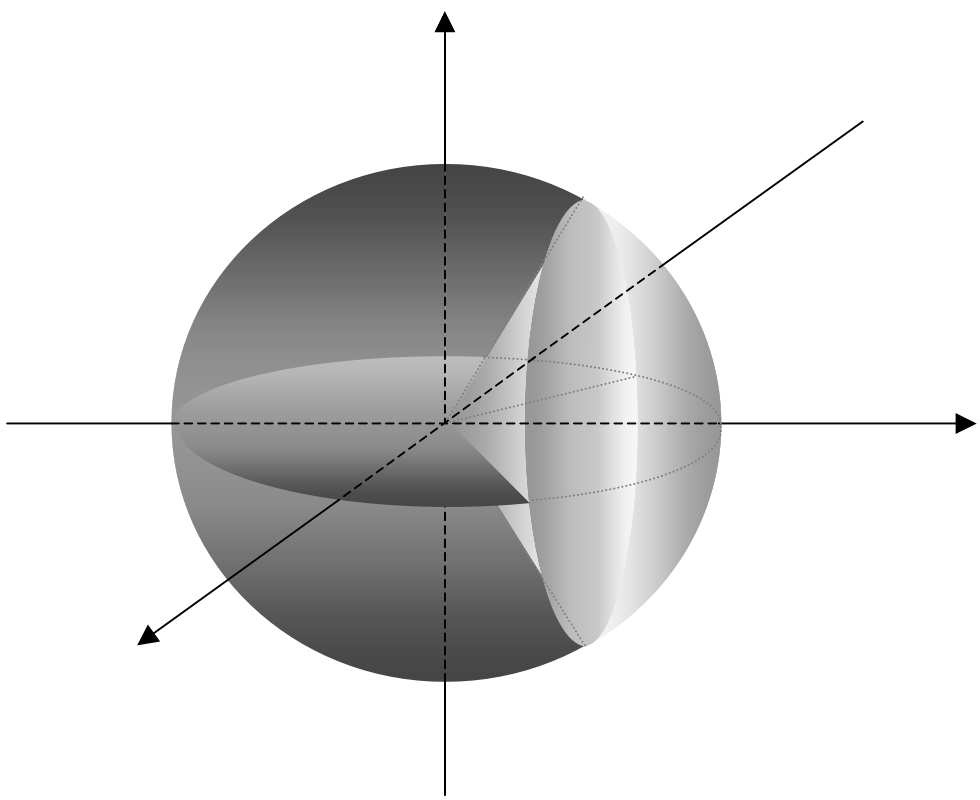
Basic Functions



김성렬

Seong R. Kim

Basic Functions



Seong R. KIM

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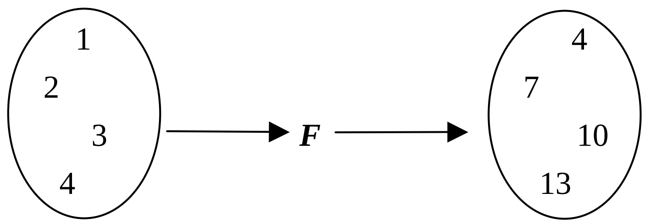
0.0. What is a function?



Though it looks like an equation, it is not. What then is it?

It's not just a math expression. We can call it an expression that creates a *correlation between data sets*. What data sets though?

The functions covered in this book are basic ones, which are thus, covered in high school math.
So such data sets are simple sets of numbers, usually real numbers.
And in elementary level as high school math, we normally use two sets of numbers.

Usually, we use as a function a math expression such as $\mathbf{2x}$, $\mathbf{x^2 + x + 2}$, $\mathbf{3^{x+2}}$, $\mathbf{\log x}$, $\mathbf{\sin x^2}$, etc. It can also, be as simple as 1, 3, -2, or 0, since a number is an expression, too.
How then does it create a correlation?

We can put a function this way, too:
A function is an expression *acting on a number set to produce another number set*.
How does a function then act on a number set?

A number from the set acted on is put into the variable in the function's expression, so a number gets produced. And the number produced belongs to the other set.
Suppose for instance, $\boldsymbol{X}$ is the set acted on, and is a set {1, 2, 3, 4}, the function is $\mathbf{3x + 1}$, and $\boldsymbol{Y}$ is the set produced, and is a set {4, 7, 10, 13}.

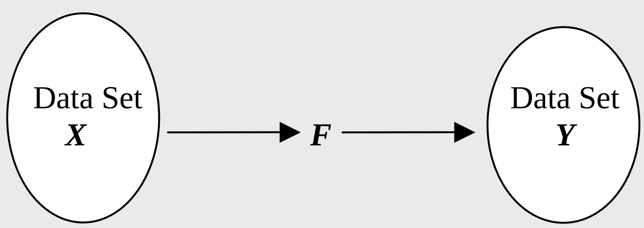
Then, if for instance, 1 from $\boldsymbol{X}$ is put into the expression $\mathbf{3x + 1}$, we get $3 \cdot 1 + 1 = 4$, so 4 gets produced. Thus, 4 belongs to $\boldsymbol{Y}$.
And the same is true, also, for all the other numbers in both sets.

That is to say that a function connects two number sets.
One is the set acted on, and the other is the set of all the numbers produced.

A function is therefore, not just an expression, but creates a *system*, where two sets of numbers are connected by the function, which is an expression as **2*x*** or ***x* + 1**.

In short, a function is an expression connecting two number sets.
And assuming ***F*** is a function, we can put it the way as follows.

Fig. 0



Through the function ***F***, the data set ***X*** gets connected to the data set ***Y***.

Such two data sets are two sets of numbers. So a function creates a correlation between two number sets.   What correlation, though?

A function creates a correlation by setting up two relationships.
One is a *dependence* relationship between the two sets.
And the other is a *mechanical* relationship between the numbers in one set and those in the other set.   What mechanical relationship?

Basically, the mechanical relationship goes the way as follows.

A pair of two numbers gets generated *at a time*. And the two numbers in each pair get connected. So number-pairs keep getting made as the function keeps running.

In each pair, one is from one set, and the other belongs to the other set. And the two numbers get connected by the function, because the function uses one to make the other.

That is, the function connects two numbers by making a number-pair as (2, 7).

And a function can generate a sequence of pairs of numbers.
That is, a function can be taken as a machine generating pairs of numbers in a sequence.

And in high school math and calculus, all the numbers in both sets have to get connected.
Thus, the function keeps running until all the numbers get connected. So?

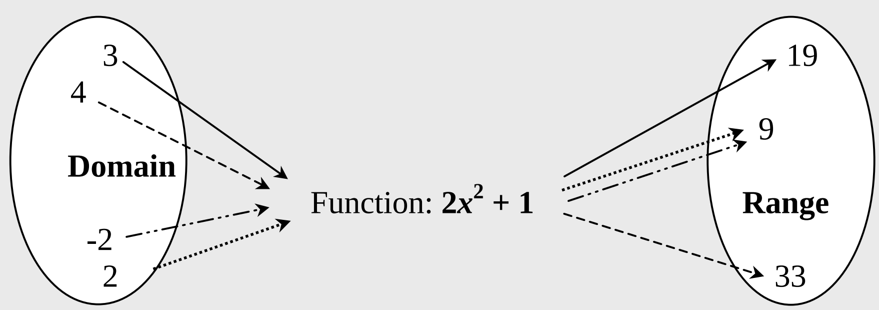
So eventually, the function connects the two sets.
What then about the dependence relationship?

Both sets can be the same or different. One can be bigger or smaller than the other. And it is often the case, both are identical. In nature though, the two sets are always different. How then are the two different?

One of the two is called a *domain*, and the other is called a *range*.
And the range is dependent upon the domain. How is it dependent, though?

We know a function is an expression acting on a number set, and producing another set of numbers. The *set acted on* is the *domain*, and the *set produced* is the *range*.

Fig. 1



A number called an *input* from the domain is put into the variable in the function, which therefore, produces a number called an *output*, which belongs thus, to the range.
And the same is true, also, for all the other numbers in both sets.

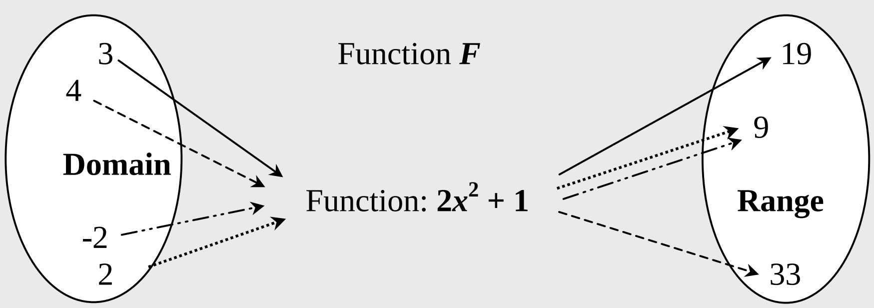
So a number in the range can get determined by a number in the domain.
And the same is true, also, for all the other numbers in the domain and the range.

