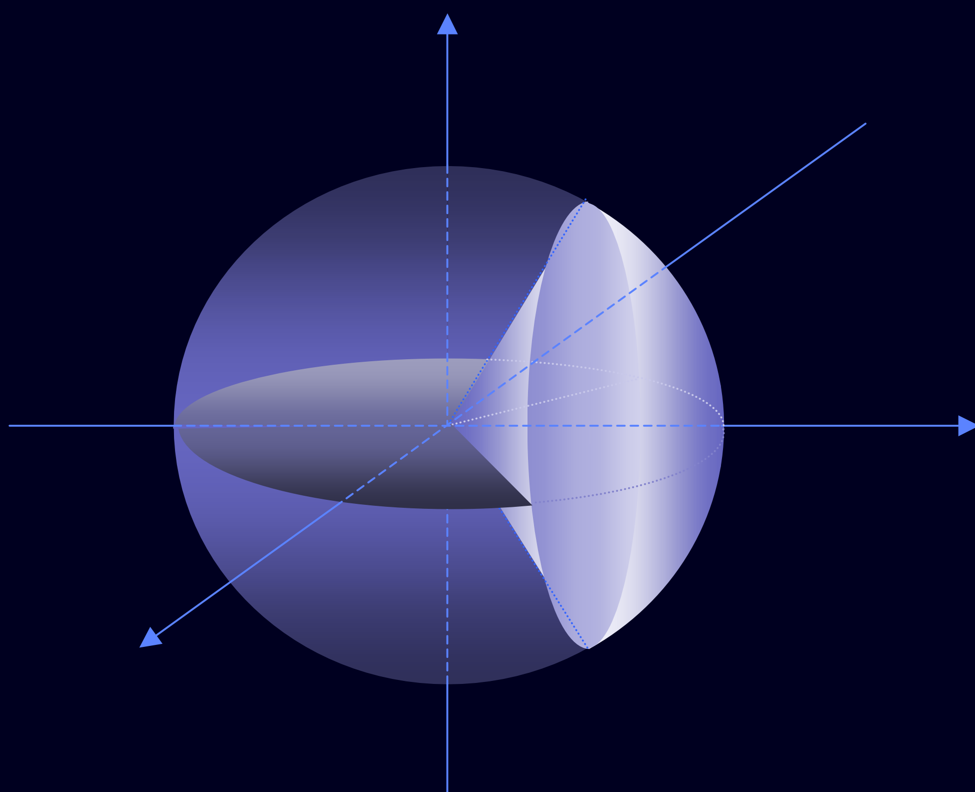
원뿔 곡선

Conic Sections



김성렬

Seong R. Kim

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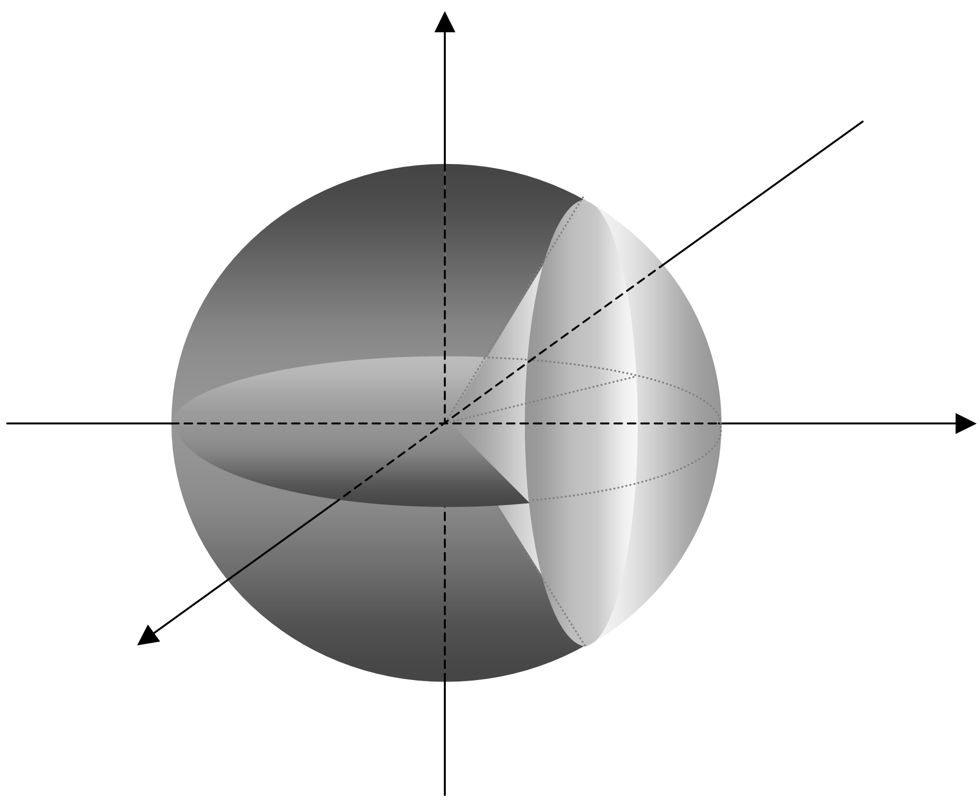

Seong R. KIM

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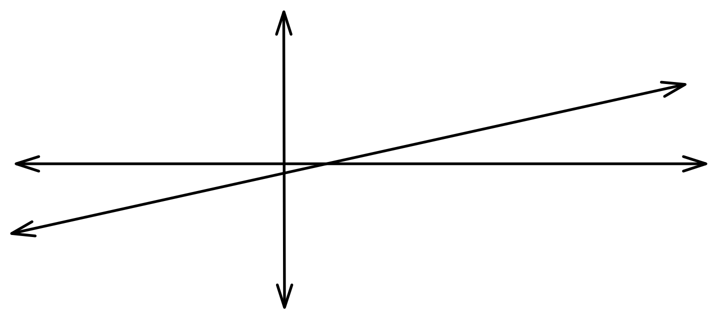
0.0. What is a line?



A line in math is in a plane, is of infinite length,
and is a set of points,
where every segment from one point to another is straight,
and has the same slope,
which is called the slope of the line.
And each segment is called a line segment, of course.

So in short, we can put a line the way as follows.

Every segment from one point to another in a line
shares the same slope as the line has.

And we can put the same this way, too:

No matter what two points we may choose in a line,
the slope of the segment between the two is constant,
that is, the same, and is the slope of the line.

So what is a line about?

It is about a slope.
Saying thus, a line in math, we can even say that we mean a slope.

Note: In this book, just saying geometry, we mean analytic geometry, often called coordinate geometry, too, and in geometry analytic, we put in a graph an object as a line, circle, triangle, etc., and study the object.

A line in math is a geometric object, and in geometry analytic, we put it in a graph. Putting an object in a graph, we graph it or make the graph of it.

Ordinarily, a straight line is simply called a line, and a curved line is just called a curve.
In math however, either curved or not, a line can be called a curve, too.
So a curve in a graph can be a line, parabola, circle, ellipse, hyperbola, etc.

And we have another object that looks like a line, but is not a line.
We can get it cutting a line. And it is called a ray or a half-line.

So we can make two different cuts in a line. One is a ray, and the other is a line segment.

A line segment has a length finite, but the length of a line or a ray is infinite. A ray is said to grow one way, but a line is said to grow two ways.

A ray begins with a point, and grows forever in one way only.
A line begins with a point, too, but grows forever in two ways, opposite of each other.

Fig. 0

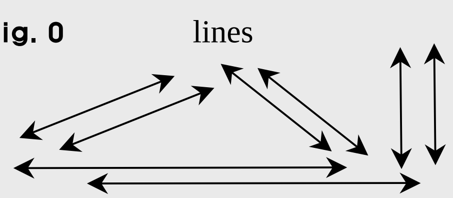
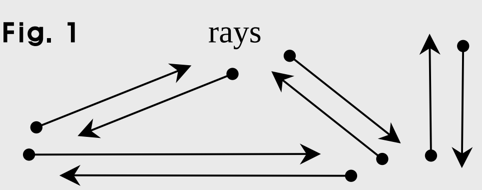


Fig. 1



How can we get a line?

A line is one of conic sections, often just called conics.

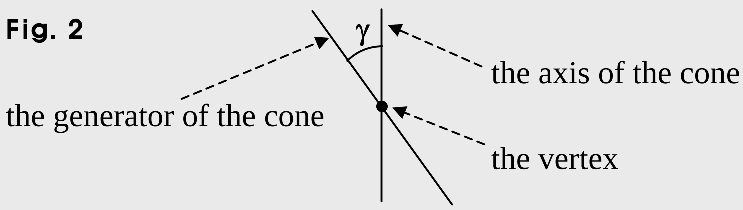
So it is a conic, and if cutting with a plane two right cones meeting at their vertices, we can get a cross section called a line. So it's a conic section, just called a conic, too.

What is a right cone though?

If a cone is a right cone, the line connecting the vertex of the cone and the center of the base is perpendicular to the base, and thus, makes a right angle (90^o) with the base. And the line stated above is called the axis of the cone. And in the study of conics, such a right cone is made or generated the way as follows.

First, take a line as the axis of the cone to be made.
Next, make another line meet the axis of the cone at a point at an angle γ, which is an acute angle, that is, $\mathbf{0 < \gamma < 90^o}$.

Fig. 2

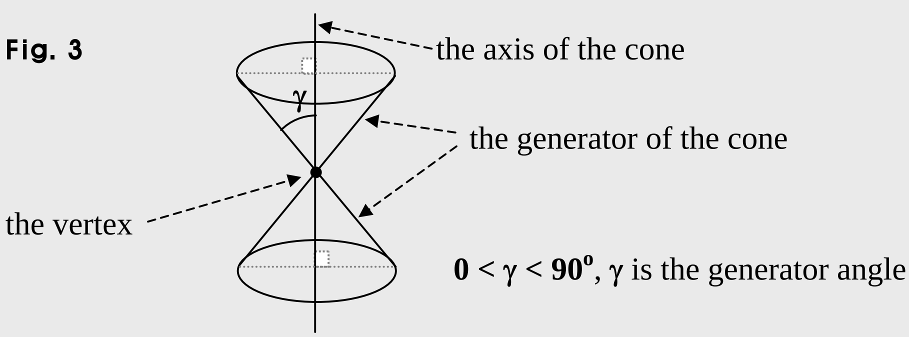


Then, we call the other line the *generator of the cone*, and the point where the generator of the cone meets the axis of the cone is the vertex of the cone.

So the vertex of the cone is the point where the generator meets the axis of the cone. And in the figure above, the angle γ is called the *generator angle*.

Then, rotating the generator around the axis fixing the generator at the vertex, we get a double-cone as shown in Fig. 3.

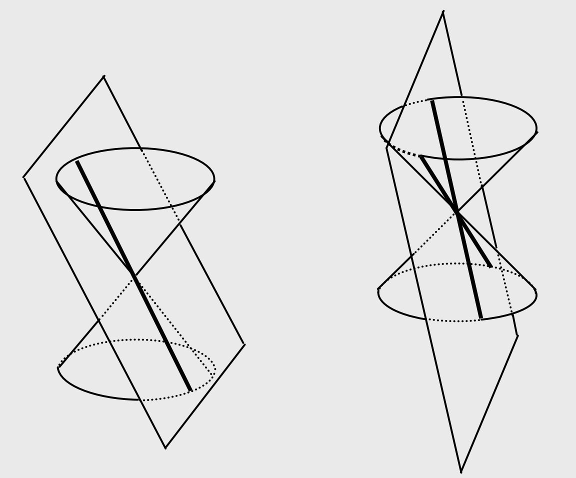
Fig. 3



And note that the generator and the axis are lines, and lines have infinite lengths, so the lengths of the cones are infinite, too. So we cannot show them all. Showing thus, such a cone or a curve in math, we just show some part of it.

How then can we get a line cutting the cone with a plane?

Fig. 4



In the figure above, the cross sections in black are lines. So if α is the angle between the plane and the axis of the cone, and $\mathbf{0} \leq \alpha < \gamma$, and if the plane includes the vertex, the cross section is a pair of lines meeting at the vertex. And if the plane is tangent to the cone's side (called the generator of the cone, too), that is, if α is γ, only one line gets made as shown on the left in the figure above, and the line made is in fact, the generator of the cone.

What if a plane cuts through one cone only?

Then, if the plane includes the vertex, and $\mathbf{0} \leq \alpha < \gamma$, we get two rays emitting from the vertex. And if the plane is tangent to the side of the cone, that is, if α is γ, only one ray gets created, and is emitting from the vertex.   What then about a line segment?

It is a part of a line, and has a finite length. In math, saying a line segment, we mean a straight line segment, and a distance means a length of a line segment.

So for instance, we indicate a distance between two points by means of the length of the line segment connecting the two points.

What then about a segment cut from a curved line?

It can be called a curved line segment.   What then is an arc?

