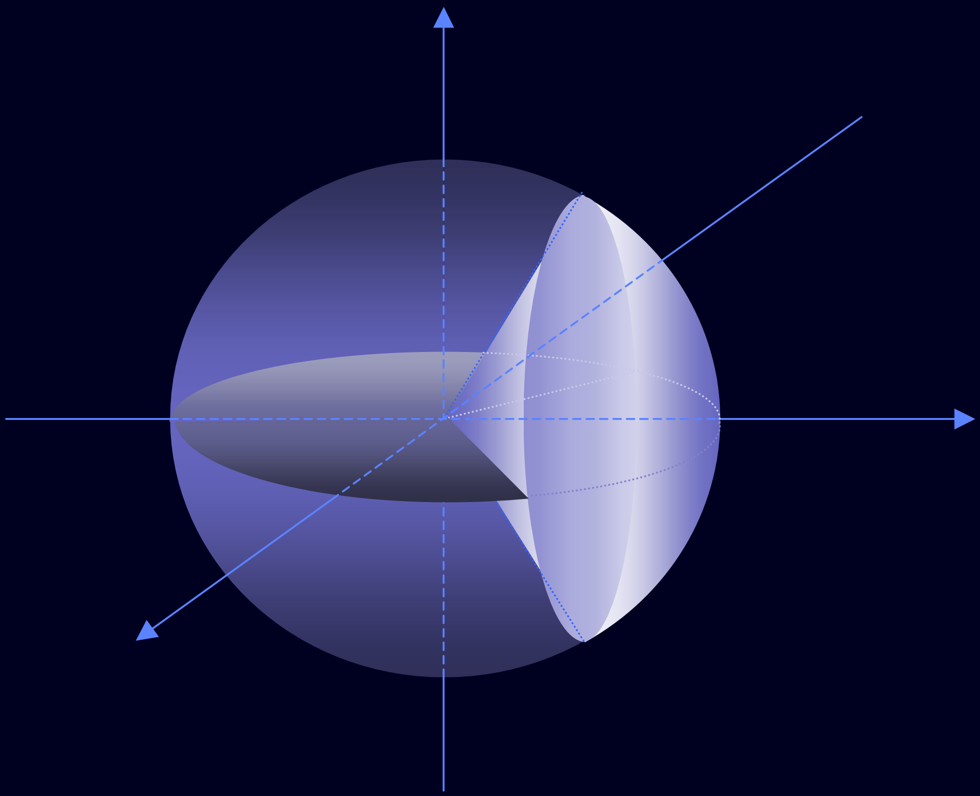
함수의 변환
Function Transformations



김성렬
Seong R. Kim

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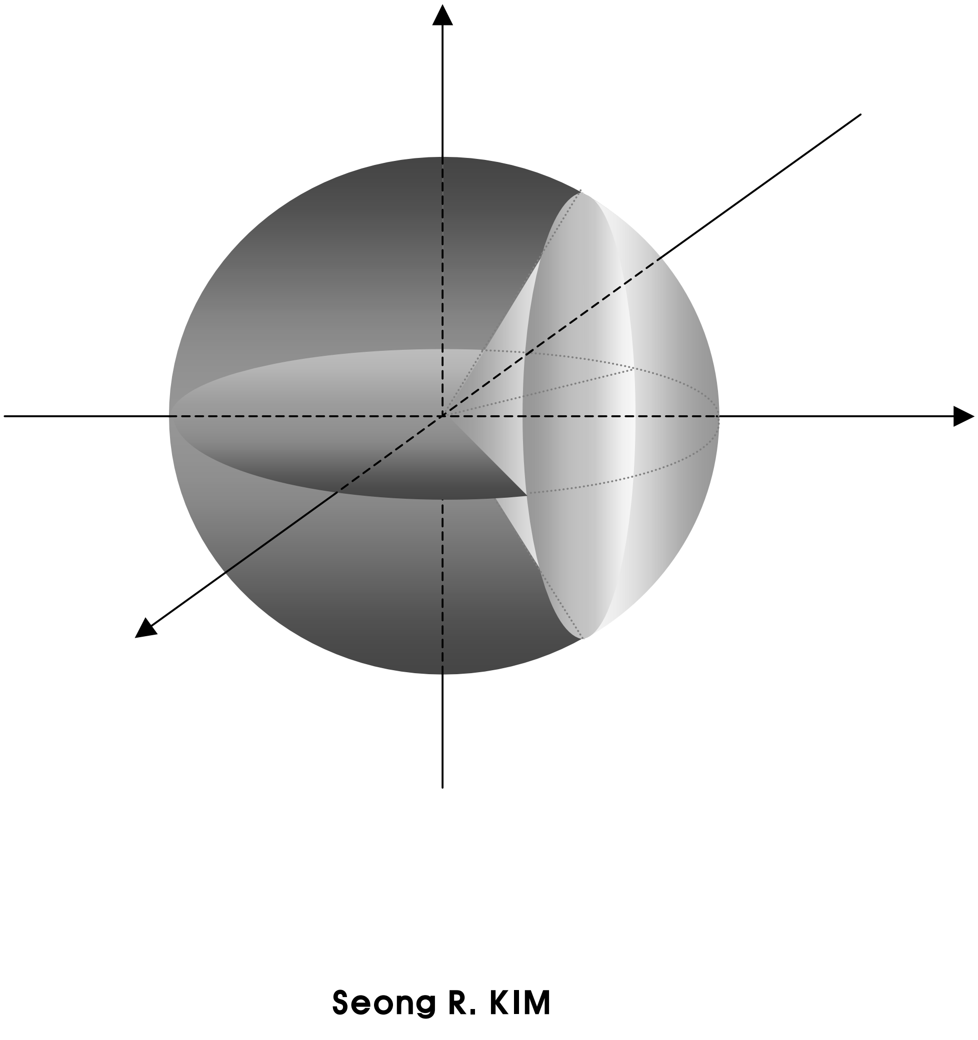


Seong R. KIM

Contents

0. What is a transformation?

A transformation is a function. What function then is it?

It is a function taking a point as an input and producing a point as an output.
What point?

It is an arbitrary point in a curve.
An arbitrary point in a curve is the representative of all the points in the curve, so it represents each and every point in the curve, and is random.
So it randomly represents each and every point in the curve. And every point in the curve works exactly the way the arbitrary point works.

And the point taken as an input is an arbitrary point in a curve to be transformed, so the curve is called the original curve.

And the point produced as an output is an arbitrary point in a curve to be made, so the curve is called the new curve.

So we can call the input the original arbitrary point, and call the output the new arbitrary point.

And using the new arbitrary point and the original curve, we can come up with the new curve.

So if using a transformation, we come up with the new curve using the new arbitrary point and the original curve.

And using a transformation, we can say we apply a transformation.
And applying a transformation, we apply it to a point, a curve, a function, or an equation.
Then, we get a new point, a new curve, a new function, or a new equation.

Why functions and equations though?

That’s because a function or an equation has the curve. So transforming a curve, we transform the function or the equation that has the curve.

So a transformation changes a function or equation to a new function or equation.