That is, if no number is chosen in the domain, no number can be determined in the range, and if a number is chosen in the domain, one number has to be determined in the range.

So each number in the range is dependent upon at least one number in the domain. In sum, the range depends on the domain. So no domain, then no range.

And for the same reason, we can say that the domain causes the range. And each number in the range can be said to *correspond to* at least one number in the domain.

Fig. 2



So we can say that the numbers in the range correspond respectively to the numbers in the domain. And in the system that has the function ***F*** above, we can say

that 19 in the range corresponds to 3 in the domain,

that -2 gets chosen in the domain, enters the function, the function runs, then 9 gets determined in the range. And the same is true for 2 in the domain, too.

and also, that 4 in the domain is connected to 33 in the range, that is, the two get connected by means of the function $\mathbf{2x^2 + 1}$.

In sum, the domain and the range get connected via the function ***F***, which is $\mathbf{2x^2 + 1}$. And as others do in math, a function has its name.

Naming a function, we usually use two letters, together with brackets or parentheses. One is called a function *designator* (another word for a name), and the other is the variable used in the expression.

So for instance, if a function is called ***f***, and the expression is $\mathbf{2x + 1}$, we can name the function this way: $\boldsymbol{f(x)}$, read as ***f*** of ***x***.

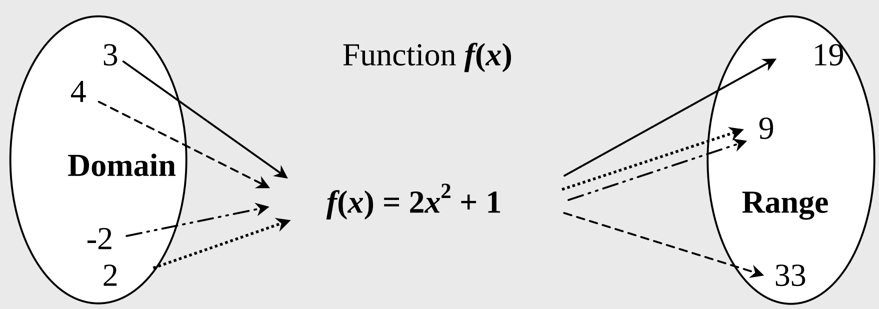
Then, we can set $\boldsymbol{f(x) = 2x + 1}$, which is the form we often use specifying a function.

And in $\boldsymbol{f(x)}$, we call $\boldsymbol{x}$ an *input variable*.

That's because it gets each of all the numbers in the domain, so each number is put *into* the variable $\boldsymbol{x}$. And for the same reason, each number in the domain is called *an input value*, often just called *an input* for short.

And for instance, putting an input 3 into $\boldsymbol{x}$ in the function $\boldsymbol{f}$, we do it this way: $\boldsymbol{f(3)}$, read as $\boldsymbol{f}$ of 3.
Then, the function $\boldsymbol{f}$ produces the output this way: $\boldsymbol{f(3) = 2 \cdot 3 + 1 = 6 + 1 = 7}$.

Fig. 3



And naming (or specifying) a function more specifically, we can use another variable that can get the value of the function. What value?

It is each and every number the function produces, and thus, is each and every output.

So for instance, if using $\boldsymbol{y}$ as the variable that gets the value of the function $\boldsymbol{f(x)}$, we can name the function $\boldsymbol{f}$ this way, too: $\boldsymbol{y = f(x)}$, and can specify the function $\boldsymbol{f}$ the way as follows. $\boldsymbol{y = f(x) = 2x + 1}$, which is a bit more specific than this: $\boldsymbol{f(x) = 2x + 1}$.

And we call $\boldsymbol{y}$ an *output variable*.
That's because it gets all the numbers (values) the function produces, and those numbers look getting *out* of the function.
And for the same reason, each of those numbers is called *an output value*, often just called *an output* for short. And of course, the output variable gets one output at a time, and the output belongs to the range.

So what is a function?

A function is a math expression connecting two number sets called a domain and a range. Connecting the two sets, the function generates a sequence of pairs of numbers as (0, 1), (3, 7), (8, 17), etc.

And for instance, in a pair (8, 17), 8 is called an input, and 17 is called the output for the input 8.   In short, in (8, 17), 8 is an input, and 17 is an output.

And putting an input 8 into x in the function $y = f(x) = 2x + 1$, we do it this way: $f(8)$.

Then, the function f produces the output this way: $f(8) = 2 \cdot 8 + 1 = 16 + 1 = 17$.
So in this case, we can call 17 the output for the input 8.

And the same is true for $f(8)$, too, since we have $f(8) = 17$.
So we can call $f(8)$ the output for the input 8, too.

And we often say that the value of the function f is 17 when $x = 8$, or that the value of the function f is 17 for $x = 8$.

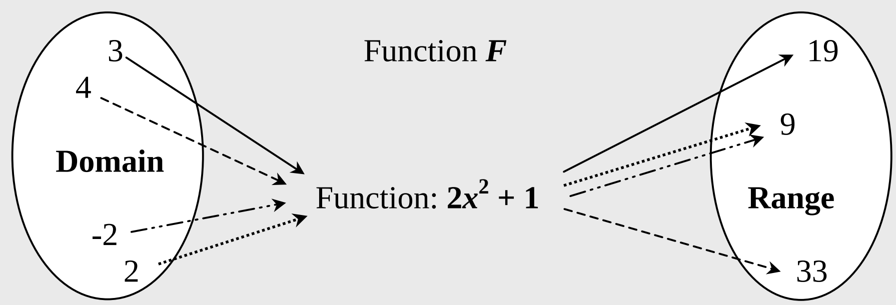
And we have $y = f(x)$.

So the output variable y is said to get the value of f for each value of x.

Thus, we can say that $y = 17$ when $x = 8$, that $y = 17$ if $x = 8$, or that $y = 17$ for $x = 8$.

0.1. How Functions Work 1

Fig. 0



So what is a function?

A function is an expression acting on a number set called a domain, and producing another set of numbers called a range.

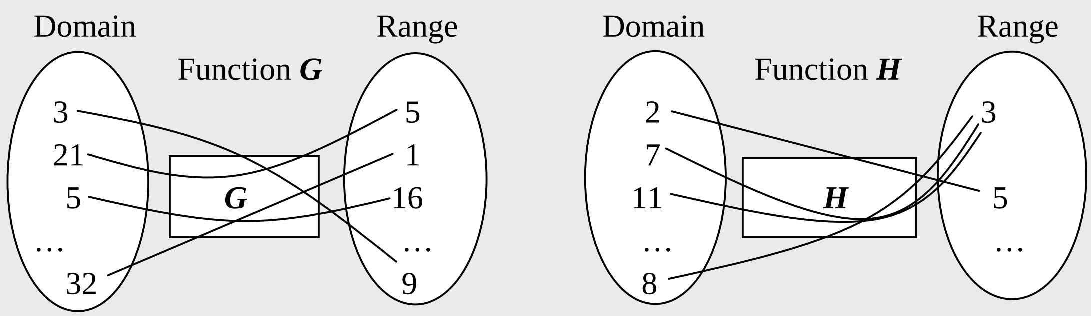
In short, a function is an expression that makes a range using a domain.

And we can notice that in the example above, some number in the range can correspond to more than one number in the domain.   One number at a time though.

Each number in a range corresponds to *at least one* number in the domain.

Therefore, for some functions, one number in the range can correspond to many numbers or even all the numbers in the domain.   So for instance, we can have the functions ***G*** and ***H*** in Fig. 1.

Fig. 1



That is because two or more numbers in the domain can cause the same number in the range.   What then, about the numbers in the range?

As mentioned above, in the range, one number can correspond to many numbers in the domain. And if the range has one number only, that only number has to correspond to all the numbers in the domain.