It is a part of a circle, and is a curved line segment, too. Technically though, a circle is a curved line segment, too. A circle is however, a closed line segment, but an arc is not. For what then do we use a line? That is, what's the purpose of it?

A line has an idea of length or distance, for which we use a line segment.
Also, a line has another idea, called a direction. In geometry analytic or not, a line often means a direction. We use a line or ray to indicate a direction rather than a length or distance, for which we use a line segment.

In fact, a line has a direction. So we can indicate a direction by means of the direction a line or a ray has, and for instance, we can use a line or a ray indicating north or west.

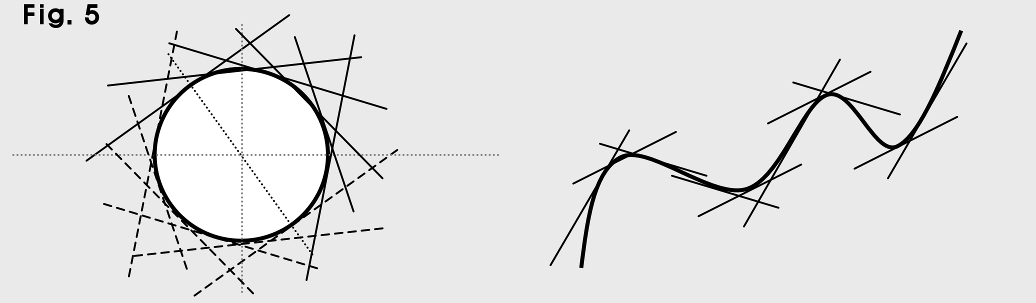
What then do we mean by a direction of a line?

Suppose a line is tangent to a circle, and for instance, the tangent line indicates two O'clock direction at the moment. Then, of course, the line can be said to indicate eight O'clock direction, too.

Suppose now, the line starts moving, and keeps moving being tangent to the circle.

Then, the line keeps changing its direction.
A line can be tangent to a circle at one point only.
So at every moment, the line is tangent to the circle at a different point. So the line gets a different direction as the line moves along the circle keeping itself tangent to the circle.
Thus, we can say that the tangent line can indicate every possible direction.

Fig. 5



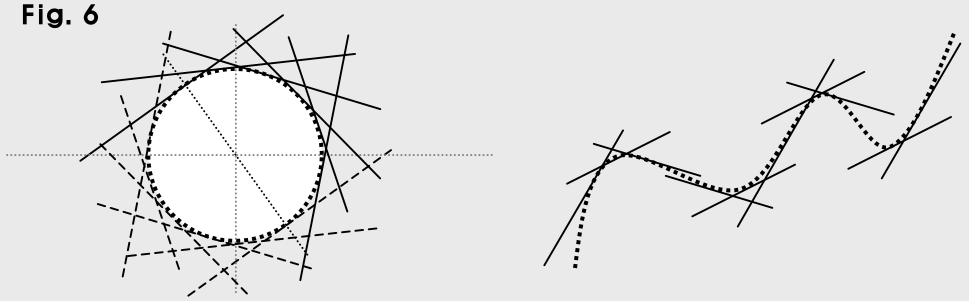
In fact, it doesn't have to be a whole circle, and can be a half circle, too. How?

In a circle, there are two points, at which the two lines tangent to the circle have the same direction. What are the two points then?

The two points are called antipodal points, and the length of the line segment connecting the two points is the diameter.

And if a line is tangent to a curve, it can be tangent to it at one point only.
In other words, the curve and the tangent line share one point only.

Fig. 6



And there is no bending, turning, twisting, or such in a line, so a line is said to be straight. How?

Suppose a line is tangent to an object, but this time, the line is tangent to the object at all points in the object at the same time. So even moving along the object keeping itself tangent to the object, the line maintains the same direction. What then is the object?

It is a line, too, and in fact, is the same line. Thus, there is no bending, turning, twisting, or such in a line, so a line is straight. It's kind of too forced, isn't it?

So let's now be more specific about a line.

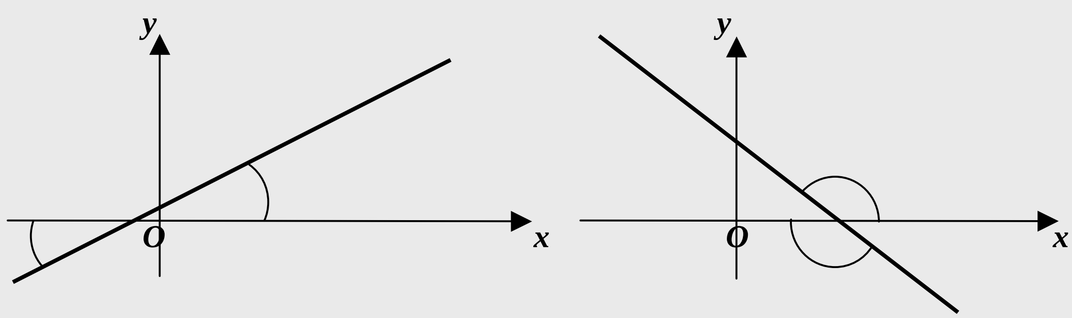
An object in a graph is a collection of points, so a line in a graph is a collection of points. So such a line or its segment is an object composed of points. What points, though?

No matter what two points we may choose in a line in a graph, a line segment connecting the two points has the same direction, which is the direction of the line.

Also, all lines parallel to each other share the same direction.
How then do we specify the direction of a line?

We can take a measurement of an angle, and can use it for the direction.
A line makes an angle against a coordinate axis in a graph, and the axis is normally horizontal. And usually, we put a graph in the ***x***-*y* plane. So for instance, a line makes an angle against the ***x***-axis, and we take the angle the way as follows.

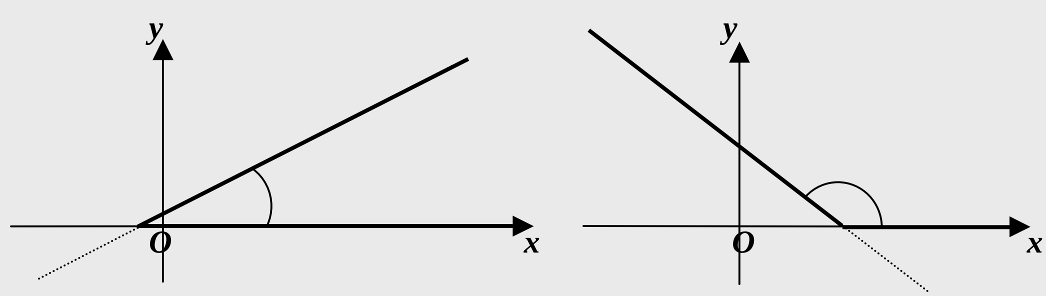
Fig. 7



Taking the measurement of such an angle though, we usually use a ray rather than a line.

Normally, we use as the ray the part of the line above the ***x***-axis, and we use a part of the ***x***-axis on the right of the point where the line meets the ***x***-axis. So we normally take the measurement of the angle between the ray and the part of the ***x***-axis as shown in Fig. 8.

Fig. 8

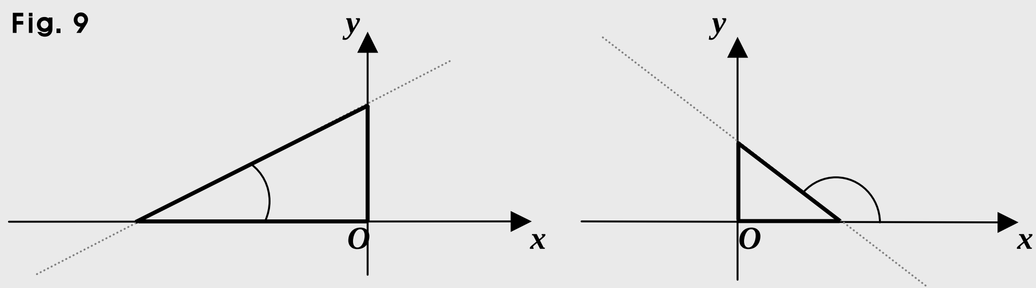


Taking such a measurement though, we do not normally use a protractor. How then can we take the measurement without such a device?

In fact, we usually specify the direction of a line with a different math tool.
The tool is called a slope, and a line has a slope. So we can specify the direction of a line by means of the slope of the line. What slope? What do we mean by the slope?

A slope is the ratio of the height to the base of a right triangle, that is, the ratio of the vertical leg to the horizontal leg.
Assuming a line is in the ***x-y*** plane, we can make a right triangle using parts of both coordinate axes and a part of the line as shown in Fig. 9 below.
So we can get a right triangle, where a part of the ***x***-axis is the base of the triangle, a part of the ***y***-axis is the height, and a part of the line is the hypotenuse.

Fig. 9



So the ratio of the height to the base is the slope of the line, and shows the degree of the steepness of the line. Therefore, the slope tells us how much the line is inclined with the ***x***-axis. However, the slope is often considered to be a rate rather than a ratio. We can take it as a rate of change, as well as a ratio. We will get to the detail in the next section.

Now, once we've got the slope, we can get the angle by means of a special ratio called a trig ratio, which is in this case called the tangent.
So the slope is the tangent of the angle. And parallel lines share the same angle. Thus, being parallel means the same slope and the same angle, as well as the same direction.

Working with a line, we can put the line in a graph. Then, we can actually see the line while working. A graph is not the only place we can put a line in, though.
Where else then can we put a line?

We can put it in an equation, too.
Putting a line in an equation, we express a line by means of an equation, that is, we indicate a line by an equation. And we call such an equation an equation of a line.
And we will get to see how we can get it in the next section.

0.1. Equations for Lines 1

Doing math, we solve problems, because solutions are what math is about. And doing problems, we often do problems with lines. A line seems to be simple, and is in fact, one of the simplest of all curves. When it comes to a problem though, it's not that simple.

Doing a problem with a line, we need to work with the line. Working with a line though, we can't just hold it in our hand, so we want to put it somewhere so that we can work with it.   Where then can we put a line?

We can put it in a graph. Then, we can actually see the line working with it. So putting it in a graph, it is much easier to work with it. So doing a problem with a line, graph it.

A graph is not the only place though, we can put a line in. We can put it in an equation, too. And we call such an equation an equation of a line.

Usually thus, putting a line in a graph, we say that we put its equation in a graph, too.
So putting an equation of a line in a graph, we put the line in a graph, and we say that we graph the equation, or make the graph of it.

How then, can we put a line in an equation, that is, how can we get an equation of a line?

Getting an equation of a line, we express a line using an equation.
And expressing a line, we want to know what it's made of.   So what is a line made of?

A line is basically a set of points, so it is made of points.   What points though?

A set of points is not necessarily a line. Every geometric object put in a graph is in fact, composed of points. So depending on what points put together, the shape of the object gets determined. What points then, do we have to put together to form a line?

As stated in the previous section, we can explain what a line is the way as follows.

> No matter what two points we may pick from a line,
> the segment between the two has the same slope,
> which is the slope of the line.

Suppose for instance, $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are three points in a line.
Then, the slope of $\overline{\boldsymbol{AB}}$ is the same as that of $\overline{\boldsymbol{BC}}$, and is the same as that of $\overline{\boldsymbol{AC}}$, too, and thus, is the slope of the line.

So forming a line, we want to put together all such points.
We can't actually do so, of course, but can do it mathematically. How?

Putting all those points in an equation, we get the equation of the line.
So we are back again, to the question that how to get an equation of a line. And in fact, we can get such an equation using the explanation above, since it explains what a line is. That is, it can be the definition for lines. And we can put it the way as follows, too

> Every segment from one point to another in a line has
> the same slope as the line has.

So we can notice that every point in a line has to do with the slope.
Thus, a line is no other than a slope if you will.

So given a particular slope, can we define a particular line, that is, can we get the equation of it?

Many lines can have the same slope, so we can't. In fact, infinitely many lines can share one slope. How many lines then, can share a particular slope and a particular point?

Not even two can do so. Of infinitely many lines sharing one slope, only one can have a particular point. That is, only one line can have a particular slope and a particular point.

So defining a line, we want to get a point the line has, together with the slope.
That is, finding a line, we need at least one point in the line and the slope.

So suppose now that we are given the slope and a point of a line. How then can we get the line, that is, the equation of the line?

As stated earlier, we can get the equation using the definition for lines, which is below.

Every segment from one point to another in a line has the same slope as the line has.

And we can put the same this way, too:

No matter what two points we may choose in a line,
the slope of the line segment between the two is constant,
and thus, is the same.

So let's assume that the slope is $\boldsymbol{a}$, and that $\boldsymbol{L}$ is the line we want to define, and has a particular point $\boldsymbol{(x_1, y_1)}$.
What do we mean by a slope, though?

A slope is the ratio of the height to the base of a right triangle.
Assuming a line is in the ***x-y*** plane, we can make a right triangle using parts from the coordinate axes and a part of the line. We will see a specific example shortly.
So now getting back to the definition again, we have this:

No matter what two points we may choose in a line, the slope of the line segment between the two is constant, and thus, is the same.
What then do we mean by the phrase, ***no matter what two points***?