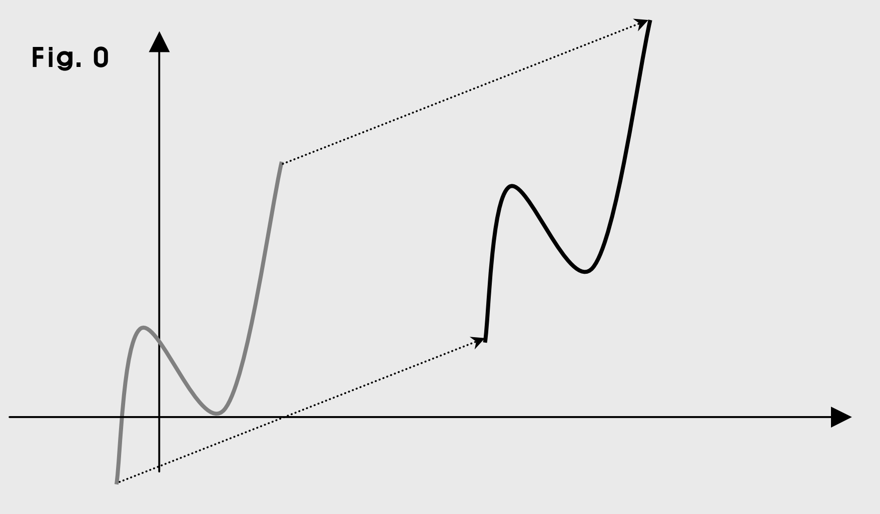
And thus, if we want to make a change to a function, an equation, or a point, we can apply a transformation to it, and get the new one, which is the function, the equation, or the point changed.

So in sum, applying a transformation to a curve, we transform the curve. And transforming it, we change it. And changing it, we change its form or its location, or do both. So we get to have a new curve, that is, we get a new equation or a new function, because a new curve gets a new equation or a new function.

And we are going to begin with a simple transformation called a parallel transformation, and get to the bottom of it, and cover all the details so that you can get the concept of it. It is called a translation, too, and is in the next section, **Parallel Transformations**.

1.1. Parallel Transformations 1

Fig. 0



Applying a transformation to a curve, we transform the curve. And transforming it, we change its form or its location, or do both. So we get a new curve, that is, we get a new equation, because a new curve gets a new equation.

What then is a parallel transformation?

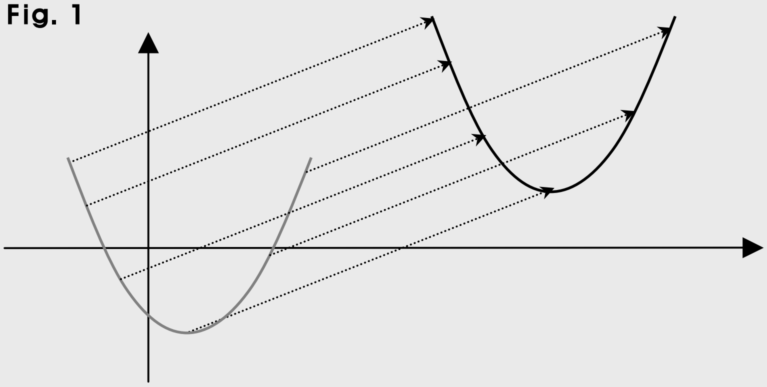
Applying a parallel transformation to a curve, we transform the curve the way as follows.

We move the curve in a particular amount along a particular line. So the curve gets a new location. Thus, we get a new curve, which is the curve that gets the new location.

Suppose now that the ***curve we transform*** is called an ***original curve***.
Then, applying a parallel transformation, we translate an original curve in a particular direction and amount, and then, get a ***new curve***, which is the ***original curve translated***.
Why parallel though?

All the points in the original curve move not only in the same amount but in the same direction, too. So all the paths all the points make are lines parallel to each other. So the transformation is said to be parallel. And we call such a transformation a translation, too.

Fig. 1

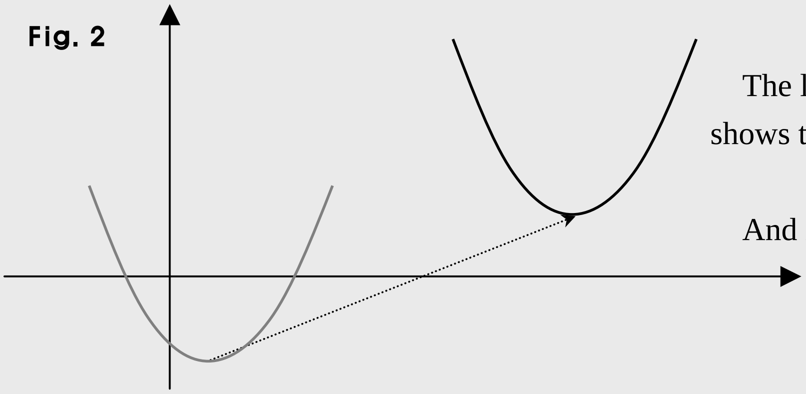


Applying thus, a parallel transformation to a curve, we translate the curve.

And translating a curve, we apply a parallel transformation to the curve.

So we can say that in the case of a parallel transformation, the original curve moves in a particular amount along a particular line without rotation and deformation. So we can simply put the graph above the way below, too.

Fig. 2



The line segment dotted
shows the amount in which
the curve translates.
And its arrowhead shows
the direction.

How then, can a curve make such a move?

Suppose now, that ***P*** is a parallel transformation, and that if we apply ***P*** to the curve of a function $y = f(x)$, we get the curve of another function $y = g(x)$.

Then, we can say that the curve of the function ***f*** is the original curve, and the curve of the function ***g*** is the new curve.

And of course, we can say that the function ***f*** is the original function, and the other function ***g*** is the new function.
So we can say that applying the transformation ***P*** to the function ***f***, we get the function ***g***.

In other words, transforming the function ***f*** by the transformation ***P***, we get the function ***g***. In short, ***g*** is the transformation of ***f*** by ***P***. And more briefly, using a transformation operator, we can put it this way: $g = P \bullet f$, in which therefore, $\bullet$ is the operator.

In sum, applying ***P*** to ***f***, we get $P \bullet f$, which is ***g***. That is, we get $g = P \bullet f$.

How then can we get the new function ***g***?

Transforming in fact, a function or an equation, we don't really change its curve directly. We change the points the curve has rather than the curve itself. We don't change though, each and every point the curve has. We can't change infinitely many points.   So?

We change one point only. If changing the one point, we make all the other points change. And they all change exactly the way the one point changes. So a new curve is made. Then, we get the new function or the new equation that has the new curve.
And the same is true for all the other transformations, too.

What then is the one point?

It is called the arbitrary point in the curve.

An arbitrary point in a curve is the representative of all the points in the entire curve, that is, it represents each and every point in the curve, and can be called a variable-point or a point-variable, too. So transforming a curve in fact, we transform the arbitrary point in the curve.

How then can we get the arbitrary point in the original curve?

It is a point, too. So we can specify it specifying the coordinates of it.
And we can specify the coordinates using the variables used in the function or equation that has (indicates) the original curve. Since the arbitrary point is in the original curve, we can just call it the original arbitrary point without mentioning the original curve.

So in the case of transforming the function f, when specifying the original arbitrary point, we specify its coordinates by the variables used in the original function $y = f(x)$, where x and y are the variables. Usually in fact, we put a function or equation in the x-y system. So we normally use x and y as the variables.

Thus, the original arbitrary point is usually specified by (x, y).
How then can we get the new function g?

Let's first, get back to a point. Changing a point, we get a new point. So changing the arbitrary point in the original curve, we get a new arbitrary point, which is of course, in a new curve, which is the very new curve we get transforming the original curve.

Changing thus, the original arbitrary point, we get the new arbitrary point.
Then, using the new arbitrary point, together with the original curve, we get the new curve. Specifically, using the coordinates of the new arbitrary point, along with the original function or equation, we can get the new function or the new equation.

How then do we get the new arbitrary point?

We set it up first. That is, we assume it to be the new arbitrary point. And setting it up, we specify the coordinates of it. And specifying the coordinates, we use the variables to be used in the new function or equation. Naming the variables though, we use names different from the ones used in the original. For instance, we can use (s, t) as the new arbitrary point, since (x, y) is used as the original. And after getting the new function or equation, we replace s and t with x and y, because the new is in the x-y system, too.

Usually, transforming, we use the definition of the transformation. And the definition has the new arbitrary point. If the definition is not given, we need to come up with the one we need, that is, we need to find it. What then is the definition about?