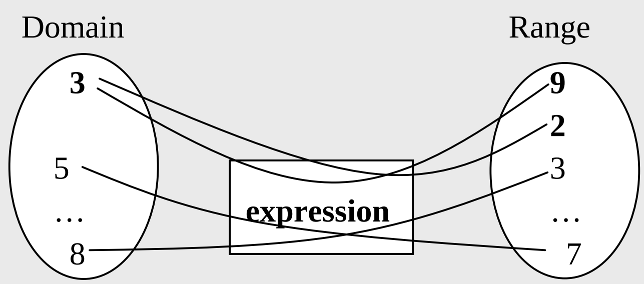
Note however, for any function, two or more numbers in a range cannot correspond to the same number in a domain.

In other words, ***no number*** in the domain ***can cause two or more*** numbers in the range.

So no matter what function it may be, it has to produce one number only at a time.

For instance, we do ***not*** have a function as in Fig. 9. If it were a function, the two numbers 9 and 2 in the range wouldn't be able to correspond to the number 3 in the domain.

Fig. 2 No function can do this:



Let's now, get back to the mechanical relationship.

We know that a machine basically repeats the same work.
A function is an expression made of math operations.

Every time a number is put into a function, the same group of operations repeats, and a number is produced.
So a function can be taken for a machine that performs a task over and over.

The task is a group of math operations as the ones in $x^2 - x + 1$.
What are the operations though, in $x^2 - x + 1$?

We can see three.
One is $x^2 = x \cdot x$, which is a multiplication.
Another is $x^2 - x$, which is a subtraction.
And the other is $(x^2 - x) + 1$, which is an addition.

Doing this x^2, we get a product. And using the product, we do this $x^2 - x$.
Then, using the difference, we do this $(x^2 - x) + 1$. And then, we get the sum.

The sum is a number that the expression $x^2 - x + 1$ produces.
That is, it's an output that the function $x^2 - x + 1$ produces for an input.

And every time the set of operations gets completed, an output gets produced.
So for instance,

if 1 gets into $x^2 - x + 1$, we get $1^2 - 1 + 1 = 1$,

if 2 gets into $x^2 - x + 1$, we get $2^2 - 2 + 1 = 3$.

and if 3 gets into $x^2 - x + 1$, we get $3^2 - 3 + 1 = 7$,
…

And we can get put it this way, too:
$1 \longrightarrow x^2 - x + 1 \longrightarrow 1$

So we get (1, 1), where 1 is an input, and 1 is the output.

$2 \longrightarrow x^2 - x + 1 \longrightarrow 3$

So we get (2, 3), where 1 is from the domain, and 3 belongs to the range.

$3 \longrightarrow x^2 - x + 1 \longrightarrow 7$ So we get (3, 7).

And assuming X is the domain, and Y is the range, and putting at once all the function executions for all the inputs in X, we can put it the way as follows.

$X \longrightarrow x^2 - x + 1 \longrightarrow Y$

And assuming F is the function $x^2 - x + 1$, we can simplify the idea the way as follows.

$X \xrightarrow{F} Y$.

Next, depending on the nature of the correspondence a function creates, the type of the function can get determined. One is *one-to-one*, and another is *many-to-one*.

What then do we mean by one-to-one?

Suppose no two numbers in the domain cause the same number in the range.

That is, every number in the domain causes a different number in the range.
In other words, every output is different.
Then, the function is one-to-one. Otherwise, it is many-to-one.

What then about one-to-many?

There is no such a function. That's because a function produces one number only at a time. In other words, in any function, no two numbers can be caused by a number.

So a function is either one-to-one or many-to-one.
And we can put the idea the way as follows, too:

If all the numbers in the range are used once only, it is one-to-one.
So every output is different. Thus, if $f(a) \neq f(b)$ where $a \neq b$, f is one-to-one.

If some or all the numbers in the range are used more than once, it is many-to-one.
So some or all the outputs can be the same. For instance:

If $f(2) = f(3) = 1$, f is many-to-one.
Also, if $g(x) = 2$ for all the values of x, g is many-to-one.
And we call such a function as the function g above a ***constant function***, since each and every output is constant, that is, the same.

And as mentioned earlier, in high school math and calculus, all the numbers in both the domain and the range have to get connected. And such a function is said to be *onto*, and thus, is called an onto function.

So more specifically, a function is either one-to-one and onto or many-to-one and onto.

Working with a function then do we need to see if a function is one-to-one or not?

Sometimes, we need to come up with a new function from a particular function. Coming up with the new function, we have to use as the domain the range of the function particular, and use as the range the domain of the particular function. And that's not it.

We need to maintain the existing correspondence between numbers in both sets, the domain and the range of the particular function.

For instance, if the particular function makes (1, 2), (4, 7), (6, -5) etc., the new function has to make (2, 1), (7, 4), (-5, 6), etc.

We call such a new function *an inverse function*.
And we call the new function stated above, the inverse of the particular function, and often just call it *the inverse* for short.

Then, the function particular has to be one-to-one. Why?

Taking the inverse of a function many-to-one, we cannot get a function.
Or rather, we cannot take the inverse of a function many-to-one. Why not though?

If a function is many-to-one, the inverse would be one-to-many, and thus, would not be a function. A function cannot be one-to-many.

Each number in a domain gets connected to one number only in the range.
So no two numbers in the domain can be connected to one same number in the range.
So what is a function?

A function is an expression that connects two number sets, called a domain and a range in a way that every time the function gets executed, a *pair* of numbers gets determined.

And number pairs get determined in either of the two ways as follows.

(2, 1), **(4, 3)**, **(7, 5)**,etc. Then, the function f is one-to-one, since $f(a) \neq f(b)$ if $a \neq b$.

(1, 3), **(5, 9)**, **(7, 3)**, etc. Then, the function f is many-to-one, since $f(1) = f(7)$.

The first one in each pair is from the domain, and the other belongs to the range.

And that's the way the two sets, that is, the domain and the range get connected.

But no function can generate number pairs the way as follows. **(1, 2)**, **(1, 9)**, **(3, 4)**, etc.

0.2. How Functions Work 2

Why is $f(x)$ though, read as f of x?

Saying f of x, we are saying that f is a function of x. So in short, we just say f of x.
And saying f is a function of x, we mean $f(x)$ changes as x changes.
In short, f varies as x varies.

More specifically, as the value of x changes, the value of $f(x)$ changes.
Suppose for instance, $y = f(x) = 2x + 5$. Then:

If $x = 1$, we get $y = f(1) = 2\cdot 1 + 5 = 7$, so we get $y = f(1) = 7$. That is, $y = 7$ if $x = 1$.

If $x = 2$, we get $y = f(2) = 2\cdot 2 + 5 = 9$, so we get $y = f(2) = 9$. That is, $y = 9$ if $x = 2$.

So as the value of x changes, the value of $f(x)$ changes, and so does the value of y, of course. And we know x gets the inputs. And as the input changes, the output changes.

Saying thus, a function f of x, or just f of x, we are saying that f is a function of x, and that x is the input variable.

And we mean that the value of f or the value of $f(x)$ changes as the value of x changes.
Also of course, since $y = f(x)$, the values of y changes as the value of x changes.

Suppose for another instance, $t = g(s) = 3s + 4$. Then:

If $s = 1$, we get $t = g(1) = 3 \cdot 1 + 4 = 7$, so we get $t = g(1) = 5$. That is, $t = 7$ if $s = 1$.

If $s = 2$, we get $t = g(2) = 3 \cdot 2 + 4 = 10$, so we get $t = g(2) = 10$. That is, $t = 10$ if $s = 2$.

So as s changes, the value of $g(s)$ changes, and so does the value of t, of course.
Thus, s gets the inputs, $g(s)$ gets the outputs, and so does t, since $t = g(s)$.

So saying a function g of s, or just g of s, we are saying that g is a function of s, and that s is the input variable.

And we mean that the value of g or $g(s)$ changes as the value of s changes.
And of course, since $t = g(s)$, the value of t changes as the value of s changes, too.

Specifying however, a function the way above, that is, with no domain specified, we just assume the largest possible domain.

Given for instance, a function this way: $y = f(x) = 2x + 5$, we are to assume that the domain is a set of all real numbers, because it is the largest possible domain.
Why is it the largest?

Every real number can be an input of the function f. In other words, the function f can be defined for every real number. So the largest domain is a set of all real numbers.

So just setting $y = f(x) = 2x + 5$, we mean $y = f(x) = 2x + 5$ for x real.

What if the domain of the function f above is a set of all numbers grater than 3?

Then, we can put it this way: $y = f(x) = 2x + 5$ for $x > 3$.

What then is the range of the function f above?

We have $x > 5$. So we can get $x > 5 \Rightarrow 2x > 10 \Rightarrow 2x + 1 > 11$.
Also, we have $y = f(x) = 2x + 1$ for $x > 5$. So we get $y = f(x) > 11 \Rightarrow y > 11$.
And we know y is the output variable, so it gets all the outputs that belong to the range.
The range is thus, a set of all numbers bigger than 11, and can be put this way: $y > 11$.