We mean an arbitrary pair of points. How then can we make an arbitrary pair?

An arbitrary point in a line is a random point, and represents all the points in the line.
So we don't just pick any two points in the ***x-y*** plane, of course.

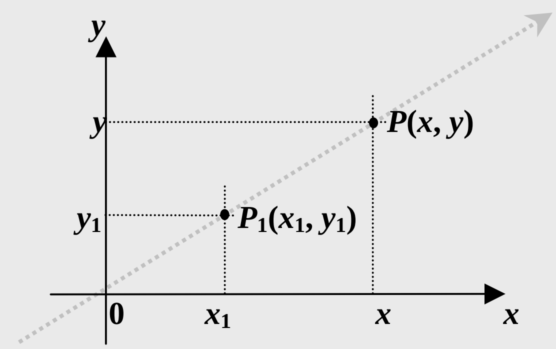
A pair of points has two points. Of the two, one is the point given.
What then is the other point?

It is the point that makes the pair arbitrary, so it has to be an arbitrary point in the line.
So since the line is in the ***x-y*** plane, we can specify the arbitrary point this way: (x, y).

Let's now, put in the ***x-y*** plane the two points stated above. One is (x_1, y_1), which is the point given, and the other is (x, y), which is the arbitrary point in the line.

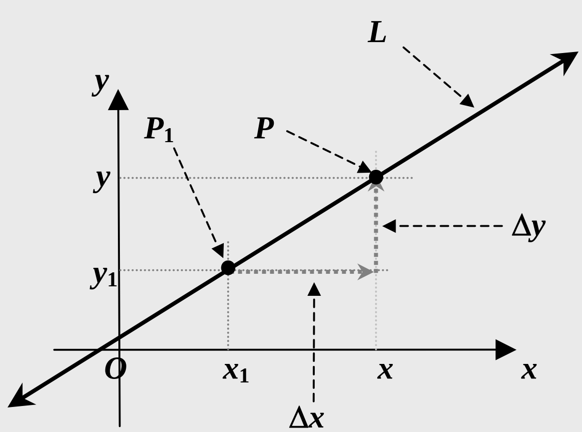
So putting the two in the ***x-y*** plane, we can get a graph as shown in Fig. 0.

Fig. 0



We know L is the line we are after, so the two points P and P_1 are in the line L.
And showing now the line L in the graph, we can show it the way as follows.

Fig. 1



In the graph above, the line segment indicated by Δx is parallel to the x-axis, and the line segment indicated by Δy is parallel to the y-axis.

So we can take Δx as a part of the x-axis, and take Δy as a part of the y-axis.
That's because we get $\Delta x = x - x_1$ and $\Delta y = y - y_1$.

So in the graph above, we can see a right triangle where the base is Δx, the height is Δy, and the hypotenuse is the line segment between P and P_1.
Now again, what do we mean by a slope?

A slope is the ratio of the height to the base of a right triangle.
We have a right triangle in the graph above, and also, a is the slope of the line L.
Thus, the slope of the line segment $\overline{P_1P}$ is a, too.

So we can put the slope of the line L the way as follows.

$\dfrac{\Delta y}{\Delta x} = \dfrac{y - y_1}{x - x_1} = a$, where a is constant.

So what do we have now?

We now have an equation, and the equation is $a = \dfrac{y - y_1}{x - x_1}$. What equation then is it?

It is the equation of the line L. How?

We know (x, y) is an arbitrary point in the line L, so it represents all the points in the line, and the equation above explains the coordinates of the arbitrary point. So?

So the equation explains each and every point in the line L.

For instance, putting a value into x, we can get the value of y, since a, x_1, and y_1 are known to us.

So using the equation, we can get each and every point in the line L. We can say thus, the equation indicates the line L. So we can call it the equation of the line L.
By the way, many people use different words for Δx and Δy in $\dfrac{\Delta y}{\Delta x} = \dfrac{y - y_1}{x - x_1} = a$.

They call Δy 'rise', and call Δx 'run', so the slope is often called 'rise-over-run'.

It makes sense, because Δy is ***the change*** in y-coordinate between two points in a line, and is the change in the vertical direction, and Δx is ***the change*** in x-coordinate, and is the change in the horizontal direction.

In the equation above, Δy is the change in y-coordinate between (x, y) and (x_1, y_1), and Δx is the change in x-coordinate between the two points, which are in the line for which the equation is made.
Usually though, we don't put the equation the way above. It has a bit different look.
Putting it in more conventional form, we get $a = \frac{y - y_1}{x - x_1} \Rightarrow y - y_1 = a(x - x_1)$.

So we usually put the equation this way: $y - y_1 = a(x - x_1)$.

And we just call the equation an equation of a line. So indicating a line in general, we can use the equation above, and just call it an equation of a line.
It's not the only form though, we use indicating a line in general.

Solving the equation above for y, we get $y - y_1 = a(x - x_1) \Rightarrow y = ax + y_1 - ax_1$.
And we know a, x_1, and y_1 are all constant. So $y_1 - ax_1$ is constant, too.

So assuming b is constant, and replacing $(y_1 - ax_1)$ with b, we can put the equation above this way too: $y = ax + b$.

So we now have two forms for the same equation. And the two are as follows.

$y - y_1 = a(x - x_1)$ $\qquad$ $y = ax + b$, where $b = y_1 - ax_1$

So all the equations above indicate the same line, and the line has a slope of a, and is passing through a point (x_1, y_1). Thus, given a slope and a point, we can get the line.
By the way, we call each equation above the ***connective equation*** between x and y, because it connects x and y, and explains specifically the relation between the two.

That is, it explains how x and y are related (or connected) in a specific manner.
So we usually call it the connective equation between the two.

And in this case, x and y are the coordinates of the arbitrary point (x, y) in a line.
What then do we need to find finding an equation of a curve as a circle or parabola?

It is the connective equation, which connects the coordinates of the arbitrary point in the curve. And we can find it using the definition for those curves as circles, for instance.

So for instance, using the definition for circles, we can find the connective equation, connecting the coordinates of an arbitrary point (x, y) in a circle. And the connective equation for circles is $(x - a)^2 + (y - b)^2 = c^2$, where (a, b) is the center, and c is the radius.

So the connective equation for certain curves indicates such a curve in general.
Thus, the connective equation for lines indicates a line in general.

And finding a connective equation using such a definition, we say that we ***derive*** the equation. Then, depending on the use of it, we change the form as we did to the connective equation for lines. And then, we give a name to each form.

Now, getting back to the forms we made for lines, we have these:

$$y - y_1 = a(x - x_1). \qquad y = ax + b$$

The first form can be called the ***point-slope*** form, because a is the slope, and (x_1, y_1) is a point in the line.
And the second can be called the ***slope-intercept*** form, because a is the slope, and b is the y-intercept of the line.
The y-intercept is the y-coordinate of the point where the line meets the y-axis.

And of course, there can be many other forms that can indicate a line in a general manner. It all depends on the use of the form, that is, the situation where the form is used.

Many options are not always good though. You might get confused, and there are quite a few things to learn besides things in math.
And another bad news is the names of the forms change from people to people, from country to country, from teachers to teachers, from time to time, etc.
It doesn’t really matter though.

What matters is you can do the derivation of the equation from the definition.

At least, you understand the derivation processes, that is, how the derivation can be done. How then can we find the equation of a particular line?

We can find it using one of the forms above, i.e., one of the equations for lines.

So assuming for instance, the slope of a line is 2, and a point (1, 3) is in the line, we can use the slope form, and can put the equation of the line this way: $(y - 3) = 2(x - 1)$.

And for another instance, if the slope is 2 and the y-intercept is 1, we can use the intercept form, and can put the equation of the line this way: $y = 2x + 1$.

In fact, the two equations above indicate the same line, so both are actually the same equations, and only have different looks.

So depending on the problem we do, or the situation where the equation is used, we choose one of the forms, and can get the equation putting the specifics into the form. And the specifics are information as the slope, the point in the line, the intercept, or something else, which depends on the form we choose.

So there are some more forms, and you will get to see them in the next section.

0.2. Equations for Lines 2

To begin with, we can put the definition for lines the way as follows.

> Every segment from one point to another in a line has
> the same slope as the line has.

And we can put the same this way, too:

> No matter what two points we may choose in a line,
> the slope of the line segment between the two is constant,
> and thus, is the same.

And using the definition above, we can get the connective equation for lines, and can put it in either of the two forms as follows.

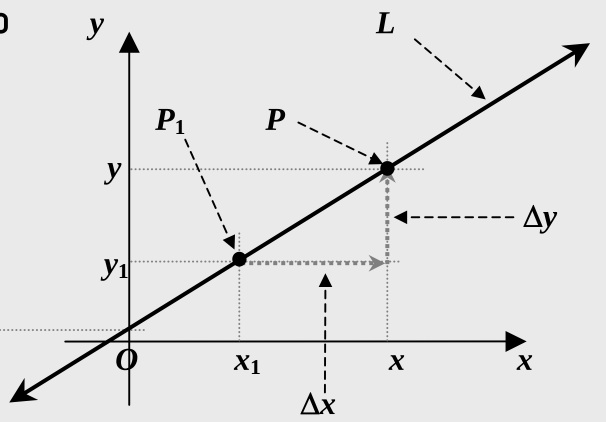
$$y - y_1 = a(x - x_1) \qquad y = ax + b, \text{ where } b = y_1 - ax_1$$

So both the two above indicate the same line, and the line has a slope of a, and is passing through a point (x_1, y_1). Thus, given a slope and a point, we can get the line.

Of the two above, taking a close look at the second one, we can notice that we get $y = b$ setting $x = 0$ in the equation $y = ax + b$. And we call b the y-intercept, which is the y-coordinate of the point where the line meets the y-axis.

So if the line passes though the origin $O(0, 0)$, the y-intercept is 0.
And assuming the equation above indicates a line L, we can graph it the way as follows.

Fig. 0



Then, we can see that the line $\boldsymbol{y = ax + b}$ cuts through the $\boldsymbol{y}$-axis at a point $\mathbf{(0,}\ \boldsymbol{b}\mathbf{)}$.

So when $\boldsymbol{x} = \mathbf{0}$, we get $\boldsymbol{y = b}$. Note however, not $\boldsymbol{b}$ but $|\boldsymbol{b}|$ is the distance from the origin to the point where the line crosses the $\boldsymbol{y}$-axis. That's because it can be the case a line can meet the $\boldsymbol{y}$-axis at a point below the origin, that is, below the $\boldsymbol{x}$-axis, so $\boldsymbol{b}$ can be negative.

What then about the $\boldsymbol{x}$-intercept?

We can see it in the graph above, and we can get it setting $\boldsymbol{y} = \mathbf{0}$ in the equation, because when the line crosses the $\boldsymbol{x}$-axis, we get $\boldsymbol{y} = \mathbf{0}$. So setting $\boldsymbol{y} = \mathbf{0}$ in $\boldsymbol{y = ax + b}$, we get

$\mathbf{0} = \boldsymbol{ax + b} \Rightarrow \boldsymbol{x} = -\dfrac{\boldsymbol{b}}{\boldsymbol{a}}$. Thus, the $\boldsymbol{x}$-intercept of the line $\boldsymbol{L}$ is $-\dfrac{\boldsymbol{b}}{\boldsymbol{a}}$.

So for now, we have two forms we can choose from finding an equation of a line.

$$\boldsymbol{y - y_1 = a(x - x_1)} \qquad \boldsymbol{y = ax + b}$$

So if given a slope and a point, we can use this form: $\boldsymbol{y - y_1 = a(x - x_1)}$.

And if given a slope and an intercept, we can use the form of this: $\boldsymbol{y = ax + b}$.

What if we are not given a slope, though?

A line is no other than a slope so to speak. So does no slope mean no line?

There is no rule without an exception. So some lines are said to have no slope.

Lines with no slope are lines vertical, and thus, are perpendicular to the x-axis.
So if lines are perpendicular to the x-axis, the lines have no slope. Why not?

It's because of the definition of the slope.
By definition, if a is a slope, we get $a = \frac{\Delta y}{\Delta x}$.
And we know Δx is the change in x-coordinate between two points in the line.
So we get $\Delta x = \mathbf{0}$ if the line is vertical, that is, perpendicular to the x-axis. How?

No matter what two points we may choose in a vertical line, we get no change in x-coordinate in the two points. So we get $\Delta x = \mathbf{0}$, which is however, the denominator.
And we know that no division by 0 is allowed. So we get no slope if the line is vertical.

So in short, a line vertical has no slope. What then about lines horizontal?

Lines horizontal have slopes, but have to share one slope only, which is 0. How?

Lines parallel to the x-axis are horizontal. So if parallel to the x-axis, their slopes are 0.