<u>The definition specifies how the new arbitrary point gets made</u>.

So we are going to see now, how the definition can be made. There are two kinds. One defines a specific parallel transformation, and the other is the definition for parallel transformations, and thus, shows what it is in general.

So let's now begin with getting the definition for parallel transformations.

First, we can apply a transformation to a point, too.
That is, we can move a point by a transformation.
So let's begin with moving a point in the x-y plane by the parallel transformation P.

Suppose now, we want to move a particular point p_1 at a position (x_1, y_1) in the x-y plane to another position (x_2, y_2) by the parallel transformation P.

Then, since a point in math means a position, we can set $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$.

Suppose next, a is the amount of change in the x-coordinate, and b is the amount of change in the y-coordinate. That is, we have $x_2 - x_1 = a$, and $y_2 - y_1 = b$.

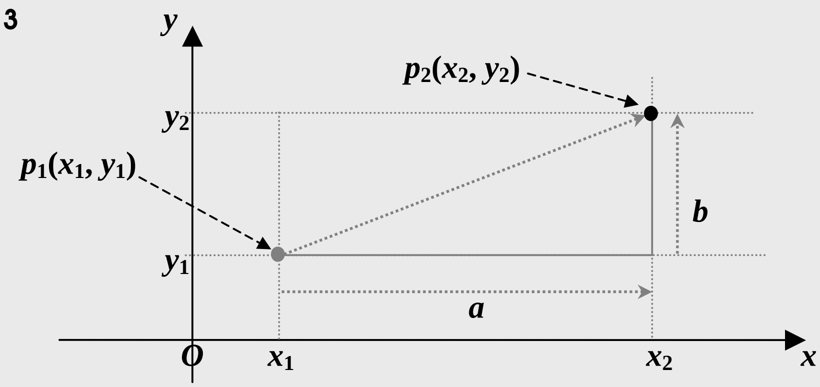
Then, if we apply the parallel transformation P to the point p_1, the point p_1 moves in the amount of $x_2 - x_1 = a$ along the x-axis, and in the amount of $y_2 - y_1 = b$ along the y-axis.

That is to say that we get $x_2 = x_1 + a$, and $y_2 = y_1 + b$.

So we can put all the ideas above this way: $(x_1, y_1) \longrightarrow (x_2, y_2) = (x_1 + a, y_1 + b)$.

And putting in a graph, the expression above, we get

Fig. 3



Then, we can say that the point p_1 has been transformed to the point p_2 by P.

Suppose now, F is the original curve, that is, the curve of the original function $y = f(x)$, and G is the new curve, that is, the curve of the new function g.
That is, we are now going to apply the transformation P to the entire curve of f.

Then, we want to make all at once every point in the curve F move exactly the way the point p_1 has been moved by P. What then do we want to work with?

It is the arbitrary point in the curve F. Since the arbitrary point represents every point in F, transforming the arbitrary point by P, we transform all the points in F in the same manner all at once. Then, the entire curve F gets transformed by P, that is, it becomes G.

So now, moving the arbitrary point in the curve F, what do we get?

The arbitrary point in the curve F, that is, the original arbitrary point is (x, y).

So moving (x, y) in the amount of a in the direction of the x-axis, and in the amount of b in the direction of the y-axis, we get $(x + a, y + b)$, which is the new arbitrary point.

Thus, we can put the parallel transformation P the way as follows.

$P: (x, y) \longrightarrow (x + a, y + b)$, where a and b are constant.

And the expression above is called the <u>definition for parallel transformations</u>.

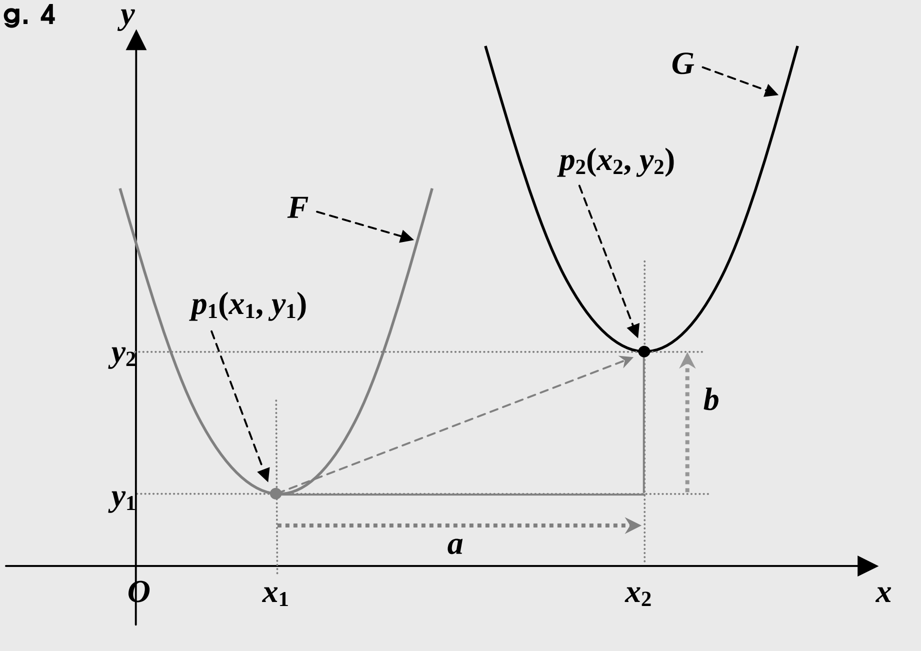
Using the new point, along with the original curve F, we get the transformation of the curve F by P, that is, we get the new curve G.

And using the definition above, we can define (or make) a particular transformation.
If for instance, Q is a parallel transformation where $a = 5$, and $b = 2$, we can put Q this way: $Q: (x, y) \longrightarrow (x + 5, y + 2)$.

Let's now, take a close look at the way parallel transformations work in general. That is, we are now going to see closely how the transformation P works.

First, assuming for instance, the original curve ***F*** stated above is a parabola, the point p_1 is in the curve ***F***, and ***G*** is the new curve we get applying ***P*** to the curve ***F***, we can put in a graph, the effect of ***P*** acted on the curve ***F***, the way as follows.

Fig. 4



Then, we get $(x_2, y_2) = (x_1 + a, y_1 + b)$, and thus, applying the transformation ***P*** to the curve ***F***, we get another curve called ***G*** in a different location but in the same graph.

Note however, the two curves themselves are identical to each other.

The two ***F*** and ***G*** are said to be though, different from each other, since they don't share the same equation. That is, the equation of ***F*** is different from the equation of ***G***.

And we can notice that the new curve ***G*** can be a curve of a function, too, because no vertical line can meet the curve ***G*** at more than one point. Applying in fact, a parallel transformation to a function, we can get a function. That's because the original curve gets translated, that is, moves along a line with no rotation and deformation.

Now, if the function that has the curve ***G*** is $y = g(x)$, the function ***g*** is the new function, which is therefore, the transformation of the original function ***f*** by ***P***. In short, $g = P \bullet f$.

How then can we get the new curve ***G***, that is, the equation of ***G***, and get the function ***g***?

We can get it using the arbitrary point in $\boldsymbol{G}$, together with the original curve $\boldsymbol{F}$. Using it in fact, so to speak, we mix it into the original curve. And getting the mixture, we get the new curve, that is, the new equation. How then do we get the mixture?

Let's first, get back to $\boldsymbol{P}$. Then, we have $\boldsymbol{P}: (\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$.

Then, $(\boldsymbol{x}, \boldsymbol{y})$ is the arbitrary point in the original curve.
So $(\boldsymbol{x}, \boldsymbol{y})$ can be just quickly called the original arbitrary point.
What then do we mean by $(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$? What does it say?