And normally, if asked to find a domain, we are to find the largest possible domain. What then is the domain if the function given is $y = g(x) = \sqrt{x+1}$?

We know what's inside the square root sign is positive or 0.

So we get $x + 1 \geq 0 \Rightarrow x \geq -1$, and thus, the (largest possible) domain is $x \geq -1$.

If any number less than -1 is put into (x in) the function g, we don't get a real number.
So for instance, we cannot make a function as $y = g(x) = \sqrt{x-2}$ for $x > 1$.
That's because if $x = 1.5$, we get $g(1.5) = \sqrt{1.5-2} = \sqrt{-0.5}$, which is not real.

What kind of functions then do we work with in high school math?

Based on expressions that functions have, we can classify or categorize functions.

An expression as $x^2 - x + 1$ is said to be algebraic, so if a function has an algebraic expression, the function is called an algebraic function. And if a function is not an expression algebraic, the function is called a transcendental function.

For instance, expressions as $\sin x$, 3^x, $\log x$, $\sqrt{x+1}$, and $\dfrac{x}{x^2+1}$ are not algebraic, so if a function is such an expression, the function is said to be transcendental. In particular:

A function $y = f(x) = \sin x$ for $x \geq \pi$ is called a trigonometric function, and is called a sine function.

A function $y = g(x) = 3^x$ for $x > -2$ is called an exponential function.

A function $y = h(x) = \log x$ for $x > 0$ is called a log function.

A function $y = p(x) = \sqrt{x+1}$ for $x \geq 2$ is called a radical (or irrational) function.

A function $y = q(x) = \dfrac{x}{x^2+1}$ for $x \geq 1$ is called a fractional (or rational) function.

So in high school math, we work with some of those transcendental functions, together with algebraic ones.
And moving on to calculus in a college or university, we get to work with a function more advanced, of which the domain can have two or more sets of numbers.

For instance, we get to work with functions as $z = r(x, y) = x + y + 1$ for $x > 0$, and $y < 1$.

The function ***r*** above has not one but two input variables, and the two are ***x*** and ***y***.
So the domain is not just one set, but is made of two sets.

One is for the input variable ***x***, and is a set of all numbers greater than 0, that is, a set of all positive numbers.
And the other is for the other input variable ***y***, and is a set of all numbers less than 1.
Of course, ***z*** is the output variable of the function ***r***.
Every function has one output variable. So every range is just one set of numbers.

Of a function basic, the domain is made of one set of numbers.

And such a basic function can be called a 2-D function, because it has two data sets, since it has two variables, one is an input variable, and the other is an output variable.

So if the domain is made of two sets, the function can be said to be 3-D, because it has three data sets, since it has three variables, two are input variables, and the other is an output variable.

And if the domain is composed of three sets, the function can be called a 4-D function, because it has four data sets, since it has four variables, three are input variables, and the other is an output variable.   And so forth.

So a function has quite a few; a name, an expression, a domain, and a range.
That’s not it though.

A function has its curve, too.
So we can put a function in a place called a graph. Putting thus, a function in a graph, we put the curve in a graph. Where do we put the graph, though?

We put a graph in a coordinate system.
If a function has two variables, an input variable and an output variable, we can say that we put the function in a coordinate system said to be 2-D. And we just call it a 2-D system, for short.
In a 2-D system, we usually put two axes perpendicular to each other, which means therefore, a plane.

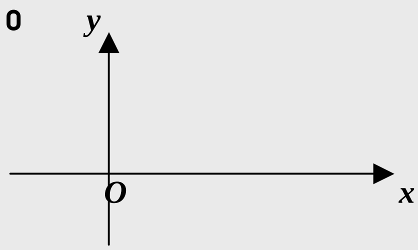
That's because we can use one axis for a length, and use the other for a width.
So usually, we just call such a 2-D system a coordinate plane.
What then about 3-D system?

We can call it a coordinate space, because in such a system, we put three axes perpendicular to each other. One is for a length, another is for a width, and the other is for a height. (The details are covered in the book **Function Transformations**.)

And we can name a coordinate system, too, and can do so using the variables used in the function to be put in the system.
So for instance, assuming we use $\boldsymbol{x}$ as the input variable, and use $\boldsymbol{y}$ as the output variable in a function $\boldsymbol{F}$, and putting the curve of $\boldsymbol{F}$ in a graph, we put the graph in the $\boldsymbol{x}$-$\boldsymbol{y}$ coordinate system, often called briefly the $\boldsymbol{x}$-$\boldsymbol{y}$ system, too, which is made of the $\boldsymbol{x}$-axis and the $\boldsymbol{y}$-axis perpendicular to each other, and thus, can be said to form a plane.

So we often just call the $\boldsymbol{x}$-$\boldsymbol{y}$ system the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, too.

Fig. 0



In a function put the $\boldsymbol{x}$-$\boldsymbol{y}$ system, $\boldsymbol{x}$ is the input variable, and $\boldsymbol{y}$ is the output variable.
For instance, putting the function $\boldsymbol{F}$ in the $\boldsymbol{x}$-$\boldsymbol{y}$ system, we can set $\boldsymbol{y = F(x)}$.
And we can say that we can put the curve of $\boldsymbol{F}$ in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane or the $\boldsymbol{x}$-$\boldsymbol{y}$ system.
Why curve though?

Putting the curve of a function in a graph, we can actually see the function, and more importantly, we can see how the function behaves. So solving problems with functions, *we can get the solutions easier and quicker* putting the functions in their graphs.

(Details on graphs and curves are covered in the book **Function Transformations**.)
So a function is an expression acting on a number set called the domain, and producing another set called the range, and can actually show itself if it's put in a graph.

Can we have though, a function that is just a number as 1?
That is, can we have a function as $y = f(x) = 1$?

Yes, we can. Technically, a number itself can be an expression, too, since it expresses a value. And from a function's point of view, we can put the idea the way as follows.

As a range, we can use a set of one number only as **{1}** or **{0}**. Why?

We can have such a function, and it is called a many-to-one function. So if the range has one number only, the only number corresponds to all the numbers in the domain.

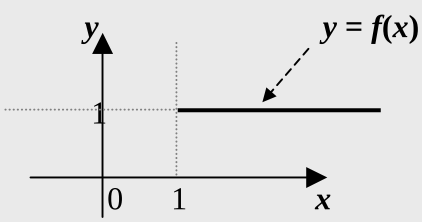
And we often call it a *constant function*.

In such a case, though the input changes, the output does not change. So in case of a constant function, the output is constant, and thus, the curve is a line parallel to the axis where all the inputs are placed. So for instance, being in the x-y plane, the line is parallel to the x-axis. And for instance, the functions below are constant functions.

$y = f(x) = 1$ for $x \geq 1$. The value of $f(x)$ is 1 if x is bigger than or equal to 1.

So for instance, we can get $f(1) = f(1.01) = f(2) = f(21) = 1$.

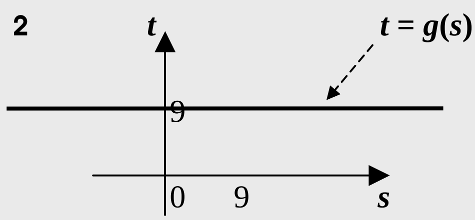
Fig. 1



The curve of f is a ray.

$t = g(s) = 9$ for s real. The value of $g(s)$ is 9 if s is a real number.

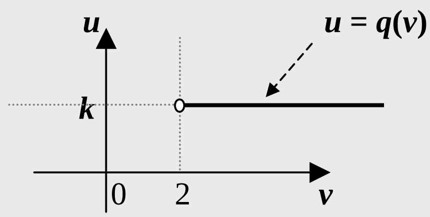
Fig. 2



The curve of g is a line.

u **=** ***q*****(*****v*****)** **=** ***k*** for ***v*** **> 2**, where ***k*** is a constant > **0**. ***q*****(*****v*****)** is ***k*** if ***v*** is bigger than 2.

Fig. 3

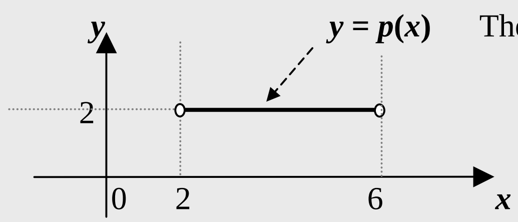


The curve of ***q*** is a ray.