It's again because of the definition of the slope. In the definition for slopes above, Δy is the change in y-coordinate in two points in the line.
So we get $\Delta y = \mathbf{0}$ if the line is horizontal, that is, parallel to the x-axis. How?

No matter what two points we may choose in a horizontal line, we get no change in y-coordinate in the two points. So we get $\Delta y = \mathbf{0}$, which is the numerator.
And a division of 0 by any nonzero is 0. So if a line is horizontal, the slope is 0.

What if we are not given any slope, and still need to find a line neither vertical nor horizontal?

If we are asked to find a line not vertical, we are given the slope anyway. That is to say that we need to find somehow the slope from the information given by the problem.

In other words, the slope is hidden somewhere in the problem.
So we've got to play some hide and seek.
Having only to have two points in fact, we can still find the line.   How?

Where are the two points?

They are in the line we want to find, and ***two points can define a slope***.
So using the two points, we can find the slope.
Then, using one of the two points, together with the slope, we can find the line.

So for instance, given two points (x_1, y_1) and (x_2, y_2), we get $\frac{y_2 - y_1}{x_2 - x_1}$, which is the slope.

And using the slope form $y - y_1 = a(x - x_1)$, since a is the slope, we get

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which can be put this way: $(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$,

which is the equation of the line passing through the two points (x_1, y_1) and (x_2, y_2), and can be called the ***two-point*** form.

And we can put it this way, too: $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$, which is closer to the definition.

What if the two points have the same x-coordinates, for instance, (1, 2) and (1, 3)?

Then, the line is vertical. A line vertical has no slope, but is still a line, so it has its equation, too.   What then is the equation of the line?

It was mentioned above that no slope means no change in x-coordinate between two points in a line.
So at all the points in a vertical line, all the x-coordinates are same,
and thus, the line gets defined to be x = x-coordinate which is given in the point given.

So if given (1, 2) and (1, 3), we get $\boldsymbol{x = 1}$, which is the equation of the line passing through the two points (1, 2) and (1, 3).
And for instance, some of the other points in the line $\boldsymbol{x = 1}$ can be (1, -2) and (1, 5).
So the arbitrary point in the line is $\boldsymbol{(1, y)}$.

What if this time, the two points given have the same $\boldsymbol{y}$-coordinates, for instance, (1, 2) and (3, 2)?

Then, the line is horizontal. So its slope is 0. Using thus, the form of $\boldsymbol{y - y_1 = a(x - x_1)}$, called the slope-point form, and using (1, 2), we get $\boldsymbol{y - 2 = 0(x - 2) \Rightarrow y = 2}$, which is the equation of the line passing through the two points (1, 2) and (3, 2).

And we can put the idea this way, too:

If the slope is 0, it means no change in $\boldsymbol{y}$-coordinate between two points in a line.
So at all the points in a line horizontal, all the $\boldsymbol{y}$-coordinates are same,
and thus, the line gets defined to be $\boldsymbol{y = y}$-coordinate which is given in the point given.

So we now have some forms we can use finding a particular line.

Given a slope and a point, we can use the form of $\boldsymbol{y - y_1 = a(x - x_1)}$ or this: $\boldsymbol{a = \frac{y - y_1}{x - x_1}}$.

Given a slope and an intercept, we can use the form of this: $\boldsymbol{y = ax + b}$.

Given two points, we can use the form of $\boldsymbol{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$, or the one below.

$$\boldsymbol{(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)}.$$

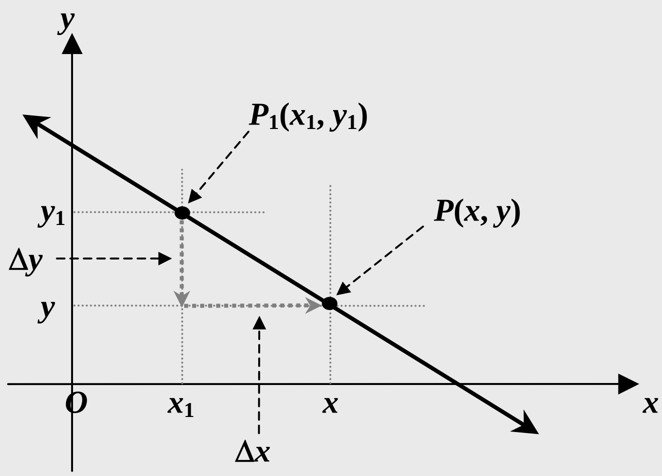
If a line is vertical and has a point $\boldsymbol{(x_1, y_1)}$, its equation is $\boldsymbol{x = x_1}$.

And if a line is horizontal and has a point $\boldsymbol{(x_1, y_1)}$, its equation is $\boldsymbol{y = x_1}$.

What if the slope $\boldsymbol{a}$ is negative?

We can put in a graph a line with a negative slope the way as follows.

Fig. 3



So a slope says how a line slants, and explains the amount and direction a line is inclined with the ***x***-axis.
The amount is the magnitude of the slope.
Taking the absolute value of the slope, we can get the magnitude.

And the direction a line slants depends on the sign of its slope.
If the slope is positive, the line slants to the left, and if the slope is negative, it slants to the right.
And if the slope is 0, the line is horizontal, and if no slope, the line is vertical.
What then is the slope?

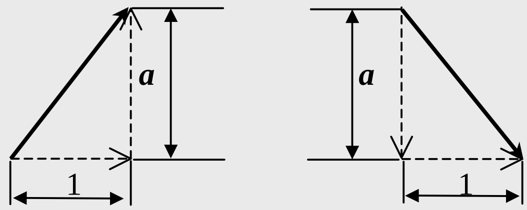
The slope is the ratio of the vertical change to the horizontal change between two points in a line.
The vertical change is the change in ***y***-coordinate between two points, and the horizontal change is the change in the ***x***-coordinate. So assuming Δy is the vertical change, Δx is the horizontal change, and a is the slope, we can put the slope a the way as follows.

$a = \frac{\Delta y}{\Delta x}$. So if (x_1, y_1) and (x_2, y_2) are two points in a line, the slope is as follows.

$$a = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

And Δy is often called 'rise', and Δx is called 'run', so it's often called rise-over-run, too.

Fig. 4 The slope $a > 0$ The slope $a < 0$



So if we put in a graph a line indicated by the equation $y - y_1 = a(x - x_1)$, depending on the sign of a, the equation indicates either of the two lines as follows.

Fig. 5 The slope $a > 0$.

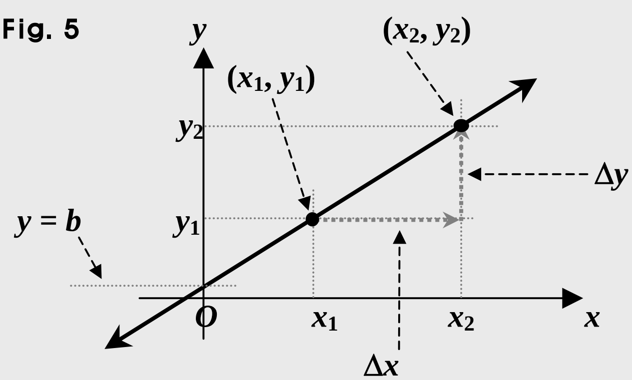
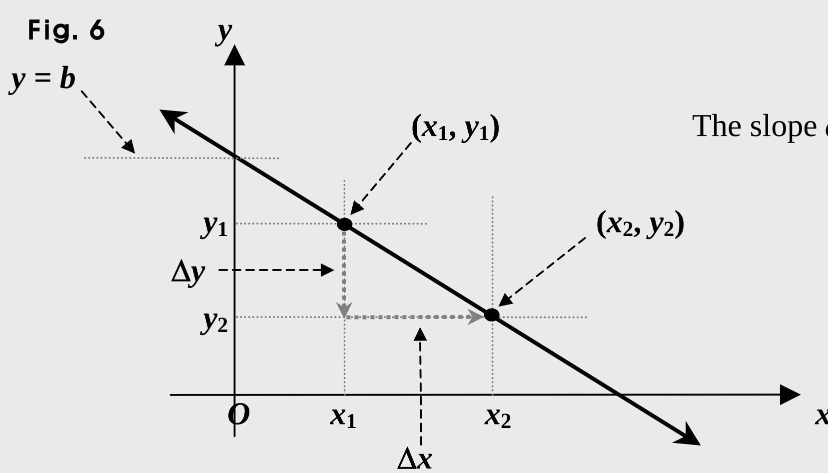


Fig. 6 The slope $a < 0$.



Normally, a line is said to proceed from the left to the right in the ***x*****-*****y*** plane.
Thus, as a line proceeds, the ***x***-coordinate at each point increases, so **Δ*x*** is positive.

Next, if the slope is positive, the ***y***-coordinate at each point increases, so **Δ*y*** is positive, too. If therefore, the slope is positive, we get the graph in Fig. 5 above.

If the slope is negative, the ***y***-coordinate at each point decreases, so **Δ*y*** is negative, too. So if the slope is negative, we get the graph in Fig. 6 above.

What then about the magnitude?

The bigger the magnitude of the slope, the steeper the line slants.

And the sign of the slope shows the direction the line slants. If the slope is positive, the line slants to the left, and if negative, the line slants to the right.

All the lines parallel to each other share the same slope, and therefore, have the same direction, so they all have the same angle (against the ***x***-axis if they are in the ***x*****-*****y*** plane).

Also, we can notice that the slope in a line is not simply a ratio but actually a rate.
How?

Each point in a line has an ***x***-coordinate and a ***y***-coordinate.
And we call the ***x***-coordinate the ***x***-value, and call a ***y***-coordinate the ***y***-value.

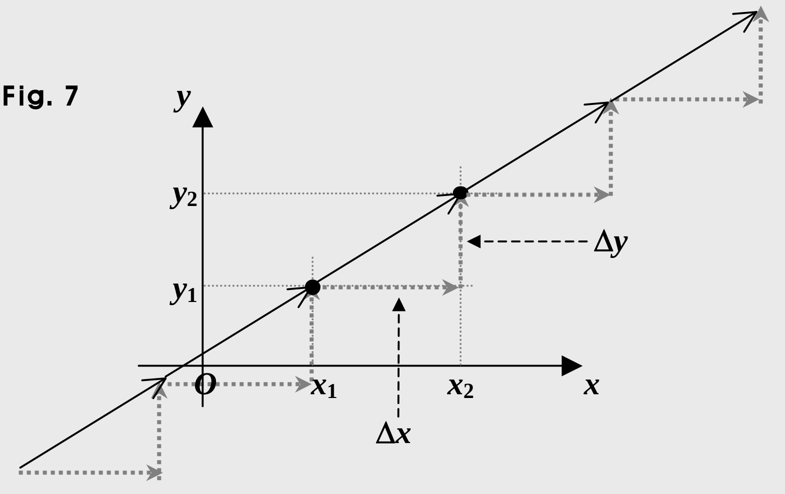
So the slope tells how fast or slow the ***y***-value changes as ***x***-value changes.
In other words, the slope is the rate of change in ***y***-value.

So for instance, assuming ***y***-value is an amount of distance, and ***x***-value is an amount of time, we can say that the slope is ***the time rate of change in distance***, because it shows how fast or slow the distance changes as the time changes.

Let's now again, put the line $\boldsymbol{y - y_1 = a(x - x_1)}$ in a graph.

Then, assuming $\boldsymbol{a > 0}$, we can graph the line the way as follows.

Fig. 7



And the slope of the line above is as follows. $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = a$, where a is constant.

Now, assuming a point (x, y) is moving along the line above, we can see that Δx is the change in the x-coordinate, that is, the change in the direction of the x-axis, and also, see that Δy is the change in the y-coordinate, that is, the change in the direction of the y-axis.

So we can put the idea above the way as follows, too.

As x-coordinate changes by Δx, y-coordinate changes by Δy.

So we can put the expression $\frac{\Delta y}{\Delta x}$ the way as follows, too.

It is the ratio of the ***change*** in the y-coordinate to the ***change*** in the x-coordinate.

What then do we mean by such a ratio?

It is a rate. A rate explains a change in one object with respect to a change in the other.

So a rate specifies how the amount of one object changes with respect to change in the other object.

For instance, a speed is a rate, which is a time rate of change in distance.

So a speed specifies how a distance changes with respect to change in time. That is, it can specify the change in distance with respect to a particular amount of change in time.

For instance, an amount of distance can increase every second as 7 m/sec.

So assuming the value of $\Delta \boldsymbol{x}$ indicates time, and that of $\Delta \boldsymbol{y}$ indicates a distance, we can see that the expression above indicates change in distance with respect to change in time, which is a speed, which is a rate, more specifically, a rate of change.

For instance, we can get $\dfrac{\Delta \boldsymbol{y} \text{ meters}}{\Delta \boldsymbol{x} \text{ seconds}} = \dfrac{\frac{\Delta \boldsymbol{y}}{\Delta \boldsymbol{x}} \text{ meters}}{\mathbf{1} \text{ second}} = \dfrac{\Delta \boldsymbol{y}}{\Delta \boldsymbol{x}} \; \dfrac{\text{meters}}{\text{second}}$

That is, the slope of the line specifies the time rate of change in distance, and thus, is a *rate of change*.