It shows how each and every point in the new curve gets made. So it explains ***how*** the arbitrary point in the new curve is made, that is, ***how*** the new arbitrary point gets made.

Given thus, the transformation we need to apply, we are given ***how*** the new arbitrary point gets made from the original arbitrary point.

So assuming now, $(\boldsymbol{s}, \boldsymbol{t})$ is the new arbitrary point, we can say that the transformation $\boldsymbol{P}$ is saying that the point $(\boldsymbol{x}, \boldsymbol{y})$ becomes or changes to $(\boldsymbol{s}, \boldsymbol{t})$ by means of $(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$.

So given the transformation definition, we are given specifically ***how*** each and every point in the new curve gets made. In short, the definition shows the way the new arbitrary point gets made. And the way is $(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$, which is in a form of a point.

So $\boldsymbol{x} + \boldsymbol{a}$ shows how the $\boldsymbol{x}$-coordinate at the new arbitrary point gets made, and $\boldsymbol{y} + \boldsymbol{b}$ shows how the $\boldsymbol{y}$-coordinate at the new point gets made.

In short, the two expressions show how the new arbitrary point gets made.

Adding $\boldsymbol{a}$ to the $\boldsymbol{x}$-coordinate at a point in the original curve, we get the $\boldsymbol{x}$-coordinate at the corresponding point in the new curve. And we can put it this way: $\boldsymbol{x} \longrightarrow \boldsymbol{x} + \boldsymbol{a}$.

And adding $\boldsymbol{b}$ to the $\boldsymbol{y}$-coordinate at a point in the original curve, we get the $\boldsymbol{y}$-coordinate at the corresponding point in the new curve. So we can put it this way: $\boldsymbol{y} \longrightarrow \boldsymbol{y} + \boldsymbol{b}$.

Thus, $(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$ is no other than a point, and shows how all the points in the new curve get made, and is random. So $(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$ is no other than the arbitrary point in the new curve, and can be set equal to the new arbitrary point set up already above.

Assuming thus, (s, t) is the new arbitrary point, we can define P the way below, too.

$$P: (x, y) \longrightarrow (s, t) = (x + a, y + b).$$

In the definition of P, $(x + a, y + b)$ is the new arbitrary point, and shows how (x, y) changes to (s, t), that is, how the original curve F changes to the new curve G.
How then can we get the equation of the new curve G?

We can get it getting the <u>connective equation</u> between the coordinates of (s, t), the new arbitrary point. How then do we get the connective equation?

We can get it, so to speak, <u>mixing</u> the new arbitrary point (s, t) <u>into</u> the original curve F.
In other words, we can get it, blending the new arbitrary point (s, t) into the original equation $y = f(x)$. And getting the mixture, we use this: $(x + a, y + b)$.
Then, the mixture is the connective equation.

In short, mixing or blending (s, t) into $y = f(x)$ using $(x + a, y + b)$, we get the equation connecting s and t.
So when mixing happens, $(x + a, y + b)$ acts as an agent, a transformation agent. And the mixture is the new curve, that is, the new equation. How then do we get the mixture?

Suppose now that the original function f is $y = f(x) = x^2 - 4x + 5$.

Then, $y = f(x)$ means $y = x^2 - 4x + 5$, which is the <u>equation</u> of the <u>original</u> curve F. So either of $y = f(x)$ and $y = x^2 - 4x + 5$ indicates the equation of F, and can be taken as the <u>original equation</u>. How then do we mix (s, t) into $y = f(x)$, and get s and t connected? That is, how do we use this: $(x + a, y + b)$?

We can use it the way as follows.

First, we have set $(s, t) = (x + a, y + b)$. So we can get $s = x + a$, and $t = y + b$.

That is, we get a simple system of two equations, and the system is for x and y.

Thus next, solving the system for x and y, we get $x = s - a$, and $y = t - b$.

So we can now mix (s, t) into $y = f(x)$. How?

We know $x = s - a$, and $y = t - b$, so mixing (s, t) into $y = f(x)$, we put $(s - a)$ into x in $f(x)$, and put $(t - b)$ into y in $y = f(x)$.

Then, we get $y = f(x) \Rightarrow t - b = f(s - a)$, which is the equation connecting s and t.
So using the agent $(x + a, y + b)$ means that we get the system of equations stated above, and solve the system for x and y, and then, do the substitutions as explained above.

What do we mean by $f(s - a)$ though?

We have $y = f(x) = x^2 - 4x + 5$. So we get $f(s - a) = (s - a)^2 - 4(s - a) + 5$.
So we can express $t - b = f(s - a)$ this way, too: $t - b = (s - a)^2 - 4(s - a) + 5$.
What equation then is it?

We know (s, t) is the arbitrary point in the new curve G. And the equation connects s and t, so it is the equation connecting s and t, and thus, indicates the new curve G.

That is, $t - b = f(s - a)$ is the very equation indicating the new curve G.
So it is the new equation, and can be put this way, too: $t - b = (s - a)^2 - 4(s - a) + 5$.
It's because $f(s - a) = (s - a)^2 - 4(s - a) + 5$.

And as mentioned earlier, s is the x-coordinate at the new arbitrary point, so s is just another name for the variable x in the new equation, and t is the y-coordinate at the new arbitrary point, so t is just also, another name for the variable y in the new equation.

So replacing s with x, and t with y, we put the new equation in the x-y system. So doing the replacement, we get $y - b = f(x - a)$, which can be put this way, too: $y = f(x - a) + b$.

So the equation of the new curve G is $y - b = f(x - a)$. And more specifically, we can put it this way too: $y - b = (x - a)^2 - 4(x - a) + 5$, since $f(x - a) = (x - a)^2 - 4(x - a) + 5$.
And of course, we can put it this way, too: $y = (x - a)^2 - 4(x - a) + 5 + b$.

How then do we get the new function g?

We have $y - b = f(x - a)$. So we get $y = f(x - a) + b$.
And we have $y = g(x)$. So we get $g(x) = f(x - a) + b$.
So the new function g is $y = g(x) = f(x - a) + b$. And we can put it the way below, too.
$y = g(x) = (x - a)^2 - 4(x - a) + 5 + b$, since $f(x - a) = (x - a)^2 - 4(x - a) + 5$.

Thus, summing up, we can put the whole idea above the way below.

Assuming first, (s, t) is the arbitrary point in the new curve, we can set

$P: (x, y) \longrightarrow (s, t) = (x + a, y + b)$.

So we can set $s = x + a$, and $t = y + b$, so next, we can get $x = s - a$, and $y = t - b$.
Thus next, mixing the new arbitrary point (s, t) into the original curve, that is, getting the connective equation between s and t, we get $y = f(x) \Rightarrow t - b = f(s - a)$, which is the equation connecting s and t, that is, the new equation.

And assuming for instance, the original equation is $y = f(x) = x^2 - 4x + 5$, we get $f(s - a) = (s - a)^2 - 4(s - a) + 5$.

Thus, we get $t - b = f(s - a) \Rightarrow t - b = (s - a)^2 - 4(s - a) + 5$.

And we know s is just another name for the variable x in the new equation, and t is just also, another name for the variable y in the new equation. So replacing s with x, and t with y, we get the new equation. Doing thus, the replacements, we get $y - b = f(x - a)$, which is the new equation. And we can put it this way, too: $y = f(x - a) + b$.
And assuming the new function is $y = g(x)$, we get $y = g(x) = f(x - a) + b$, where $f(x - a) = (x - a)^2 - 4(x - a) + 5$.

So we can put the new function g the way below, too.

$y = g(x) = (x - a)^2 - 4(x - a) + 5 + b$.

And assuming for instance, Q is a parallel transformation where $a = 5$, and $b = 2$, we can put Q this way: $Q: (x, y) \longrightarrow (s, t) = (x + 5, y + 2)$.