Note that the point **(2, *k*)** doesn't belong to the ray, which is the curve of ***q***, of course. And of course, if ***k*** **< 0**, the ray will be below the ***x***-axis.

y **=** ***p*****(*****x*****)** **= 2** for **2 <** ***x*** **< 6**. The value of ***p*****(*****x*****)** is 2 if ***x*** is between 2 and 6.

Fig. 4



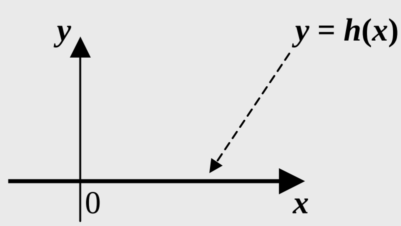
The curve of ***p*** is a line segment.

Note that the two points **(2, 2)** and **(6, 2)** do not belong to the curve of ***p***.

y **=** ***h*****(*****x*****)** **= 0**. In particular, we call such a function as ***h*** a zero function.

And of course, the curve of the zero function ***h*** is the ***x***-axis itself.

Fig. 5



1. Why variables?

Working with functions, we usually work with variables, because functions are math expressions usually made of variables, together with numbers, of course.
Why though, are they called *variables*, and what's the use?

Doing math, we work with values. What values, though?

All kinds of values. Doing math, we do operations with values. Adding for instance, we do additions. Doing such operations, we get values, too. What values though?

Quite ambiguous, isn't it?
So we don't just do such operations. We use tools called *numbers* to specify values. So adding for instance, we add numbers together, and get the sum. So?

A number has a value. How does it get a value, though?

We assign a value to it.
So for instance, we assign seven to 7, which has thus, a value called seven. And that's it.
For another instance, 1 has a value called one, and is thus, of course, read as one.

And adding 7 to 1, we get the sum. Then, the sum gets a value called eight.
We have 8 that has a value called eight. So we can say that the sum is 8.

We have 2, also, so adding 2 and 7 together, we get the sum, which is 9. So this time, the sum is 9. Until we do another addition, the sum remains 9.

And we can give a name to the sum. So we can name it using a letter, and it can be called ***s***, for instance. Then, ***s*** can change its value every time the sum gets a new value.

Every time therefore, we add two numbers together, the sum gets a new value, so ***s*** gets the new value. Until however, we get the next sum, ***s*** has to keep the new value. That is to say that the value of ***s*** has to remain the same until it is assigned another new value.

In other words, the value of ***s*** has to be constant until we get the next sum.
In short, we say that ***s*** is *constant*.

And ***s*** is an object, so we just call ***s*** *a constant*.
So a constant is an object that keeps a value until it gets assigned a new value.

And we can make as many constants as we need.   How then, do we make them?

We just assume those, or declare some letters constants, in a proper manner, of course.

So for instance, we can set up an operation this way: ***a* + *b* = *c*** where ***a***, ***b***, and ***c*** are constants. Or just setting ***a* + *b* = *c***, we normally assume that ***a***, ***b***, and ***c*** are constants.

Then, giving a value to each of ***a*** and ***b***, we get the sum, which is the value of ***c***.

So we have tools called *numbers* as 0, 1, 2, 3, etc., and tools called *constants* as ***a***, ***b***, etc.
Doing math therefore, we can conveniently use those tools doing operations as additions.
What is the difference though, between a number and a constant?

A number cannot change its value once it's been given a value.
So for instance, a number 7 has a value called seven forever, and thus, we cannot change the value of 7. That's because we do not give a new value to a number with a value.

A constant can however, change its value if we give a new value to it.
So we can give a new value to a constant with a value.
It has to keep though, a certain value until it gets assigned a new value.

So we can readily show a math operation in a general manner using such tools called constants. For instance, setting ***a* = 2*b*** where ***a*** and ***b*** are constants, we can say that the value of ***a*** is twice the value of ***b*** no matter what the value of ***b*** may be if $b \neq 0$.

How then, do we know if ***c*** is a constant of a *particular kind*, for instance?

We can define it to be such a constant by declaring it a constant of the particular kind.

So for instance, declaring ***c*** a constant *integer*, we take ***c*** as an integer, and can use it as if it were an integer. And *at a time*, we can assign one number (value) only to a constant. Thus, we can assign one integer to ***c***, and can use ***c*** as the integer.

So for instance, ***c*** can get one of -2, 0, 1, 3, etc.
It has to however, keep the integer given until we assign another integer to it, so it cannot change its value by itself. So for instance:

Assuming now, ***c*** is given 3, we use ***c*** as 3.
And if we want to give –1 to ***c***, we give –1 to ***c***, and use ***c*** as –1, and of course, we do not use ***c*** as 3.
If however, we want to give 3 to ***c*** again, we give 3 to ***c***, and use ***c*** as 3, and of course, we do not use ***c*** as –1.

For another instance, declaring ***d*** a constant *real*, we can just declare ***d*** a constant.
So just declaring ***d*** a constant, we can use ***d*** as a real number,
and can assign at a time, one real number to ***d***, which has to keep the real number assigned until we assign another real number to it.

Suppose for instance, we get an equation $y = dx^2 + 1$ where ***d*** is a nonzero constant.
Then, we use ***d*** as a nonzero real number, and ***d*** can be given one real number nonzero, but has to keep the real number given until it gets assigned another real number nonzero.

So the equation above can represent all the equations such as $y = x^2 + 1$, $y = 2x^2 + 1$, $y = -x^2 + 1$, $y = \frac{x^2}{2} + 1$, etc.

Suppose for another instance, we get an equation as follows.
$y = bx^2$ where ***b*** is a constant integer
Then, ***b*** is an integer constant. So we use ***b*** as an integer, and ***b*** can get one integer, but has to keep the integer given until we give another integer to it.

Suppose for another instance, we get a statement as follows.

- Assuming that ***a*** is constant, find the point where a line ***y* = 2*x* + *a*** meets the ***x***-axis.

Then, we use ***a*** as a real number, and ***a*** can represent all real numbers, but has to keep a real number assigned until it gets assigned another real number.

Let's take one more example. Suppose we get a statement as follows.

Assuming ***d*** is a constant > 3, and ***A*** is a line ***y* = *x* + *d***,
find the point where ***A*** meets the ***x***-axis.

Then, we use ***d*** as a real number > 3, so ***d*** can be assigned one real number > 3, but has to keep the real number assigned until it gets assigned another real number > 3.

So a constant itself cannot change its value, but we can change the value of a constant if we need to.

And we have another kind in tools that work like numbers or constants, and thus can have values.

Such a tool is very different from a number, but seems quite close to a constant. It's certainly not a constant, of course. What then, is the tool?

It *looks* no other than a constant.
So like a constant, it can have one value at a time, and cannot have two or more values at a time. Unlike a constant though, it keeps changing its value. Or rather, it has to constantly change its value.

So its value varies continuously. And thus, we call it *a variable*.

So a variable varies its value continuously, and thus, its value keeps changing.
What values then, can a variable have?

A variable covers all values in a set. For instance, it can cover a set of all values ≥ 0.

And we use numbers to show values. So for instance, a variable can cover a set of all real numbers, a set of all nonzero real numbers, or a set of all real numbers less than 1.

And for instance, if a variable covers all real numbers positive, we only know the fact that the variable keeps changing its value covering all positive real numbers.

However, a variable cannot have two or more numbers at a time, and has to have one number only at each moment.

How then can we get a variable? That is, how can we make one?

Like other tools in math, defining a particular variable, we can make one.
So by a variable definition, we can make one and use it.

A variable varies covering all the values in a particular kind or set, which is a number set, of course. So we can define one by declaring it that way. So let's now make one.

Note:
We can use '$\in$' to show that particular objects *belong to* or *are elements of* other object. For instance, setting $\boldsymbol{a} \in \boldsymbol{B}$, we mean that ***a*** belongs to or is an element of ***B***, and also, setting ***u*** and $\boldsymbol{v} \in \boldsymbol{W}$, we mean that ***u*** and ***v*** both belong to or are elements of ***W***.

Suppose now, for instance, we get a statement as follows. ***d* is real**.

Then, the statement can be a variable definition, and it can be assumed that ***d*** is a variable that can take any real number, and covers all real numbers.

So ***d*** keeps changing its value, and has one real number at each moment.

Suppose for another instance, we get an *equation* as follows. $\boldsymbol{y = x^2 + x + 1}$

Then, it is assumed that ***x*** is a variable that can take any real number, and covers all real numbers, because ***x*** can get any real number in this case. What then about ***y***?