Now, the *slope* itself can be positive, 0, or negative, and thus, can be taken as a *velocity*. So the *magnitude* of the slope can be called a *speed*.

In fact, a rate of change, a velocity, and a slope can be synonyms in math.

What then about acceleration?

It's a time rate of change in velocity. So it is a rate of change in rate of change.
So it can indicate how fast or slow a rate changes, that is, how fast or slow a velocity changes.
If the rate of change increases, the velocity increases, so the slope increases, too.
And if the slope decreases, the velocity decreases, so the rate of change decreases, too.

So no acceleration is in a line. There is no change in slope in a line, so the velocity doesn't change, that is, we have no change in the rate.
And no change in the rate of change means that acceleration is 0.

So if we collect data a moving body produces, and put the data in a graph, we can get a line as the curve if the body is moving at a constant velocity, and thus, has 0 acceleration.

In sum:

A line is a collection of points in a plane, and the points are as follows.

No matter what two points we may pick from a line, the line segment between the two points has the same slope as the line has. In short, every line segment from one point to another in a line has the same slope as the line has. If a line is not vertical, it has a slope.

The magnitude of the slope shows how steep the line slants.
The bigger the amount of the slope, the steeper the line slants.

The sign of the slope shows the direction the line slants. If the slope is positive, the line slants to the left, and if negative, the line slants to the right.

All the lines parallel to each other have the same slope, and therefore, have the same direction, so they all have the same angle (against the x-axis if they are in the x-y plane).

And we have some forms we can use finding a particular line.

Given a slope and a point, we can use this form: $y - y_1 = a(x - x_1)$ or this: $a = \frac{y - y_1}{x - x_1}$.

Given a slope and an intercept, we can use this form: $y = ax - b$.

Given two points, we can use the form of $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, or the one as follows.

$$(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1).$$

0.3. Equations for Lines 3

To begin with, we can put the definition for lines this way: every segment from one point to another in a line has the same slope as the line has.

And we can put the same this way, too: no matter what two points we may choose in a line, the slope of the line segment between the two is constant, and thus, is the same.

And using the definition above, we can get the connective equation for lines, and can put it in either of the two forms as follows.

$$\boldsymbol{y - y_1 = a(x - x_1)} \qquad \boldsymbol{y = ax + b}\text{, where } \boldsymbol{b = y_1 - ax_1}$$

So all the equations above indicate the same line, and the line has a slope of $\boldsymbol{a}$, and is passing through a point $\boldsymbol{(x_1, y_1)}$. Thus, given a slope and a point, we can get the line.

And if given two points $\boldsymbol{(x_1, y_1)}$ and $\boldsymbol{(x_2, y_2)}$, using one of the first two forms above, we can get $\boldsymbol{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$, which can be put this way, too: $\boldsymbol{\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}}$, or this way: $\boldsymbol{(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)}$.

So putting threads together, we have three forms as follows.

$$\boldsymbol{y - y_1 = a(x - x_1)} \qquad \boldsymbol{y = ax + b} \qquad \boldsymbol{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$$

So using one of the three forms above, together with the given information as the slope and a point, or two points, we can get the equation of a particular line.

And usually, we call the three forms the equations for lines, or call each form an equation of a line, which means each indicates a line in general.

None of the three forms however, cannot indicate lines of a particular kind.
What kind then is it?

The lines of the kind are vertical lines as $\boldsymbol{x = 1}$.
Examining each of the three forms above, we can notice that for any values of the constants, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{x_1}$, $\boldsymbol{y_1}$, $\boldsymbol{x_2}$, and $\boldsymbol{y_2}$, no form can indicate a vertical line.

It is in fact, because a vertical line has no slope. Each of the three forms has a slope.
In the first two forms, $\boldsymbol{a}$ is the slope, and in the third, the slope is $\dfrac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}}$.
Isn't there any form then, that can indicate a vertical line, as well as a line with a slope?

We have one, and it is called the general form, which is thus, called a general equation of a line. And we can put the general form the way as follows.

$\boldsymbol{ax + by + c = 0}$,
where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are constant,
but $\boldsymbol{a}$ and $\boldsymbol{b}$ cannot be 0 at the same time.

So for instance, setting $\boldsymbol{a = 1}$, $\boldsymbol{b = 0}$, and $\boldsymbol{c = -1}$, we get

$$\boldsymbol{ax + by + c = 0 \Rightarrow x + 0 - 1 = 0 \Rightarrow x = 1},$$

which is a line perpendicular to the $\boldsymbol{x}$-axis, and thus, is vertical.

And for another instance, if putting $\boldsymbol{4x + 2y + 1 = 0}$ in the slope-intercept form, we can simply put it this way: $\boldsymbol{4x + 2y + 1 = 0 \Rightarrow y = 2x -}\frac{1}{2}$.

Assuming for instance, $ax + by + c = 0$ indicates a line L, we can put it this way, too:

$L = \{(x, y)|\ ax + by + c = 0$, where $a^2 + b^2 \neq 0\}$ Not quite sure?

We know a line is a set of points. So L is a set of points, too, and we can indicate L the way above using a set notation. So the set expression above is saying that L is the set of every point (x, y) that satisfies the equation $ax + by + c = 0$.

So for instance, if $K = \{(x, y)|\ 2x - y + 1 = 0\}$, K is a line $2x - y + 1 = 0$.

And we can put the line K this way, too, of course: $y = 2x + 1$.

What do we mean by this though: $a^2 + b^2 \neq 0$?

It means that a and b cannot be 0 at the same time.
And using the set notation, we can indicate some points in a line, too.
For instance, we can indicate a set of three points in such a way as follows.

$A = \{(x, y)|\ y = 2x$, and $x = 2, 3$, or $4\} = \{(2, 4), (3, 6), (4, 8)\}$.

Also, we can indicate the line L this way, too: $L(x, y) = ax + by + c = 0$.

We can read $L(x, y)$ as 'L of x and y', or 'L of xy' for short.
So $L(x, y)$ is the name of the equation of the line L, and the equation is $ax + by + c = 0$.
And in this case, the letter L is called the equation designator, which means, of course, the name of the equation.
So note that $L(x, y)$ is not a function but an equation, made of two variables.

And the conics is in fact, about equations of degree 2 with two variables.
So putting an equation of a line in full general manner, we can put it the way as follows.

$$ax^2 + by^2 + cxy + ux + vy + w = 0$$

For ***some*** and not all values of the constants, the equation above can indicate a line or two lines at the same time.

For instance, $x^2 + 3y^2 + 4xy + 3x + 7y + 2 = 0$ is an equation of two lines.

One is $x + 3y + 1 = 0$, and the other is $x + y + 2 = 0$.

So we have quite a few ways to put an equation of a line.

Taking one more example, we can put a line in this form, too: $\frac{x}{a} + \frac{y}{b} = 1$.

Then, of course, a and b are nonzero constants, since they are denominators.

In the form above, a is in fact, the x-intercept, and b is the y-intercept.   How?

Setting $x = 0$ in the equation, we get the y-intercept, and setting $y = 0$, we get the x-intercept. So setting $x = 0$ in $\frac{x}{a} + \frac{y}{b} = 1$, we get $y = b$, and setting $y = 0$, we get $x = a$.

So the form above can be called the ***intercept*** form.

Note however, that the form above cannot indicate a line passing through the origin.

Modifying the form though, we can get $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow bx + ay = ab$.

Then, in the form $bx + ay = ab$, setting $a = 0$, we get $x = 0$, which is the y-axis, and setting $b = 0$, we get $y = 0$, which is the x-axis.

So if knowing the x-intercept is a, and the y-intercept is b, using this form: $bx + ay = ab$, we can quickly get the equation of the line.

Usually though, if a line passes through the origin, we put the line in this form: $y = ax$, where a is the slope, of course. We can in fact, get the form simply putting (0, 0) into the form as follows. $y - y_1 = a(x - x_1)$. That is, setting $(x_1, y_1) = (0, 0)$, we get $y = ax$.

And in fact, we can get this: $y - y_1 = a(x - x_1)$ applying to the form $y = ax$, a graph operation called a parallel transformation.

In such a parallel transformation, we translate the line $y = ax$ in the amount of x_1 in the direction of the x-axis, and in the amount of y_1 in the direction of the y-axis.

Then, we get $y - y_1 = a(x - x_1)$, of course.

For the details on transformations, refer to the book **Function Transformations**.

So now, putting threads together, we have five forms, and the forms are as follows.

The point-slope form, $y - y_1 = a(x - x_1)$, which needs the slope and a point.

The slope-intercept form, $y = ax + b$, which needs the slope and x or y intercept.

The two-point form, $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which needs two points.

The general form, $ax + by + c = 0$, which is used in general, and can be put in one of the forms above.

And the intercept form, $\frac{x}{a} + \frac{y}{b} = 1$, where a is the x-intercept, and b is the y-intercept.
Or another intercept form, $bx + ay = ab$, where a is the x-intercept, of course, and b is the y-intercept.

So depending on the problem with a line or the situation where the equation is used, we can choose one of the forms above, and can get the equation of the line we want.

For instance, if the slope is 3 and a point (1, -2) is given,
we can use $y - y_1 = a(x - x_1)$,
and then, we get $y - (-2) = 3(x - 1) \Rightarrow y + 2 = 3(x - 1)$,
which can be put this way, too: $y = 3x - 5$.

Next, if the slope is –2, and the x-intercept is 3,
we can use $y = ax + b$.
Then, since the x-intercept is 3, we get $y = 0 \Rightarrow x = 3$.
And $a = -2$. So we get $y = ax + b \Rightarrow 0 = -2 \cdot 3 + b \Rightarrow b = 6$.
Thus, the line is $y = -2x + 6$.

Next, if the slope is 3, and the y-intercept is 2, we can use this again $y = ax + b$.
And we know that we get the y-intercept when $x = 0$.
So in the form above, b is the y-intercept, since we get $y = b$ if $x = 0$ in $y = ax + b$.

Thus, since the slope is 3, we get $a = 3$, and since the y-intercept is 2, we get $b = 2$.
So the line is $y = 3x + 2$. And of course, we can get the line the way as follows, too.

Since the y-intercept is 2, we get $x = 0 \Rightarrow y = 2$.
So we get $y = ax + b \Rightarrow 2 = a \cdot 0 + b \Rightarrow b = 2$. And $a = 3$. So the line is $y = 3x + 2$.

Next, if we are given (1, -2) and (-3, 4), we can use this form: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

Then, setting $(x_1, y_1) = (1, -2)$, and $(x_2, y_2) = (-3, 4)$, we get

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \Rightarrow y - (-2) = \frac{4 - (-2)}{-3 - (-1)}(x - 1) \Rightarrow y + 2 = \frac{4 + 2}{-3 + 1}(x - 1)$$

$\Rightarrow y + 2 = \frac{6}{-2}(x - 1) = -3(x - 1) \Rightarrow y + 2 = -3(x - 1)$, which is the line.

For some reason though, some people or teachers prefer this form: $y = ax + b$.
So if that is the case, you can keep them happy the way as follows.

$y + 2 = -3(x - 1) \Rightarrow y + 2 = -3x + 3 \Rightarrow y = -3x + 1$, which is the line.

And next, if 2 is the x-intercept, and -3 is the y-intercept, we can use either of the two as follows. $\frac{x}{a} + \frac{y}{b} = 1$, or $bx + ay = ab$, where a is the x-intercept, and b is the y-intercept.

Then, using the first, we get $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{x}{2} + \frac{y}{-3} = 1 \Rightarrow -3x + 2y = -6$, which is the line,

and can be put this way: $2y - 3x + 6 = 0$, can be put this way, too, of course: $y = \frac{3}{2}x - 3$.

And using the second form, we get $-3x + 2y = -6$, which is the one above.

0.4. Equations for Lines 4

Normally, we put one line in one equation, and yet several lines can be put in one equation, too. In other words, an equation can indicate several lines at once.

In fact, we can specify by one equation as many lines as we want. Putting a line in an equation, we often use either of two forms, one is the point-slope form, and the other is the general form. Putting however, in an equation more than one line, we use the general form, which is $ax + by + c = 0$ where a, b, and c are constant.

If an equation indicates more than one line, the equation is composed of as many equations as the number of the lines, and each of the equations is an equation of a line. So let's first, put two lines in one equation.

Suppose U is a line $ax + by + c = 0$, and V is another line $px + qy + r = 0$.

Then, we can put the two lines U and V in one equation the way as follows.

$(ax + by + c)(px + qy + r) = 0$.   How though, does it indicate two lines?

Assuming for instance, $AB = 0$, we get $A = 0$ or $B = 0$. Likewise, we can get this:

$(ax + by + c)(px + qy + r) = 0 \Rightarrow ax + by + c = 0$ or $px + qy + r = 0$.