Then, taking $y = f(x) = x^2 - 4x + 5$ as the original function, and taking $y = g(x)$ as the new function, we get $y = g(x) = (x - 5)^2 - 4(x - 5) + 5 + 2$.

In other words, assuming $g = Q \bullet f$, where $f(x) = x^2 - 4x + 5$, and finding the new function $g(x)$, we get $y = g(x) = (x - 5)^2 - 4(x - 5) + 7$.

And also, we can put the idea the way below, too.

Moving the curve of an equation $y = f(x) = x^2 - 4x + 5$ in the amount of 5 in the direction of the x-axis, and in the amount of 2 in the direction of the y-axis, we get the curve of a new equation, and the new equation is as follows.
$y - 2 = f(x - 5)$, where $f(x - 5) = (x - 5)^2 - 4(x - 5) + 5$.

So we can put it this way: $\mathbf{y - 2 = f(x - 5) \Rightarrow y = f(x - 5) + 2 = (x - 5)^2 - 4(x - 5) + 5 + 2}$.

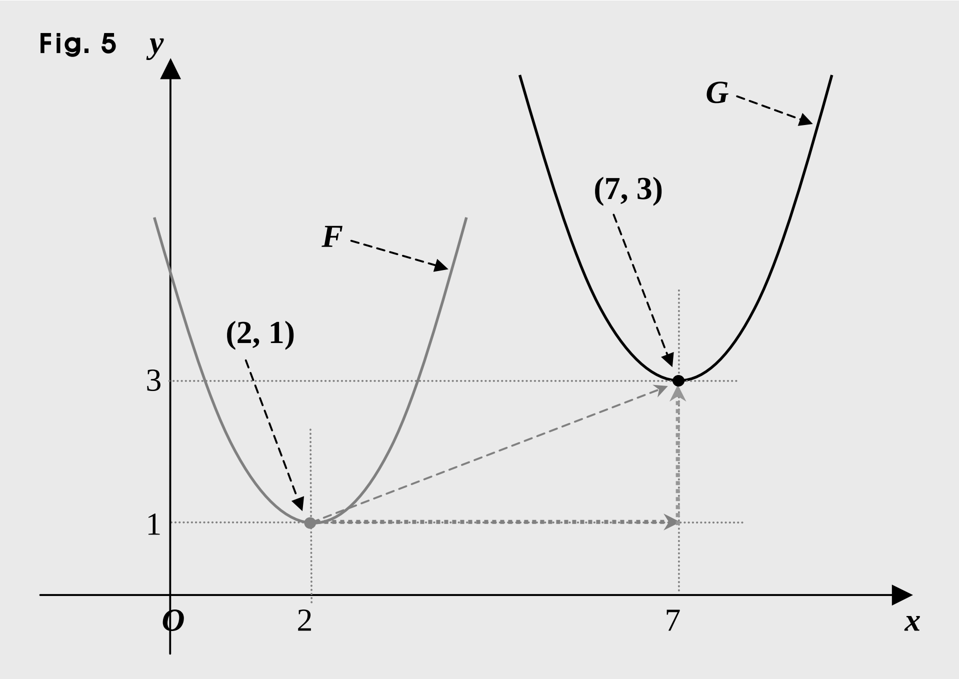
So we can put the new equation this way, too: $\mathbf{y = (x - 5)^2 - 4(x - 5) + 7}$.

And also, we can put the idea the way below, too.

Moving the curve of a function $\mathbf{y = f(x) = x^2 - 4x + 5}$ in the amount of 5 along the ***x***-axis, and in the amount of 2 along the ***y***-axis, we get the curve of a new function ***g***, and the new function ***g*** is $\mathbf{y = g(x) = (x - 5)^2 - 4(x - 5) + 7}$.

And putting the whole idea above in a graph, we get

Fig. 5



In the graph above, we can see that not only the point **(2, 1)** but all the other points in the curve ***F***, too, have moved in the amount of 5 along the ***x***-axis, and in the amount of 2 along the ***y***-axis. In fact, they all have moved diagonally.

And we can put the original equation $\mathbf{y = x^2 - 4x + 5}$ this way:

$\mathbf{y = x^2 - 4x + 5 = (x^2 - 4x) + 5 = (x^2 - 4x + 4 - 4) + 5 = (x^2 - 4x + 4) - 4 + 5}$
$\mathbf{= (x - 2)^2 + 1 \Rightarrow y = (x - 2)^2 + 1}$, which is now in the vertex form. And the vertex is at **(2, 1)**.

And we know in the parallel transformation $\boldsymbol{Q}$, every point in the original curve moves in the amount of 5 along the $\boldsymbol{x}$-axis, and in the amount of 2 along the $\boldsymbol{y}$-axis.

So the vertex $\mathbf{(2, 1)}$ gets transformed to $\mathbf{(2 + 5, 1 + 2)}$, which is $\mathbf{(7, 3)}$.
Thus, moving a parabola, we can easily move it moving its vertex first.
That is, we can just move the parabola exactly the way the vertex moves.

And of course, the transformation can be horizontal only or vertical only, too.

If it's horizontal, we get $\boldsymbol{b} = \mathbf{0}$ in $(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$ as in the case of $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{x} + \mathbf{3}, \boldsymbol{y})$. Then, the curve moves in the amount of 3 along the $\boldsymbol{x}$-axis only, that is, horizontally only.

And if it's vertical, we get $\boldsymbol{a} = \mathbf{0}$ as in the case of $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{x}, \boldsymbol{y} - \mathbf{2})$. Then, the curve moves in the amount of -2 along the $\boldsymbol{y}$-axis only, that is, it only goes downward.

And if necessary, the diagonal distance and direction can be easily found by means of the distance formula (called Pythagorean theorem, too) and a trigonometric ratio called the tangent. So the direction is indicated by an angle.

Assuming the distance is $\boldsymbol{d}$, and using the distance formula, we get $\boldsymbol{d}^2 = (\Delta \boldsymbol{x})^2 + (\Delta \boldsymbol{y})^2$.

And assuming the angle is θ, we get $\tan \theta = \dfrac{\Delta y}{\Delta x}$.

So in the case of the transformation $\boldsymbol{Q}$, we have $\boldsymbol{a} = \Delta \boldsymbol{x} = \mathbf{5}$, and $\boldsymbol{b} = \Delta \boldsymbol{y} = \mathbf{2}$.

Thus, we get $\boldsymbol{d} = \sqrt{29}$, and $\tan \theta = \dfrac{2}{5}$.

And finding the angle θ using a calculator, we get $\theta = \tan^{-1} \dfrac{2}{5} \approx 21.8^{\circ}$.

Then, we can say that by the parallel transformation $\boldsymbol{Q}$, the curve $\boldsymbol{F}$ itself has moved to a new location in the amount of $\boldsymbol{d}$ at the angle of θ against the $\boldsymbol{x}$-axis, and we can say that the curve at the new location is the new curve $\boldsymbol{G}$.

And notice that applying $\boldsymbol{Q}$ to $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$, we get $\boldsymbol{y} - \mathbf{2} = \boldsymbol{f}(\boldsymbol{x} - \mathbf{5})$, which is the new one.

So applying $\boldsymbol{Q}$ to $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ to get the new equation, we can get the new just replacing $\boldsymbol{x}$ with $\boldsymbol{x} - \mathbf{5}$, and $\boldsymbol{y}$ with $\boldsymbol{y} - \mathbf{2}$ in the original $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$. That is, the new is $\boldsymbol{y} - \mathbf{2} = \boldsymbol{f}(\boldsymbol{x} - \mathbf{5})$.

Thus in general, if a parallel transformation is $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{x} + \boldsymbol{p}, \boldsymbol{y} + \boldsymbol{q})$, and the original is $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$, then the new is $\boldsymbol{y} - \boldsymbol{q} = \boldsymbol{f}(\boldsymbol{x} - \boldsymbol{p})$.