It is a variable, too, and covers a set of some real numbers. And if no declaration is done to the letters as x and y, we just assume that the letters are variables.
So x and y keep changing their values, and each has one real number unknown at a time.
Why unknown?

It's because a variable keeps changing its value. We only know what kind of values it can have at a time.

And we can notice that the values of the variables x and y are dependent upon each other. That is, the value of one variable determines the value of the other.

So for instance, when x is 1, that is, if $x = 1$, we get $y = 1 + 1 + 1 = 3$.
Also, when x is -2, that is, if $x = -2$, we get $y = (-2)^2 + (-2) + 1 = 3$, too.
In other words, if $y = 3$, we get $x = 1$ or -2.

And in fact, the value of y is limited in this case. That's because we get

$$y = x^2 + x + 1 = x^2 + x + \frac{1}{4} - \frac{1}{4} + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} \geq \frac{3}{4}.$$ So we get $y \geq \frac{3}{4}$.

Thus, we can say that y is a variable that covers all the numbers $\geq \frac{3}{4}$.
Also, we can limit the value a variable can take this way, too: $y = x^2 + x + 1$ for $x > 1$.

Then, it is assumed that x is a variable that can take any real number > 1.
What then about y?

It is assumed that y is a variable, too, of course, but all the real numbers y can take are subject to the condition that $x > 1$. So let's now find the set of all the numbers y can take.

To begin with, we have $y = x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4}$. And we have $x > 1$.

So we get $y > (1 + \frac{1}{2})^2 + \frac{3}{4} = \frac{9}{4} + \frac{3}{4} = 3$ Thus, we get $y > 3$.

So it is assumed that y is a variable that can take any real number > 3.

If defining a constant though, we need to declare it a constant. So for instance, if we just get $y = cx^2 + x + 1$, we have to assume that not only x and y but c is a variable, too.

If however, we get $y = cx^2 + x + 1$ where c is a constant, we just assume that x and y only are variables.

And for another instance, declaring $0 < c < 1$ where c is constant, we mean that c is a constant that can be assigned at a time, one real number between 0 and 1.

Note that we can change the value of a constant if we want to, so a constant itself cannot change its value unlike a variable, and thus, a number as 7 *can be* technically *taken for* a constant, too, since 7 cannot change its value anyway. Normally however, we don't call a number a constant, of course.

• Let's next, define variables using set notation.

Setting $A = \{e|\ 1 < e < 2\}$, and $c \in A$, we mean that c is a variable that can take any real number between 1 and 2. So it is just as good as setting $1 < c < 2$.

For another instance, setting $A = \{e|\ 1 < e < 2\}$, $B = \{t|\ 3 < t < 4\}$, and $s \in A$ or B, we mean s is a variable that can take any number between 1 and 2 or between 3 and 4. And of course, s cannot take any number ≤ 1, any number from 2 to 3, and any number ≥ 4.

So for instance, s can take 1.5 or 3.8, but cannot take 2.
Also, it is just as good as setting $1 < s < 2$ or $3 < s < 4$.

• Let's this time, define constants using set notation.

Stating $A = \{e|\ 1 < e < 2\}$, and c is a constant $\in A$, we mean that c is a constant that can take at a time, one real number between 1 and 2.

So it is just as good as declaring or stating $1 < c < 2$ where c is constant.

And next, stating $A = \{e|\ 1 < e < 2\}$, $B = \{t|\ 3 < t < 4\}$, and v is a constant $\in A$ or B, we mean that v is a constant that can take at a time, one real number between 1 and 2 or between 3 and 4. And of course, v cannot take any number ≤ 1, any number from 2 to 3, and any number ≥ 4. So for instance, v can take 1.2 or 3.1, but cannot take 4 or 5.

Also, it is just as good as declaring $1 < v < 2$ or $3 < v < 4$ where v is constant.

Now, why do we frequently have to make and use, or work with variables?

Doing math, we often work with objects that change. What change, though?

Such an object changes by changing the value of itself.
Suppose for instance, an apple is falling.
Then, the position of the apple changes.

And we can put it the way as follows, too.

The distance from the position where the apple starts falling changes.
In short, the distance changes.
Then, what do we need to keep track of the distance?

We need an object that can change its value.
More specifically, we need an object that can indicate a distance, and can keep changing its own value. How then can we get the object?

We may want to begin with naming the object, and for instance, can name it ***d***.
Then, ***d*** is the distance, and keeps changing its value. How then do we call ***d***?

We call ***d*** a variable, and can have ***d*** get all real numbers it can get.

Since d is a distance, d has to be greater than or equal to 0. So we want d to be able to get any real number ≥ 0. Thus, we can simply declare that $d \geq 0$.

How then does d change?

The distance changes as time changes, so we need to keep track of the time, too. We don't just take the measurement of time though. We need an object that can indicate time, and can keep changing its own value. So what do we need?

We need another variable that indicates time, and for instance, can name it t.

Since t is time, t has to be greater than or equal to 0. So we want t to be able to get any real number ≥ 0. Thus, we can simply declare that $t \geq 0$, too.

So we now can use d as the distance, which keeps changing its value along the flow of time called t, which keeps changing, of course, its value, too.

So we now have two variables, and one is subject to the other.

Suppose now, we want to see where the apple is at a particular moment of time.

That is, we want to know a particular value of d after a particular amount of time t. What then, do we need to find such a value of d?

We need a particular relation between the distance and time. That is, we want a specific correlation between d and t. In other words, we need the *connection* between the two. How then can we get the connection?

We can find it doing some experiments and collecting data. That is, taking distances after some amounts of time, we can come up with the connection.
How then can we come up with it?

We know that the distance d keeps changing its value as the time t changes its value.

Finding thus, the math expression that is expressed in terms of t, and produces a value of d for each value of t, we get the connection.

For instance, the expression can be $4.9t^2$ or $t^2 + 3t + 1$. What then is the connection?

The connection is of course, *the expression* in terms of t.
We know that *the expression* produces a value of d for each value of t.

So setting d = *the expression*, we get an equation, and we call in fact, the equation the *connective expression* between d and t.

And we call such an *expression* a function.
Also, we can say that the distance d is a function of time t.

That's because the value of d, that is, the value of the function changes as time t changes.

And in fact, the function is $\frac{1}{2}gt^2$, where g is a constant, and more specifically, is called the gravitational constant on earth, and is approximately 9.8 (m/sec^2).

So for each value of t, the value of $\frac{1}{2}gt^2$ is the value of the distance d.

And the connective expression between d and t is $d = \frac{1}{2}gt^2$

So we can now, define a function that specifies the correlation between the amount of distance d and the amount of time t.

And the inputs are the values of t, and the outputs are those of d.
What then is the domain?

The domain of the function is the set of all the values of t, of course.
How then can we get the domain?

Assuming that the apple is initially h meters away from the ground, we can say that the maximum distance the apple can travel is h meters. So finding t when $d = h$, we get

$d = \frac{1}{2}gt^2 \Rightarrow h = \frac{1}{2}gt^2 \Rightarrow t^2 = \frac{2h}{g} \Rightarrow t = \sqrt{\frac{2h}{g}}$, since we have $t \geq 0$.

So the domain is $0 \leq t \leq \sqrt{\frac{2h}{g}}$.

Thus, assuming the function is called ***D***, we can set $\boldsymbol{d = D(t) = \frac{1}{2}gt^2}$ for $\boldsymbol{0 \le t \le \sqrt{\frac{2h}{g}}}$.

And we know that the distance ***d*** can have values from 0 to ***h***, and is the output variable. So the range is $\boldsymbol{0 \le d \le h}$.

And we can apply such an idea as above to somewhere else.

For instance, we can use a variable as a speed, inventory, amount of time, price, height or altitude, depth, volume, force, area, etc.
In other words, we can take as a variable any object that has to change its value covering all the values defined. And we can keep track of its change. So a variable is a pretty handy tool for many uses, isn't it? We do math therefore, to make life easier.

And we want to note that a change does not just happen.
Where there is a change, there is a rate.   What change though?

It can be a change in distance, a change in area, etc.   What rate then?

It's a rate of change.   What rate of change though?

It usually is a time rate of change, and for instance, we often use a time rate of change in distance, which is called a velocity, which has a magnitude called a speed.

For instance, light travels at approximately three hundred thousand kilometers a second, which is a magnitude of a rate of change, and the magnitude is the speed.
So the speed of light is about 300,000 km/sec.

A rate of change can be however, not only a velocity but something else, too.