So the one equation $(ax + by + c)(px + qy + r) = 0$ can indicate two lines, one of which is $ax + by + c = 0$, which is the line U, and the other is $px + qy + r = 0$, which is the line V.

And yet, there is another way.
We can put the two lines U and V in this equation: $U^2 + V^2 = 0$.
Then, we get $U = 0$ and $V = 0$.
So the one equation $(ax + by + c)^2 + (px + qy + r)^2 = 0$ can indicate two lines, one of which is $ax + by + c = 0$, which is the line U, and the other is $px + qy + r = 0$, which is the line V.

Assuming next, $ABC = 0$, we get $A = 0$, $B = 0$, or $C = 0$.
So assuming W is another line $ux + vy + w = 0$, we can get this:

$(ax + by + c)(px + qy + r)(ux + vy + w) = 0 \Rightarrow ax + by + c = 0$, $px + qy + r = 0$, or $ux + vy + w = 0$.
So the one equation $(ax + by + c)(px + qy + r)(ux + vy + w) = 0$ can indicate three lines.

And of course, we can use the other way.
Assuming now, $A^2 + B^2 + C^2 = 0$, we get $A = 0$, $B = 0$, and $C = 0$.
So assuming W is another line $ux + vy + w = 0$, we can get this:

$(ax + by + c)^2 + (px + qy + r)^2 + (ux + vy + w)^2 = 0 \Rightarrow ax + by + c = 0$, $px + qy + r = 0$, and $ux + vy + w = 0$.
So the one equation $(ax + by + c)^2 + (px + qy + r)^2 + (ux + vy + w)^2 = 0$ can indicate three lines.

And by the same token, we can put in an equation more than three lines, too.
So let's this time, put n lines in one equation.

Suppose first, $L_1L_2L_3 \ldots L_n = 0$, where $L_k = a_kx + b_ky + c_k$, where a_k, b_k, and c_k are constant, where $k = 1, 2, \ldots n$.

That is, we have $L_1 = a_1x + b_1y + c_1$, $L_2 = a_2x + b_2y + c_2$, . . ., and $L_n = a_nx + b_ny + c_n$.

Then, we get

$L_1L_2L_3 \ldots L_n = 0 \Rightarrow (a_1x + b_1y + c_1)(a_2x + b_2y + c_2) \ldots (a_nx + b_ny + c_n) = 0$.

So we get $L_1 = 0$, $L_2 = 0$, . . . or $L_n = 0$.
That is, we get $L_k = 0$, where $k = 1, 2, \ldots n$.

In other words, we get $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0 \ldots$ or $a_nx + b_ny + c_n = 0$.

Therefore, $L_1L_2L_3 \ldots L_n = 0$ is an equation that indicates n lines.

And of course, we can use the other way, too.

We can get $L_1^2 + L_2^2 + L_3^2 + \ldots + L_n^2 = 0$

$\Rightarrow (a_1x + b_1y + c_1)^2 + (a_2x + b_2y + c_2)^2 + \ldots + (a_nx + b_ny + c_n)^2 = 0$.

So we get $L_1 = 0$, $L_2 = 0$, . . . and $L_n = 0$.

That is, we get $L_k = 0$, where $k = 1, 2, \ldots n$.

In other words, we get $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0 \ldots$ and $a_nx + b_ny + c_n = 0$.

Therefore, $L_1L_2L_3 \ldots L_n = 0$ is an equation that indicates n lines.

Then, can we put two or more lines in one set of points, too?

Yes, we can. We can put in one set of points, as many lines as we want.
For instance, putting a line L in a set of points, we can put the line L in a set of points the way as follows.
$L = \{(x, y)|\ ax + by + c = 0$ where a, b, and c are constant, but $a^2 + b^2 \neq 0.\}$.

What do we mean by $a^2 + b^2 \neq 0$, though?

It means that a and b can be any real number, but cannot be 0 at the same time. So either of a and b is not 0. If $a^2 + b^2 = 0$, a and b both have to be 0 at the same time.

And next, assuming for instance, that A is a set of points indicating the two lines U and V stated earlier, we can put the set A the way as follows.

$A = \{(x, y)|\ (ax + by + c)(px + qy + r) = 0$ where a, b, c, p, q, and r are constant.$\}$

And of course, $a^2 + b^2 \neq 0$, and $p^2 + q^2 \neq 0$, either.
And by the same token, we can put in a set of points more than two lines, too.
Suppose B is a set of points that indicates all those n lines stated above. Then, we get

$B = \{(x, y)|\ (a_1x + b_1y + c_1)(a_2x + b_2y + c_2)\ldots(a_nx + b_ny + c_n) = 0$ where a_k, b_k, and c_k are constant, $k = 1, 2, \ldots n$, and $a_k^2 + b_k^2 \neq 0$.$\}$

Of course, $a_k^2 + b_k^2 \neq 0 \Rightarrow a_k$ and b_k for $k = 1, 2, \ldots n$ can be any real number, but cannot be 0 at the same time. So neither of a_k and b_k is 0.

And lines can make a line, too.

When making, indicating, finding, or defining a line, we put it in an equation.
Then, defining or finding a particular line, what do we need at least?

Two points in the particular line, or one point and the slope of the line
So two points can define a line, and thus, can make a line if you will.

Given two points though, we can define (or make) one line only, because one particular line only can pass through the two points, and no other line can do so.

Let's now, go over briefly how to define a line with two points.

Suppose in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, two points $\boldsymbol{(s, t)}$ and $\boldsymbol{(u, v)}$ are in a line.
Then, we can put the line in an equation the way as follows.

$$\boldsymbol{y - t = a(x - s)}, \text{ or } \boldsymbol{y - v = a(x - u)}, \text{ where } \boldsymbol{a = \frac{v-t}{u-s} = \frac{t-v}{s-u}}.$$

Both equations above are equivalent to each other, that is, they only look different.
So for instance, given **(-1, 3)** and **(-2, 5)**, we can get the slope the way as follows.

$$\mathbf{\frac{5-3}{-2+1} = \frac{2}{-1} = -2 = \frac{3-5}{-1+2} = \frac{-2}{1}}.$$

So we get $\boldsymbol{y - 3 = -2(x + 1) \Rightarrow y = -2x + 1}$, and also, $\boldsymbol{y - 5 = -2(x + 2) \Rightarrow y = -2x + 1}$.

Thus, two points can define one line only, which is one particular line.

Next, a point and a slope can define a particular line, too.

Given a point and a slope, we can define or make one line only, since one particular line only can have the slope and pass through the point. So for instance, if a line of slope $\boldsymbol{m}$ has a point $\boldsymbol{(s, t)}$, we can define the line by an equation as follows.

$$\boldsymbol{y - t = m(x - s)}$$

Thus, two points can define one line only, and so do a point and a slope.

So we can make a line by means of two points or a point and a slope if you will.
That's not the only way though, we can make a line.
We can make a line by means of lines, too. That is, lines too, can make a line if you will.
Putting lines together, we can make another line.

We can use two different lines to make another line. Calling the two lines, original lines, we can make a new line by means of the two original lines.

Suppose first, the original lines are parallel to each other. Then, the new line, too, is parallel to the lines original. That is, all the three lines are parallel to each other.

Suppose next, the original lines meet each other at a particular point. Then, the new line, too, meets the originals at the particular point. That is, all the three lines meet each other at the particular point altogether. So they share the particular point altogether.
How then do we make the new line?

If we take the average of two original lines, the average can be the new line.
What do we mean by the average though?

Taking the sum of two originals, and then dividing the sum by 2, we get a line, which is the average.
So for instance, making a new line with $\boldsymbol{y = a_1x + b_1}$, and $\boldsymbol{y = a_2x + b_2}$, we get

$$\boldsymbol{y + y = (a_1x + b_1) + (a_2x + b_2) \Rightarrow 2y = (a_1 + a_2)x + (b_1 + b_2) \Rightarrow y = \frac{a_1 + a_2}{2}x + \frac{b_1 + b_2}{2}}.$$

So we get a new line, which is the average of the two lines original.
However, the new line above is not the only new line we can get.
That is, such a new line doesn't have to be the average. How?

Suppose $\boldsymbol{A}$ is a line $\boldsymbol{ax + by + c = 0}$, and $\boldsymbol{B}$ is another line $\boldsymbol{ux + vy + w = 0}$.
Then, taking the sum of the two, we get $\boldsymbol{A + B}$, which is $\boldsymbol{(a + u)x + (b + v)y + c + w = 0}$, which, too, can be a new line. Also, taking $\boldsymbol{2A + B}$, we get

$$\boldsymbol{2(ax + by + c) + ux + vy + w = (2a + u)x + (2b + v)y + 2c + w = 0},$$

which can be a new line, too. What do we mean by such a new line, though?

If ***A*** is parallel to ***B***, the new line is parallel to ***A***, too, that is, if ***C*** is the new line, ***A***, ***B***, and ***C*** are parallel to each other. If however, ***A*** meets ***B*** at a point ***P***, the new line ***C*** meets ***B*** at the point ***P***, too, that is, all the three lines ***A***, ***B***, and ***C*** share the point ***P***.

Suppose next, ***A*** is a line $y = ax + b$, and ***B*** is another line $y = ax + c$.
Then, ***A*** and ***B*** are parallel to each other, and taking the sum of the two, we get

$(y = ax + b) + (y = ax + c) \Rightarrow 2y = 2ax + b + c \Rightarrow y = ax + \frac{b+c}{2}$, which is a new line,

which is a line of slope a, too. So the new line, also, is parallel to the original two lines.
Suppose this time, ***A*** is a line $y = x + 1$, and ***B*** is another line $y = -2x + 5$.
Then, first, finding the point where the two originals ***A*** and ***B*** meet each other, we get

$$x + 1 = -2x + 5 \Rightarrow 3x = 4 \Rightarrow x = \frac{4}{3} \Rightarrow y = x + 1 = \frac{4}{3} + 1 = \frac{7}{3}$$

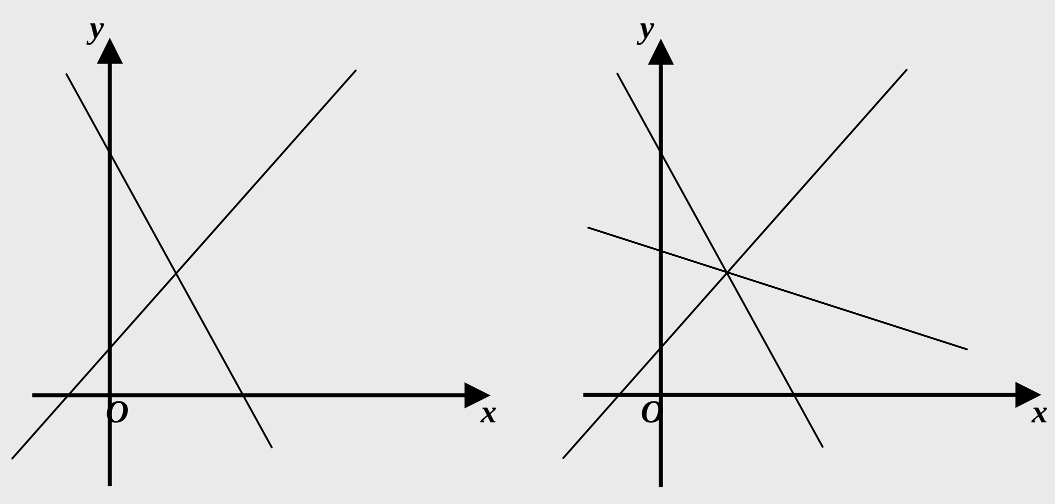
So the original two meet at a point $(\frac{4}{3}, \frac{7}{3})$. Next, taking the average of the two originals,

we get $(y = x + 1) + (y = -2x + 5) \Rightarrow 2y = -x + 6 \Rightarrow y = -\frac{1}{2}x + 3$, which is a new line.

Now, checking to see if the new line passes through the point $(\frac{4}{3}, \frac{7}{3})$, too, we get

$y = -\frac{1}{2}x + 3 \Rightarrow \frac{7}{3} = -\frac{1}{2}\cdot\frac{4}{3} + 3 = -\frac{2}{3} + 3 = \frac{7}{3}$. It does, so the three lines share the point.

Fig. 0



In fact, any multiples of the two lines ***A*** and ***B*** can make a new line, and the new line, too, includes the point $(\frac{4}{3}, \frac{7}{3})$.

For instance, taking the sum of the line ***A*** and twice the line ***B***, we get a new line passing through the point $(\frac{4}{3}, \frac{7}{3})$. Not quite sure?

Making the new one stated above, we get

$(y = x + 1) + (2y = -4x + 10) \Rightarrow 3y = -3x + 11 \Rightarrow y = -x + \frac{11}{3}$

Checking to see if it includes the point $(\frac{4}{3}, \frac{7}{3})$, too,

we get $y = -x + \frac{11}{3} \Rightarrow \frac{7}{3} = -\frac{4}{3} + \frac{11}{3} = \frac{7}{3}$.