And in the next section, we will go over the idea of parallel transformation, and will do it with another example. If sure of the transformations however, skip the next two sections.

1.2. Parallel Transformations 2

In this section, we are going to go over the idea of parallel transformation covered in the previous section, and will do so with another example.

So to begin with, suppose for instance, we want to move a curve F in the amount of -4 in the direction of the x-axis, and in the amount of -2 in the direction of the y-axis, and the curve F is the curve of a function $y = f(x) = x^3 - 14x^2 + 63x - 88$.

Then, we want to make at once all the points in the curve F move in the amount of -4 in the direction of the x-axis, and in the amount of -2 in the direction of the y-axis.

Or briefly, we can put it this way, too: all the points in the curve F move at once by -4 along the x-axis, and by -2 along the y-axis.

Then, the curve F translates, so it moves along a line with no turning and deformation. And making a curve translate, we apply a parallel transformation to the curve. So we want to make a parallel transformation and apply it to the function f or to the curve F. How then, do we make such a transformation?

Beginning with the definition for parallel transformations, we can put it the way below.

Assuming P is a transformation where a curve gets moved in the amount of a in the direction of the x-axis, and in the amount of b in the direction of the y-axis, we call P a parallel transformation, and can put P the way below.

$P: (x, y) \longrightarrow (u, v) = (x + a, y + b)$. And just briefly, $P: (x, y) \longrightarrow (x + a, y + b)$.

Then, we call each set of expressions above the definition for parallel transformations, and can call (x, y) the original (arbitrary) point, and call (u, v) the new (arbitrary) point. So $(x + a, y + b)$ is the new (arbitrary) point, and shows the way (x, y) changes to (u, v), that is, how each and every point in the new curve gets made, that is, how the new curve gets made. If not sure of the original and new points above, refer to the previous section.

And in (u, v), u is the x-coordinate at the new (arbitrary) point, so u is just another name for the variable x in the new equation, and v is the y-coordinate at the new point, so v is just also, another name for the variable y in the new equation. And the new equation is the equation of the new curve, of course.

Next, if defining a particular transformation, we make the particular transformation.

And in the parallel transformation we want to apply to the curve F, a is -4, and b is -2.

So assuming Q is the transformation we want, and defining Q, we can define it in such a way as follows. $Q: (x, y) \longrightarrow (u, v) = (x - 4, y - 2)$.

How then, do we apply the transformation Q to the function f (or to the curve F)?

Applying a transformation to a curve, we can use an operator, which is a dot, •.
And applying the transformation Q to the function f, we put it this way: $Q \bullet f$, which is the new function. So assuming g is the new function, we can set $g = Q \bullet f$.

And applying the transformation Q to the function f, we make at once every point in the curve F move by -4 along the x-axis, and by -2 along the y-axis.

In other words, we want to move at once all the points of F in the amount of 4 to the left, and in the amount of 2 downward. What then do we want to work with?

It is the arbitrary point in the curve F. Since the arbitrary point represents every point in F, transforming the arbitrary point by Q, we transform all the points in F all at once exactly the way we transform the arbitrary point. Thus, if we transform by Q the arbitrary point in the curve F, the entire curve F gets transformed by Q.

And in this book, the arbitrary point above is called the original arbitrary point or just the original point, for short. And the same is true for the arbitrary point in the new curve, too. So we call it the new arbitrary point or just the new point, for short. Note however, in other books, they may not be called that way.

How then do we get the new curve?

Let's now begin with the original curve ***F***.

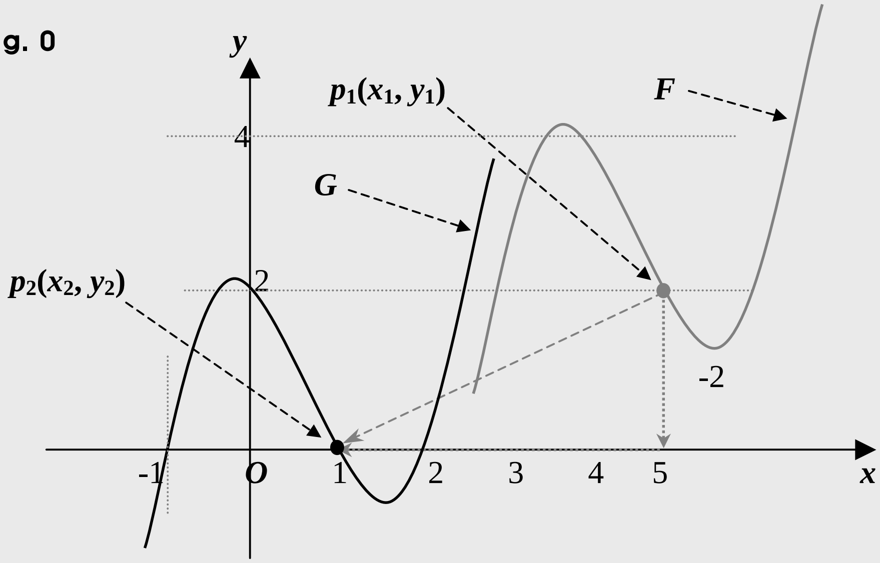
The original curve ***F*** is the curve of the function $\boldsymbol{y = f(x) = x^3 - 14x^2 + 63x - 88}$.
So we can call the function ***f*** the original function.

Assuming thus, the new function is ***g***, and applying the parallel transformation ***Q*** to the original function ***f***, we get the new function ***g***.

That is to say that if we apply the transformation ***Q*** to the function ***f***, the curve of ***f*** moves in the amount of –4 along the ***x***-axis, and in the amount of –2 along the ***y***-axis, and gets a new position, and then, the curve located at the new position is the curve of ***g***.

Assuming thus, ***G*** is the new curve, we can put all the ideas in a graph the way below.

Fig. 0



So we get another curve called ***G*** in a different location but in the same graph.
The two curves themselves are identical to each other.
The two ***F*** and ***G*** are however, said to be different from each other, since they don't share the same equation. That is, the equation of ***F*** is different from the equation of ***G***. Also, we can notice that the curve ***G*** can be a curve of a function, too, since every vertical line meets the curve ***G*** at one point only. Applying to a function in fact, a parallel transformation, we can get a function. That's because the original curve is translated, that is, moves along a line with no rotation and deformation.

So if the function that has the curve G is $y = g(x)$, the function g is the new function, which is therefore, the transformation of the original function f by Q. In short, $g = Q \bullet f$.

How then can we get the equation of the new curve G, and also, the new function g?

We can get it getting the connective equation between the coordinates of (u, v), the new arbitrary point.
And we can get it mixing the new arbitrary point into the original curve, if you will.
And getting the mixture, we get the connective equation.
How then do we get the mixture?

We have set $(u, v) = (x - 4, y - 2)$. So we can get $u = x - 4$, and $v = y - 2$.
Thus, we get $x = u + 4$, and $y = v + 2$. So we can now mix (u, v) into $y = f(x)$. How?

Getting the mixture, since $x = u + 4$, and $y = v + 2$, we put $(s + 4)$ into x in $f(x)$, and put $(v + 2)$ into y in $y = f(x)$.

Then, we get $y = f(x) \Rightarrow v + 2 = f(u + 4)$, which is the equation connecting u and v.

And we have $y = f(x) = x^3 - 14x^2 + 63x - 88$.
So we get $f(u + 4) = (u + 4)^3 - 14(u + 4)^2 + 63(u + 4) - 88$.

So we can put $v + 2 = f(u + 4)$ the way below, too.
$v + 2 = (u + 4)^3 - 14(u + 4)^2 + 63(u + 4) - 88$. What equation then is it?

We know (u, v) is the arbitrary point in the curve G. And the equation connects u and v, so it is the connective equation between u and v, and thus, indicates the new curve.