It can be an acceleration, which is a rate of change in a rate of change, if you will.
And it can be called in fact, a time rate of change in velocity.
So a velocity can change as time changes, and we call such a rate an acceleration.
There can be thus, a rate of change in velocity. So a velocity can be a function of time, too.

And we will get to see and work with a lot of those in the math called calculus. Basically thus, calculus is about how functions work, and explains how things change in amount.

2. Formal Definitions

Defining an object in math, we produce a description of it. And the description has specifics on it, and is based on rules, principles, or theories relevant.

And we usually use a formal definition.   What is a definition formal though?

Even if a definition is not formal, it is not informal.
What then is a formal definition?

Defining an object in math, we mainly use symbols, and use them as many as possible. So defining a function, too, we use symbols as many as possible. Thus, we minimize the use of natural language as English, and maximize the use of symbols.

And producing a definition the way above, we can say that we make a formal definition. Also, we can call a *formal* definition a *symbolic* definition, too, because of heavy use of symbols.
And the more formal it is, the easier it gets.

The worst thing in learning things in math is ambiguity, and we can reduce ambiguity by means of a formal definition. That is because it is concise and reduces the weaknesses in meaning or limitation of natural language, because we use symbols. So it can make a definition laconic, and remove room for ambiguity. We can therefore, alleviate the difficulty of understanding the definition.

And before we get into the discussion of formal definitions, we want to note and be familiar with the ideas as follows.

• We use '≡' to show that particular objects are identical to each other.
So for instance, $\boldsymbol{A} \equiv \boldsymbol{B}$ means that ***A*** and ***B*** are identical to each other.

• A symbol ∅ indicates an empty set,
which has no element in itself and can be put in **{}**.

• And we use '⊆' to indicate that a particular set is a subset of another set.
For instance, $\boldsymbol{S} \subseteq \boldsymbol{T}$ says that a set ***S*** is a subset of a set ***T***.

Let's now make some function definitions, that is, define or make some functions.
We can make functions making formal definitions of those functions.
To begin with, we can define a function called ***G*** the way as follows.

Suppose that the function designator is ***G***.

$\boldsymbol{X = \{s \mid 1 \leq s \leq 2\}}$, which is the domain,
and $\boldsymbol{Y = \{t \mid 14 \leq t \leq 17\}}$, which is the range.

And the expression is $\boldsymbol{3x + 11}$.

Then, $\boldsymbol{y = G(x)}$ for $\boldsymbol{x \in X}$.

The set of statements in the box above can be a function definition, and there is nothing wrong with the definition, the definition of the function ***G***. It is quite verbose, though.

So making or defining a function, we may want to use as many symbols as possible, and make use of words minimal. Then, we can make the definition clearer and succinct.

So let's now try defining the function ***G*** with less use of words.
That is, we are making a formal (symbolic) definition of the function ***G***.

Then, we can produce the formal definition of ***G*** either of the ways as follows.

$\boldsymbol{X = \{s|\ 1 \le s \le 2\}}$, and $\boldsymbol{Y = \{t|\ 14 \le t \le 17\}}$.
$\boldsymbol{G: X \longrightarrow Y}$.
$\boldsymbol{y = G(x) = 3x + 11}$ for $\boldsymbol{x \in X}$.

$\boldsymbol{X = \{s|\ 1 \le s \le 2\}}$, and $\boldsymbol{Y = \{3s + 11|\ s \in X\}}$.
$\boldsymbol{G: X \longrightarrow Y}$, and $\boldsymbol{y = G(x)}$.

$\boldsymbol{X = \{s|\ 1 \le s \le 2\}}$, and $\boldsymbol{Y = \{3s + 11|\ 1 \le s \le 2\}}$.
$\boldsymbol{G: X \longrightarrow Y}$, and $\boldsymbol{y = G(x)}$.

So all the three definitions above are the same, and thus,
indicate one same function as follows. $\boldsymbol{y = G(x) = 3x + 11}$ for $\boldsymbol{1 \le x \le 2}$

"$\boldsymbol{y = G(x) = 3x + 11}$ for $\boldsymbol{1 \le x \le 2}$" is in fact, another formal definition of $\boldsymbol{G}$, and is the simplest.

Let's anyway now, take a look at what each component of the definition is about.

To begin with, the domain is $\boldsymbol{X = \{s|\ 1 \le s \le 2\}}$, which is telling us $\boldsymbol{X}$ is a set of all the real numbers that are greater than or equal to 1 and less than or equal to 2.

Next, the range is $\boldsymbol{Y = \{t|\ 14 \le t \le 17\}}$, which is saying that $\boldsymbol{Y}$ is made of all the real numbers ≥ 14 and ≤ 17. So using symbols, we can make communication simple and fast.

And describing a function, we can also say that a function is from a domain to a range.

So next, the expression below is saying that ***G*** is a function from ***X*** to ***Y***.

$$\boldsymbol{G}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$$

So we can readily and quickly see that the domain is ***X*** and the range is ***Y***.
The arrow says that the values in ***X*** cause the values in ***Y***.
That is to say that the function ***G*** produces the values in ***Y*** using the values of ***X***.

So in a symbolic definition, we don't have to declare the domain and the range.
That is, we can still distinguish each by means of the direction of an arrow.

So the statement, "$\boldsymbol{G}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$." is clearly saying that ***G*** is a function from ***X*** to ***Y***.

> And next, the expression $\boldsymbol{y = G(x) = 3x + 11}$ is saying that
> ***y*** is the output variable, gets each output, which belongs to ***Y***,
> ***x*** is the input variable, gets each input from ***X***,
> and the expression is $\boldsymbol{3x + 11}$,
> together with the name of the function, which is $\boldsymbol{G(x)}$ or just ***G***.

What do we mean by though, $\boldsymbol{\{3s + 11 |\ s \in X\}}$ and $\boldsymbol{\{3s + 11 |\ 1 \le s \le 2\}}$?

First of all, both are the same sets, and each is saying that it is a set of all the values that can be made from the expression $\boldsymbol{3s + 11}$ for $\boldsymbol{1 \le s \le 2}$.

So it is a set of all the values that are made from the expression $\boldsymbol{3x + 11}$ for $\boldsymbol{1 \le x \le 2}$, and thus, the set is the range, and is $\boldsymbol{Y = \{e |\ 14 \le e \le 17\}}$.

So $\boldsymbol{\{3s + 11 |\ s \in X\}}$, $\boldsymbol{\{3s + 11 |\ 1 \le s \le 2\}}$, and $\boldsymbol{\{e |\ 14 \le e \le 17\}}$ are identical to each other.

That is, we have $\boldsymbol{\{e |\ 14 \le e \le 17\} \equiv \{3s + 11 |\ 1 \le s \le 2\} \equiv \{3s + 11 |\ s \in X\}}$**,** because we have $\boldsymbol{X = \{e |\ 1 \le e \le 2\}}$.

Also, we can see from the fact above, $\boldsymbol{3s + 11}$ is in fact, equal to $\boldsymbol{3x + 11}$, so we can say that we can choose any letter for a variable insofar as the consistency is maintained.

Now, from the definition of the function ***G***, we can say this:

> The function is called ***G***, which is a function of ***x***,
> and from the domain ***X***, each input gets into ***x*** in the expression **3*x* + 11**,
> in which the operations get performed,
> then the output for the input gets produced,
> is put into ***y***,
> and thus, is an element of the range ***Y***.

Besides, we can put the definition of ***G*** in any of the ways as follows, too.

$\boldsymbol{X = \{u \mid 1 \le u \le 2\}}$, $\boldsymbol{Y = \{v \mid 14 \le v \le 17\}}$, $\boldsymbol{X \xrightarrow{G} Y}$, and $\boldsymbol{y = G(x) = 3x + 11}$.

$\boldsymbol{X = \{s \mid 1 \le s \le 2\}}$, $\boldsymbol{Y = \{3t + 11 \mid t \in X\}}$, $\boldsymbol{X \xrightarrow{G} Y}$, and $\boldsymbol{y = G(x)}$.

$\boldsymbol{X = \{p \mid 1 \le p \le 2\}}$, $\boldsymbol{Y = \{3q + 11 \mid 1 \le q \le 2\}}$, $\boldsymbol{X \xrightarrow{G} Y}$, and $\boldsymbol{y = G(x)}$.

In the definitions above, we put the function designator above the arrow, and other than that, they are all the same as the previous ones.

> That is, $\boldsymbol{G: X \longrightarrow Y}$ is the same as $\boldsymbol{X \xrightarrow{G} Y}$.

Now, let’s have a look at another example.

$\boldsymbol{X}$ and $\boldsymbol{Y}$ are sets of all real numbers, $\boldsymbol{F}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{x} + \mathbf{1}$.