Let's multiply the line ***A*** by -10, and the line ***B*** by 20, and take the sum of the product.

$(-10y = -10x - 10) + (20y = -40x + 100) \Rightarrow 10y = -50x + 90 \Rightarrow y = -5x + 9$, a new one.

Checking to see if the new one above includes the point $(\frac{4}{3}, \frac{7}{3})$, too,

we get $y = -5x + 9 = -5 \cdot \frac{4}{3} + 9 = \frac{7}{3}$.

Let's take the sum of the line ***A*** and 20 times the line ***B***, and check it again.

$(y = x + 1) + (20y = -40x + 100) \Rightarrow 21y = -39x + 101 \Rightarrow y = -\frac{39}{21}x + \frac{101}{21}$, another new.

Checking to see if the other new one above includes the point $(\frac{4}{3}, \frac{7}{3})$, too,

we get $y = -\frac{39}{21}x + \frac{101}{21} = -\frac{39}{21} \cdot \frac{4}{3} + \frac{101}{21} = \frac{-13 \cdot 4 + 101}{21} = \frac{49}{21} = \frac{7}{3}$.

Let's next, take the sum of the line ***B*** and twice the line ***A***.

$(\boldsymbol{y = -2x + 5}) + (\boldsymbol{2y = 2x + 2}) \Rightarrow \boldsymbol{3y = 7} \Rightarrow \boldsymbol{y = \frac{7}{3}}$, which is a line parallel to the ***x***-axis and passing through all the points $\boldsymbol{(x, \frac{7}{3})}$, so the line includes the point $\boldsymbol{(\frac{4}{3}, \frac{7}{3})}$, too.

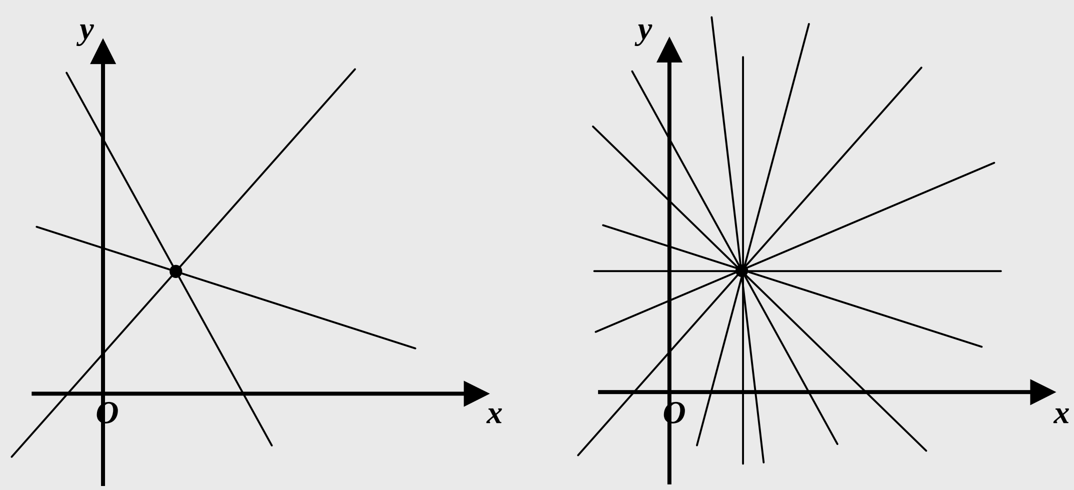
Let's this time, take the sum of the line ***A*** and the line ***B*** multiplied by -1.

$(\boldsymbol{y = x + 1}) + (\boldsymbol{-y = 2x - 5}) \Rightarrow \boldsymbol{0 = 3x - 4} \Rightarrow \boldsymbol{x = \frac{4}{3}}$, which is a line parallel to the ***y***-axis and passing through all the points $\boldsymbol{(\frac{4}{3}, y)}$, so the line includes the point $\boldsymbol{(\frac{4}{3}, \frac{7}{3})}$, too.

Now, it seems that given two lines meeting each other at a point ***P***, we can make as many new lines as we want, each of which passes through the point ***P***, too. In fact, every linear combination of two lines meeting at a point ***P*** passes through the point ***P***, too.

So infinitely many such new lines can be made.
We will cover such a fact in one of the examples followed by this section.

Fig. 1



The same is true for the lines parallel to each other, too. So given two lines parallel to each other, we can make infinitely many new lines parallel to the two parallel lines.

That is to say that with any linear combinations of two lines parallel to each other, we can make a new line, and the new line is parallel to the two, also.

So for instance, taking the average of three lines $y = ax + b$, $y = ax + c$, and $y = ax + \frac{b+c}{2}$, which is the average of the first two, we get

$$(y = ax + b) + (y = ax + c) + (y = ax + \frac{b+c}{2}) \Rightarrow 3y = 3ax + \frac{3(b+c)}{2} \Rightarrow y = ax + \frac{b+c}{2}.$$

Of course, adding the average back to the sum and taking the average again, we get the same average. The same is true, too, for two lines meeting at a point. So for instance

$$(y = ax + b) + (y = cx + d) + (y = \frac{a+c}{2}x + \frac{b+d}{2})$$

$$\Rightarrow 3y = \frac{3(a+c)}{2}x + \frac{3(b+d)}{2} \Rightarrow y = \frac{a+c}{2}x + \frac{b+d}{2}.$$

Let's this time, make a new line with three lines parallel to each other as follows

$$y = ax + b,\ y = ax + c, \text{ and } y = ax + d.$$

Then, we get $\{y = ax + b\} + \{y = ax + c\} + \{y = ax + d\} \Rightarrow 3y = 3ax + (b + c + d)$

$\Rightarrow y = ax + \frac{b+c+d}{3}$, which is a line of slope a, which is therefore, parallel to the original three.

And the same is true, also, for three lines meeting each other at a particular point. Suppose s, t, a, b, and c are constant.
Then, three lines $y - t = a(x - s)$, $y - t = b(x - s)$, and $y - t = c(x - s)$ pass through a point (s, t) in the x-y plane, so the three lines meet each other at (that is, share) the point (s, t).

Now, setting $Y = y - t$ and $X = x - s$, we get $Y = aX$, $Y = bX$, and $Y = cX$.

So we get $(Y = aX) + (Y = bX) + (Y = cX) \Rightarrow 3Y = (a + b + c)X \Rightarrow Y = \frac{a+b+c}{3}X$.

Thus, we get $y - t = \frac{a+b+c}{3}(x - s)$, which, too, is a line passing through the point (s, t).

Now, the same idea applies to four or more lines meeting at a particular point, too.
In fact, with any linear combinations of lines meeting at a particular point, we can make a new line, and the new line, also, passes through the particular point.
And the same is true for four or more lines parallel to each other, too.
What then is the point of all the stories above?

The point is ‘a point and a slope’. So there are two main ideas in a line.

One is that we can have infinitely many lines meeting at (sharing) a particular point, and each of all those many lines can be made (a sum) of two or more lines, which pass through the particular point, too.   This is about a point in a line.

And the other is that we can have infinitely many lines parallel to each other (sharing a particular slope), and each of all those many lines can be made (a sum) of two or more lines, which are parallel to (share the particular slope with) those many lines, too.

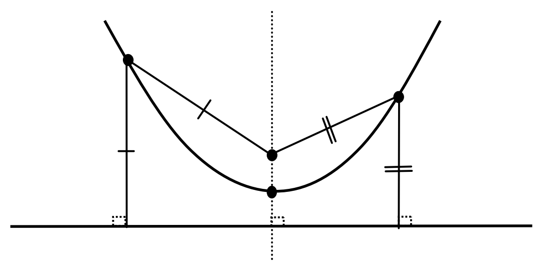
There is another important idea, too.
There is only one line with a particular slope passing through a particular point.
So a point and a slope can define a line.
Thus, a line in math is made of a point and a slope, so to speak.

As other objects in math, a line is a concept, and is not like a string on a guitar or a rope in real world no matter how thin it may be. A line in math has no thickness, and its length is infinite.

Getting the concept of a line can help, and it does very often when we do algebra solving problems with lines when learning not only basic math at high school but higher math at a collage, too, as calculus.

The strong foundation on lines is vital to the early stage of learning calculus.
As a matter of fact, calculus begins with a line.

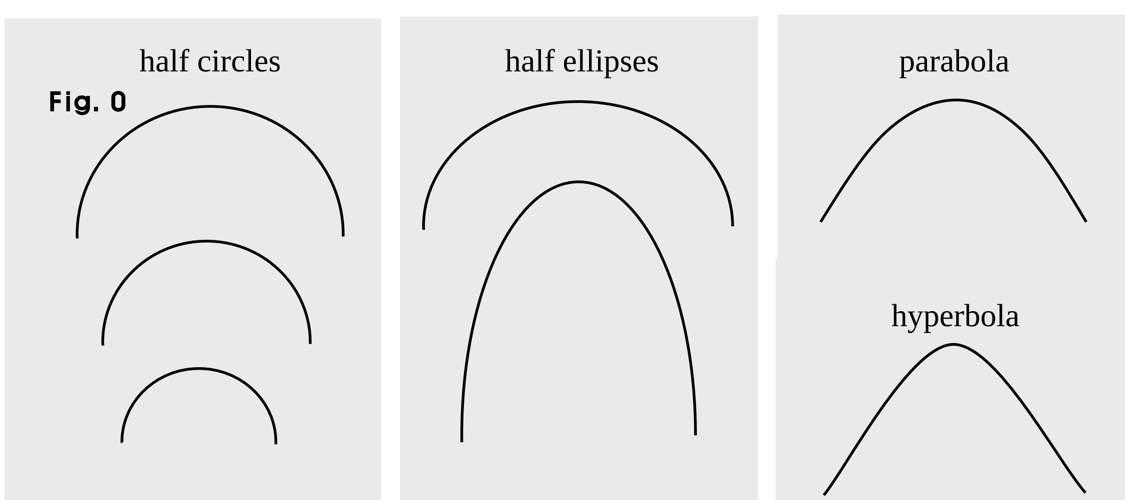
1.0. What is a parabola?



A parabola is in a plane, and is a set of points, from each of which, the distance to the focus is the same as the distance to the directrix.

A parabola is one of the basic curves called conic sections, often called conics for short. So it is a conic, and we can call it a path a baseball makes. So it looks like a half circle, but is not, of course. What curve then is it?

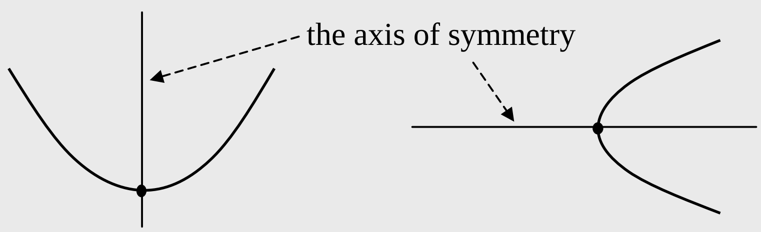
A parabola can also be quite close to a half ellipse or a hyperbola. And it can even look like a line or a wedge. So it's often the case it's hard to tell if we just look at it.



Next, unlike other conics, each and every line tangent to a parabola has a different direction (slope). Like other conics though, a parabola is symmetric, too.

It is however, symmetric about one line only, and the line is passing though its vertex. So it has only one axis of symmetry, and the axis of symmetry passes through the vertex.

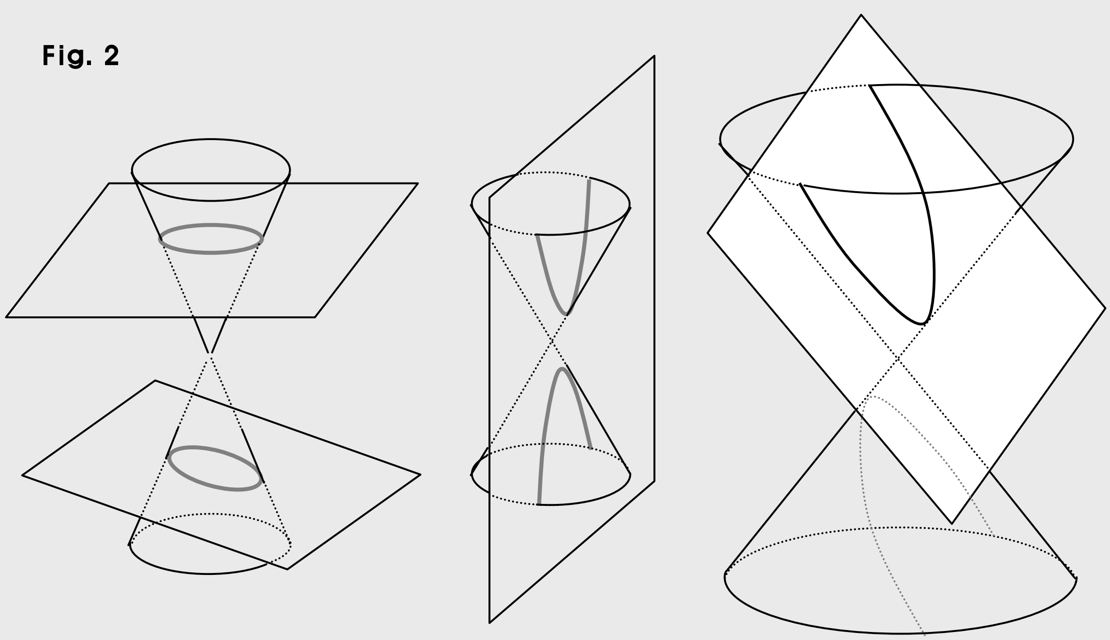
Fig. 1



As shown above, the axis of symmetry passes through the point called the vertex.
So the axis of symmetry has the vertex, which is important.
How then can we get a parabola?