That is, $v + 2 = f(u + 4)$ is the very equation indicating the new curve G.

And we know $f(u + 4) = (u + 4)^3 - 14(u + 4)^2 + 63(u + 4) - 88$. So we can put the equation above this way, too: $v + 2 = (u + 4)^3 - 14(u + 4)^2 + 63(u + 4) - 88$.

And as mentioned above, u is the x-coordinate at the new arbitrary point, so u is just another name for the variable x in the new equation, and v is the y-coordinate at the new arbitrary point, so v is just also, another name for the variable y in the new equation.

So replacing u with x, and v with y, we put the new equation in the x-y system. So doing the replacement, we get $v + 2 = f(x + 4)$, which can be put this way, too: $y = f(x + 4) - 2$.

So the equation of the new curve G is $y + 2 = f(x + 4)$.

And of course, we can put it this way, too: $y + 2 = (x + 4)^3 - 14(x + 4)^2 + 63(x + 4) - 88$, since $f(x + 4) = (x + 4)^3 - 14(x + 4)^2 + 63(x + 4) - 88$.

How then do we get the new function g?

We have $y = f(x + 4) - 2$. And we have $y = g(x)$. So we get $g(x) = f(x + 4) - 2$.

So the new function g is $y = g(x) = f(x + 4) - 2$.

And we can put it this way, too: $y = g(x) = (x + 4)^3 - 14(x + 4)^2 + 63(x + 4) - 88 - 2$, since $f(x + 4) = (x + 4)^3 - 14(x + 4)^2 + 63(x + 4) - 88$.

Thus, summing up, we can put the whole idea above the way below.

Assuming first, (u, v) is the arbitrary point in the new curve, we can set
$Q: (x, y) \longrightarrow (u, v) = (x - 4, y - 2)$.

So we can set $u = x - 4$, and $v = y - 2$, so next, we can get $x = u + 4$, and $y = v + 2$.

Thus next, mixing the new point (u, v) into the original curve, that is, getting the connective equation between u and v, we get

$y = f(x) \Rightarrow v + 2 = f(u + 4)$, which is the connective equation between u and v.

And we have $y = f(x) = x^3 - 14x^2 + 63x - 88$.

So we get $f(u + 4) = (u + 4)^3 - 14(u + 4)^2 + 63(u + 4) - 88$.

Thus, we get $v + 2 = f(u + 4) \Rightarrow v + 2 = (u + 4)^3 - 14(u + 4)^2 + 63(u + 4) - 88$.

And we know u is just another name for the variable x in the new equation, and v is just also, another name for the variable y in the new equation. So replacing u with x, and v with y, we get the new equation. Doing thus, the replacement, we get $y + 2 = f(x + 4)$, which is the new equation. And we can put it this way, too: $y = f(x + 4) - 2$.

And assuming the new function is $\boldsymbol{y = g(x)}$, we get $\boldsymbol{y = g(x) = f(x + 4) - 2}$, where $\boldsymbol{f(x + 4) = (x + 4)^3 - 14(x + 4)^2 + 63(x + 4) - 88}$.

So we can put the new function $\boldsymbol{g}$ the way below, too.
$\boldsymbol{y = g(x) = (x + 4)^3 - 14(x + 4)^2 + 63(x + 4) - 90}$.

And simplifying the right hand side, we get $\boldsymbol{y = g(x) = x^3 - 2x^2 - x + 2}$.

Also, factorizing it, we get $\boldsymbol{y = g(x) = (x + 1)(x - 1)(x - 2)}$.

And putting the whole idea in a graph, we get

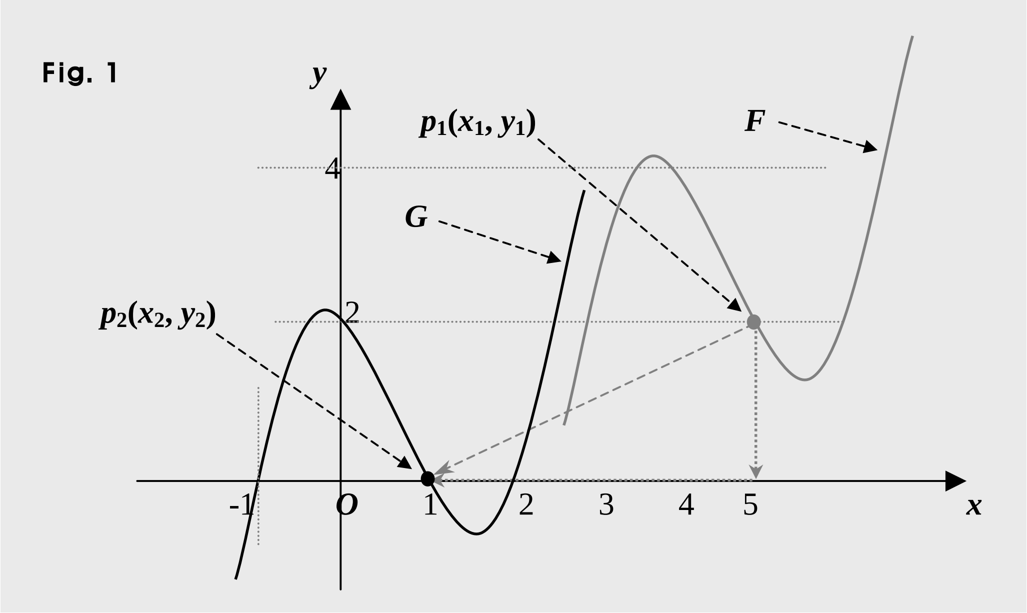


Fig. 1

1.3. Parallel Transformations 3

Translating a curve, we apply a parallel transformation to the curve. So moving a curve along a line with no rotation and deformation, we apply to the curve a parallel transformation.

And we can put the definition for parallel transformations the way below.

$P: (x, y) \longrightarrow (s, t) = (x + a, y + b)$, where a and b are constant.

Usually though, it is put briefly the way as follows. $P: (x, y) \longrightarrow (x + a, y + b)$.

In the definition above, (x, y) is the arbitrary point in the original curve, and is called briefly the original point in this book, and (s, t) is the arbitrary point in the new curve, and is thus, quickly called the new point. What then about $(x + a, y + b)$?

It shows the ***way*** the original point becomes the new point, and is in a form of a point. So it is the new point, and we set it equal to the new point (s, t).

So given a transformation definition, we are given the new point.

And in the definition P above, a indicates the change in the x-coordinate, and b indicates the change in the y-coordinate. How then, do we get the new curve or the new equation?

Suppose first, applying the transformation P to a function f, we get a function g. Then, the original curve is the curve of the function f, and the new curve is the curve of the function g, and we can say that the function g is the transformation of the function f by P.

And using the transformation operator •, we can put it the way as follows. $g = P \bullet f$.

Assuming next, (s, t) is the new point, we can set $(s, t) = (x + a, y + b)$.

So we get $s = x + a$, and $t = y + b$. Thus, we get $x = s - a$, and $y = t - b$.

Next, putting $x = s - a$ and $y = t - b$ into the original equation, we can connect s and t.

And getting the equation connecting s and t, and then, replacing s and t with x and y, we get the new equation.

So for instance, finding the new function $g = P \bullet f$, where $f(x) = x^3 - 2x^2 - x + 2$, we can find the new function g the way below.

Assuming first, (s, t) is the new arbitrary point, we can set $(s, t) = (x + a, y + b)$.

So we get $s = x + a \Rightarrow x = s - a$, and $t = y + b \Rightarrow y = t - b$.

So next, putting $x = s - a$ and $y = t - b$ into the original equation, we can connect s and t. Putting thus, $x = s - a$ and $y = t - b$ into $y = f(x) = x^3 - 2x^2 - x + 2$, we get

$t - b = f(s - a) = (s - a)^3 - 2(s - a)^2 - (s - a) + 2$, which connects s and t.