$\boldsymbol{R}$ is a set of all real numbers, $\boldsymbol{X}$ and $\boldsymbol{Y} \subseteq \boldsymbol{R}$, $\boldsymbol{F}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{x} + \mathbf{1}$.

$\boldsymbol{X}$ is a set of real numbers, $\boldsymbol{Y} = \{\mathbf{2}\boldsymbol{x} + \mathbf{1} | \ \boldsymbol{x} \in \boldsymbol{X}\}$, $\boldsymbol{F}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$.

$\boldsymbol{R}$ is a set of all real numbers, $\boldsymbol{X} \subseteq \boldsymbol{R}$, $\boldsymbol{Y} = \{\mathbf{2}\boldsymbol{s} + \mathbf{1} | \ \boldsymbol{s} \in \boldsymbol{X}\}$, $\boldsymbol{F}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$.

$\boldsymbol{R}$ is a set of all real numbers, $\boldsymbol{X} \subseteq \boldsymbol{R}$, $\boldsymbol{Y} = \{\boldsymbol{F}(\boldsymbol{x}) | \ \boldsymbol{x} \in \boldsymbol{X}\}$, $\boldsymbol{F}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{x} + \mathbf{1}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$.

Though looking different from each other, all the five definitions above are the same. In each of the definitions above, we can see these:

$\boldsymbol{F}$ is a function from $\boldsymbol{X}$ to $\boldsymbol{Y}$, so $\boldsymbol{X}$ is the domain of $\boldsymbol{F}$, and $\boldsymbol{Y}$ is the range.
The input variable is $\boldsymbol{x}$, and the output variable is $\boldsymbol{y}$, and the expression is $\mathbf{2}\boldsymbol{x} + \mathbf{1}$.

- Also, we often say that $\boldsymbol{y}$ is the *image* of $\boldsymbol{x}$ by the function $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$.

So $\boldsymbol{Y}$ can be said to be the set of the images of all the elements in $\boldsymbol{X}$ by the function $\boldsymbol{F}$.
And of course, the domain and range cannot be empty sets, that is, $\boldsymbol{X} \neq \varnothing$, and $\boldsymbol{Y} \neq \varnothing$.

Besides, we have another way where we can define a function.

We can express directly the expression, along with the arrow. So for instance, we can define the function $\boldsymbol{F}$ above in such a way as follows.

$\boldsymbol{X}$ is a set of all real numbers, $\boldsymbol{Y} = \{\boldsymbol{F}(\boldsymbol{x}) | \ \boldsymbol{x} \in \boldsymbol{X}\}$, $\boldsymbol{F}: \boldsymbol{x} \longrightarrow \mathbf{2}\boldsymbol{x} + \mathbf{1}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$.

And we can put it this way, too:

$\boldsymbol{X}$ is a set of all real numbers, $\boldsymbol{Y} = \{\boldsymbol{F}(\boldsymbol{x}) | \boldsymbol{x} \in \boldsymbol{X}\}$, $\boldsymbol{x} \xrightarrow{\boldsymbol{F}} \mathbf{2}\boldsymbol{x} + \mathbf{1}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$.

That is, $(\boldsymbol{F}: \boldsymbol{x} \longrightarrow \mathbf{2}\boldsymbol{x} + \mathbf{1}) \equiv (\boldsymbol{x} \xrightarrow{\boldsymbol{F}} \mathbf{2}\boldsymbol{x} + \mathbf{1})$.

So from the definition of $\boldsymbol{F}$ above, too, we can see $\boldsymbol{X}$ is the domain, and $\boldsymbol{Y}$ is the range, and for each value of $\boldsymbol{x}$, the output gets generated by the expression $\mathbf{2}\boldsymbol{x} + \mathbf{1}$, and therefore, we can set $\underline{\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{x} + \mathbf{1}}$ <u>for $\boldsymbol{x}$ real</u>, or just $\underline{\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{x} + \mathbf{1}}$, because in this case, it is assumed the domain is a set of all real numbers.

And there can be many other ways we can define a function.
For instance, we can sometimes use parameters, too.
And such a parameter can be called a medium.

Using a parameter to make a function, we parameterize a variable to produce a function.
Then, the variable is the output variable in the function produced, and the parameter becomes the input variable.

So the function is a function of the parameter, and is called a parametric function.

Assuming for instance, $\boldsymbol{x} = \boldsymbol{t} + \mathbf{1}$ when $\boldsymbol{t} > \mathbf{0}$, and parameterizing $\boldsymbol{x}$ to make a function called $\boldsymbol{f}$, we can get $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{t}) = \boldsymbol{t} + \mathbf{1}$ for $\boldsymbol{t} > \mathbf{0}$.

Then, we can say $\boldsymbol{x}$ is parameterized, and $\boldsymbol{t}$ is the parameter, and $\boldsymbol{f}$ is called a parametric function, where $\boldsymbol{x}$ is the output variable, so $\boldsymbol{f}$ is a function of $\boldsymbol{t}$, which is thus, the input variable.

Parameterizing a variable to produce a function, we say we do parameterization.
Doing parameterization though, we usually parameterize two or more variables.
So we get two or more functions that are parametric.

And we use the same parameter as the input variable of each parametric function.
So the parameter can be called the common input variable of the functions.

That is, the parameter goes between the functions.
Thus, we can connect the functions by means of the parameter.

That is, all the output variables get connected through the medium called the parameter, and then, a new function or an equation gets produced.

In other words, we get a connective expression between the output variables by means of the parameter, and the expression is the expression of the new function or the equation we get.

Suppose for instance, $x = f(t) = t + 1$ for $t > 0$, and $y = g(t) = t^2$ for $t > 0$.

Then, f and g both are functions of t, and can be called parametric functions, and x and y are output variables, so connecting the two output variables, we can come up with another function.

To begin with, from $x = f(t) = t + 1$, we can get $t = x - 1$ for $t > 0$.

So we get $t = x - 1 > 0 \Rightarrow x > 1$, and thus, we get $t = x - 1$ for $x > 1$.

And we have $y = g(t) = t^2$, too, so we get $y = t^2 = (x - 1)^2 \Rightarrow y = (x - 1)^2$ for $x > 1$.

Naming the function above, $h(x)$, we get a function $y = h(x) = (x - 1)^2$ for $x > 1$.

Connecting functions through a parameter, we say we do a parametric transformation, which is covered in the book, **Function Transformations**.

Note:
We can choose any letter for a variable insofar as the consistency is maintained. In other words, it doesn't matter what letter we take as the variable as long as the consistency is maintained.

So we can freely choose any letter provided we keep the consistency. Thus, for instance, "$y = g(x) = 8(3 - x)$ where $x \geq 1$." is the same as "$t = h(s) = 8(3 - s)$ where $s \geq 1$."

That is, the two functions $g(x)$ and $h(s)$ are the same functions.

In other words, though both have different names, both are the same, because both have the same domain, and the same expression.

The input variable s is no more than a container for one input value at a time, and covers all the values in the domain of the function g. And the output variable y is a container for one output value at a time, and covers all the values in the range.

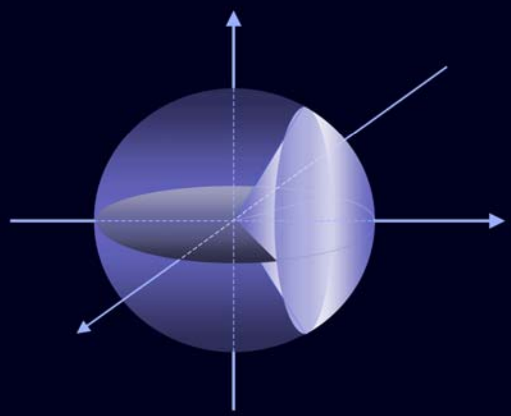
By the way, defining a function, we define a particular function. Defining a function particular, we make a function. Making a function, we can say we make a function definition, too.

And making such definitions, we can have two choices. One is a full definition, and the other is a short definition. It depends on how much detail we show on the function we define. And also, we can call a short definition a brief definition or a semi-definition, too.

A full definition is composed of the name, the domain and the expression. Quite often though, the expression is missing. Then, we can call it a short definition.

If the expression is unavailable or unnecessary for the moment, we can make a short definition using the name and domain only, and can call it the minimal definition.

So for instance, a function definition can be $y = f(x)$ for $x > 2$, or $t = g(s)$ for s real.
And if we get $t = g(s)$ only, it is assumed that the domain is a set of all real numbers.



기본 함수

Basic Functions

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