As shown in Fig. 2 below, if cutting a right cone with a plane, we can get a cross section called a parabola. So it's called a conic section, just called a conic for short.

Fig. 2



What is a right cone though?

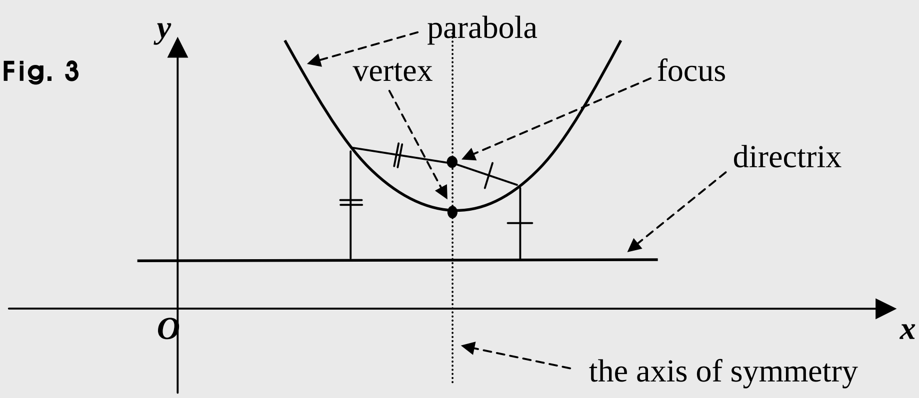
If a cone is a right cone, the line connecting the vertex of the cone and the center of the base is perpendicular to the base, and thus, makes a right angle (90^o) with the base.

And in the figure above, the cross section in black is a parabola. So if the plane is parallel to the side of the cone (called the generator of the cone, too), and does not include the vertex, the cross section is a parabola.

And we can explain a parabola the way as follows.

A parabola is a collection of points as in the case of a line.
Suppose we designate a particular point and a line in the ***x-y*** plane as shown in Fig. 3 below. Then, a parabola is a set of points, from each of which, the distance to the particular point is the same as the distance to the line. And the particular point is called the focus of the parabola, and the line is called the directrix.

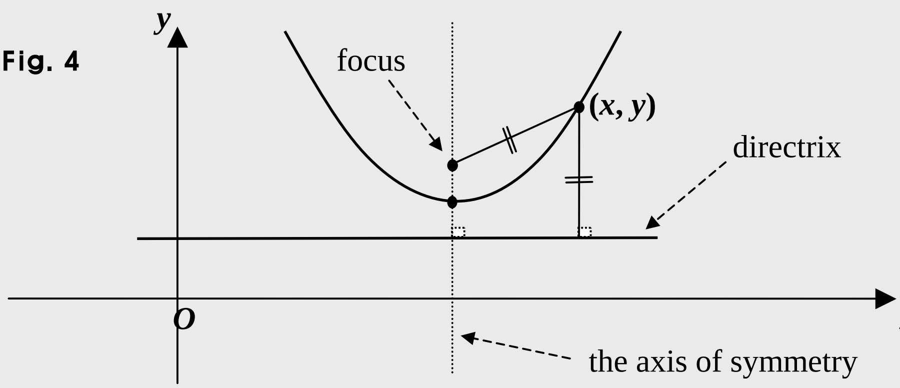
Fig. 3



And we can also explain a parabola the way as follows.

Suppose a point is moving along a curve, and the distance from the moving point to a particular point is the same as the distance to a particular line. Then, the particular point is the focus, the particular line is the directrix, and the curve is a parabola.
So if in the ***x-y*** plane, a point **(*x*, *y*)** is in a parabola, no matter where the point **(*x*, *y*)** may be in the parabola, the distance from the point **(*x*, *y*)** to the point called the focus is the same as the distance to a line called the directrix.

Fig. 4

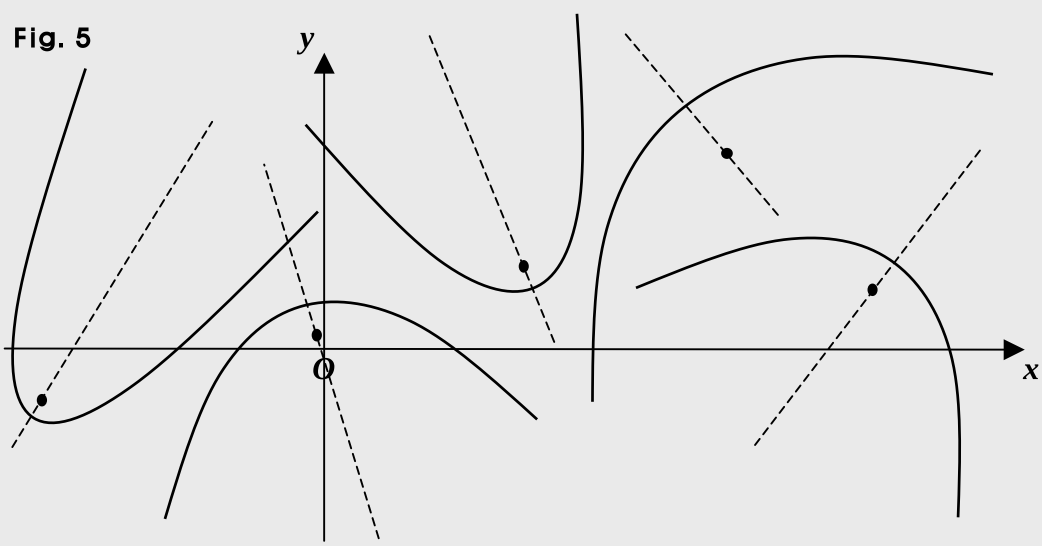


And we can notice that the axis of symmetry is perpendicular to the directrix.

In the figure above, the axis of symmetry is perpendicular to the ***x***-axis. And in that case, of course, the directrix is parallel to the ***x***-axis.

It can be the case, of course, the axis of symmetry is neither perpendicular nor parallel to the ***x***-axis. Such a parabola can be said to be skewed or tilted, and some examples can be as follows.

Fig. 5



Usually though, in high school math or basic courses in college math, if we use a parabola, the axis of symmetry is perpendicular to a coordinate axis. So such a parabola can be called a perpendicular parabola. And in this book, such a parabola is said to be perpendicular, and thus, is called a perpendicular parabola. Note however, in other books, it may not be called a perpendicular parabola, and can be called differently.

Anyway, putting a perpendicular parabola in the ***x-y*** plane, we can get this:

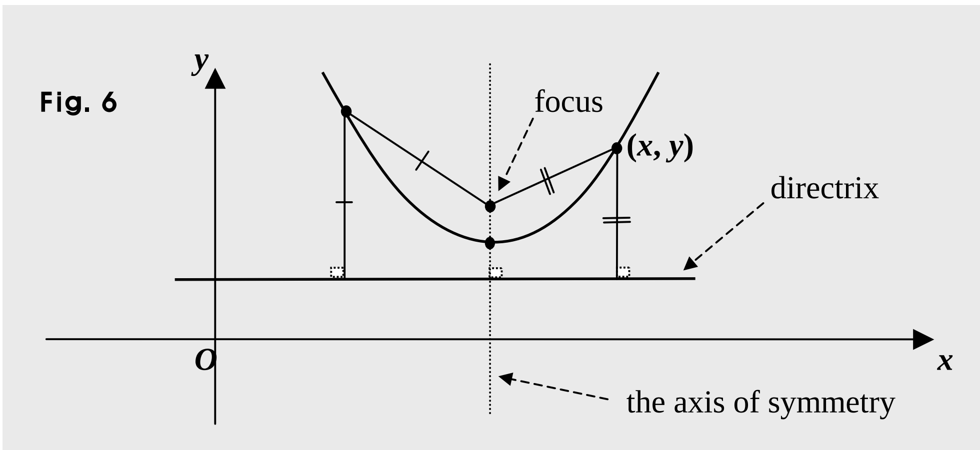


Fig. 6

So no matter what point we may take in a parabola, the distance from the point to the focus is the same as the distance to the line called the directrix.

And the directrix is perpendicular to the axis of symmetry.
And the axis of symmetry passes through the focus and the vertex.
So the focus and the vertex are in a line called the axis of symmetry.

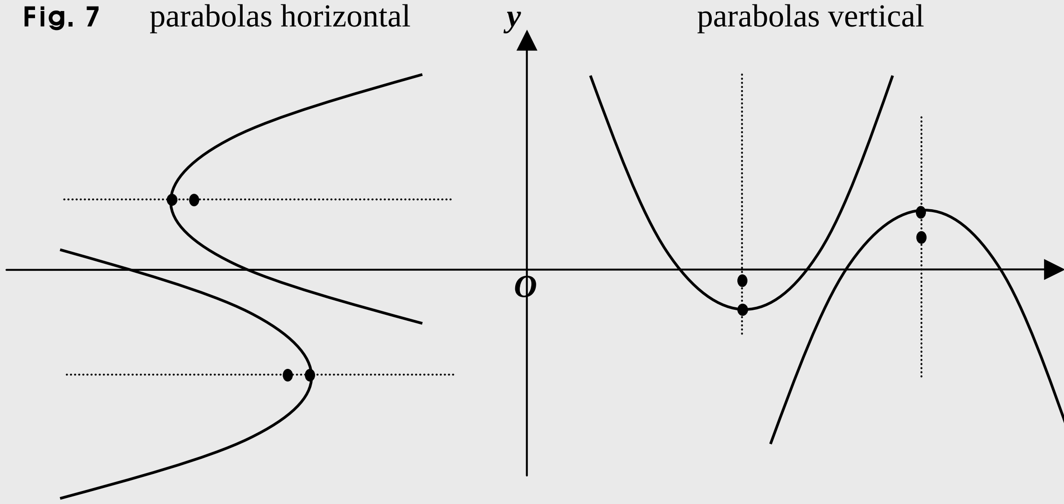
And more importantly, we can see that a curve called a parabola changes its behavior at the point called the vertex. If a parabola is perpendicular, the vertex is the point where a maximum or a minimum occurs.

And using the fact above, we solve many problems with parabolas. That is to say that we often use the idea of parabolas doing problems with maximums or minimums.

And in this book, we have two kinds in parabolas perpendicular.

One is horizontal, and the other is vertical.

Fig. 7



So in a parabola horizontal, the axis of symmetry is horizontal, that is, parallel to the ***x***-axis, and in a parabola vertical, the axis of symmetry is vertical, that is, perpendicular to the ***x***-axis.
Thus, if the axis of symmetry is horizontal, the parabola is horizontal, and if the axis of symmetry is vertical, the parabola is vertical.

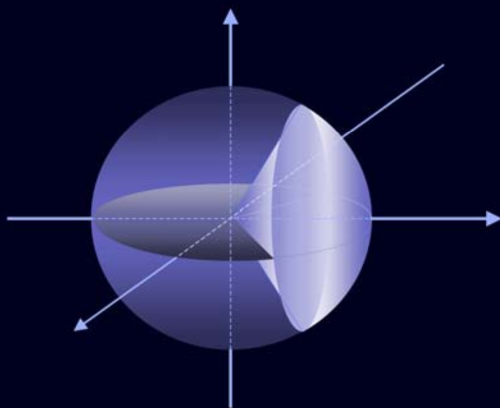
(Note that though, in other books, parabolas may not be classified the way above, and thus, may not be said to be horizontal or vertical.)

Either way, the vertex is in the axis of symmetry, and is the key point in a parabola.

Where can we find a parabola though?

We use a parabola a lot. Parabolas have much to do with simple trajectories. That is, a parabola is often used for a path a projectile makes through space under the action of a constant force as gravity. Such a path is called a trajectory, and is a parabola segment. For instance, if we throw a basketball, the basketball travels in the air along a path that is a parabola segment.

For another instance, we use a parabola when we design a tool for watching satellite TV. Such a tool is called a satellite dish. If a plane cuts the dish, the cross section is a parabola segment. And the same is true, too, for a reflector in a flashlight or such. That's because of the property of a parabola. And the property is covered in the next section.



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