So next, replacing s and t with x and y to get the new equation, we get

$y - b = f(x - a) = (x - a)^3 - 2(x - a)^2 - (x - a) + 2$, which is the new equation.

If for instance, $a = 4$, and $b = 2$, the new equation is $y - 2 = f(x - 4)$.

More specifically, it is put this way: $y - 2 = f(x - 4) = (x - 4)^3 - 2(x - 4)^2 - (x - 4) + 2$.

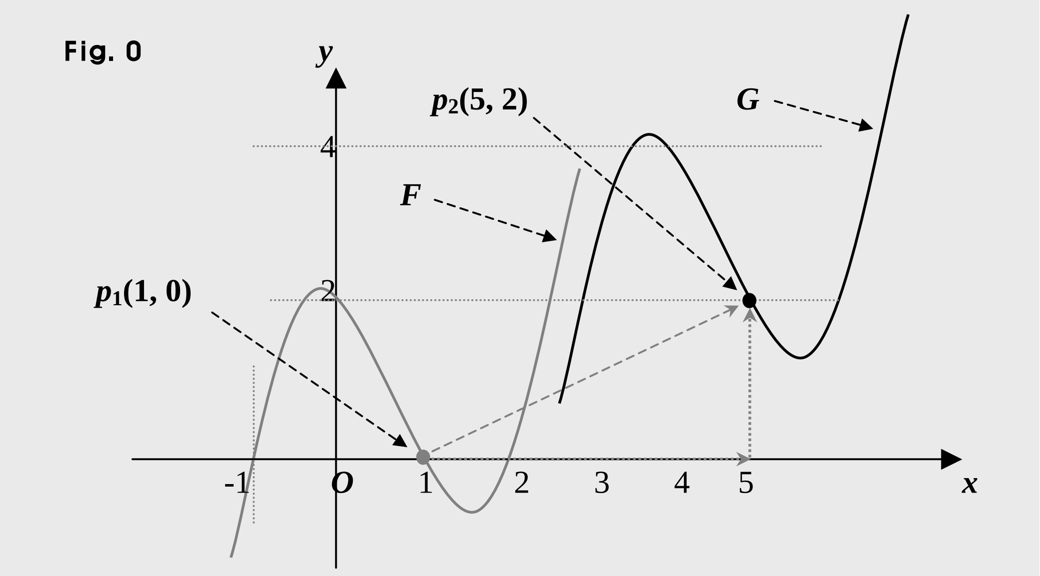
And of course, we can put it this way, too: $y - 2 = (x - 4)^3 - 2(x - 4)^2 - (x - 4) + 2$.

Or this way: $y = (x - 4)^3 - 2(x - 4)^2 - (x - 4) + 4$.

And simplifying the right hand side, we get $y = x^3 - 14x^2 + 63x - 88$.

And assuming the new function is g, we get $y = g(x) = x^3 - 14x^2 + 63x - 88$.

And assuming ***F*** is the original curve, ***G*** is the new curve, we can put them in a graph the way below.



All the points in ***F*** move exactly the way the point p_1 moves to p_2.

And for another instance, finding the new function $g = P\bullet f$, where $f(x) = (x - 1)^2 + 1$ for $0 < x < 2$, we can find the new function g the way below.

Assuming first, (s, t) is the new point, we can get

$s = x + a \Rightarrow x = s - a$, and $t = y + b \Rightarrow y = t - b$.

Next, we want to get s and t connected, so putting $x = s - a$ and $y = t - b$ into the original equation, that is, putting $x = s - a$ and $y = t - b$ into $y = f(x) = (x - 1)^2 + 1$, we get

$t - b = f(s - a) = \{(s - a) - 1\}^2 + 1$, which connects s and t.

So next, replacing s and t with x and y to get the new equation, we get

$y - b = f(x - a) = \{(x - a) - 1\}^2 + 1$, which is the new equation.

And if for instance, $a = 4$, and $b = 2$, the new equation is $y - 2 = f(x - 4)$.

More specifically, it is put this way $y - 2 = f(x - 4) = \{(x - 4) - 1\}^2 + 1$.

And of course, we can put it this way, too: $y - 2 = (x - 5)^2 + 1$.

Or this way: $y = (x - 5)^2 + 3$.

And simplifying the right hand side, we get $y = x^2 - 10x + 28$.

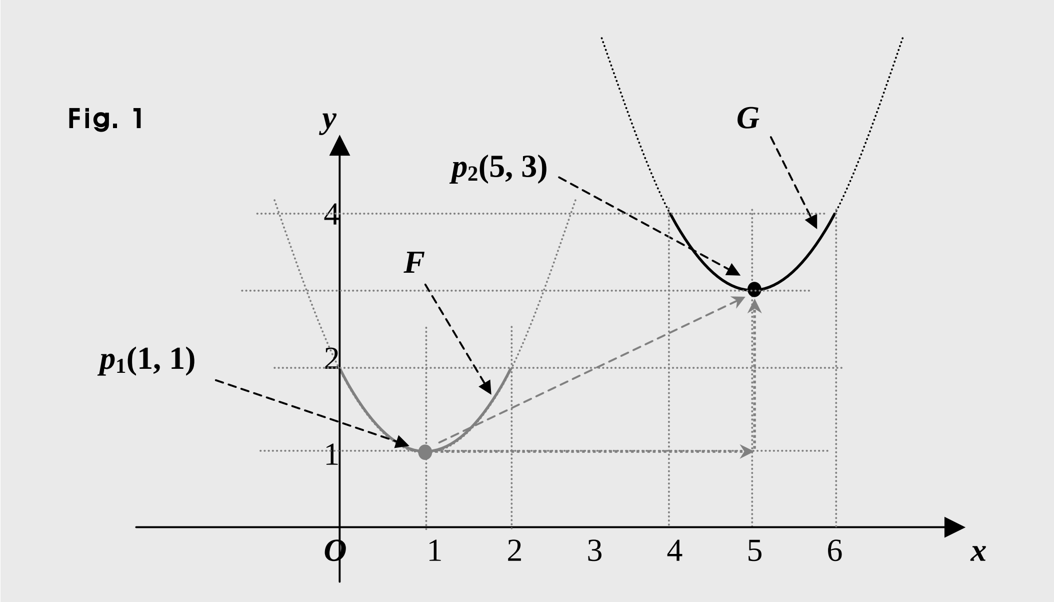
And assuming the new function is g, we get $y = g(x) = x^2 - 10x + 28$.

It is not the case however, the original function is defined for all real numbers.
The domain, that is, the original domain is $0 < x < 2$. What then about the new domain?

If the curve translates, the domain does, too. The domain is the set of all the x-values.
So finding the new domain, we count the translation in the direction of x-axis only.
And the amount is 4 to the right. So the new domain is $0 + 4 < x < 2 + 4 \Rightarrow 4 < x < 6$.

And assuming F is the original curve, G is the new curve, we can put them in a graph the way below.

Fig. 1



And notice that the range gets translated, too. The range is the set of all the y-values.

So since the original is $1 < y < 2$, the new is $1 + 2 < y < 2 + 2 \Rightarrow 3 < y < 4$.

And for another instance, finding a new equation $g = P \bullet f$, where $f(x, y) = x^2 + y^2 - 1 = 0$, we can find the new equation g the way below.

Assuming first, (s, t) is the new arbitrary point, we can get

$s = x + a \Rightarrow x = s - a$, and $t = y + b \Rightarrow y = t - b$.

Next, we want to get s and t connected, so putting $x = s - a$ and $y = t - b$ into the original equation, that is, putting $x = s - a$ and $y = t - b$ into $f(x, y) = x^2 + y^2 - 1 = 0$, we get

$f(s - a, t - b) = (s - a)^2 + (t - b)^2 - 1 = 0$, which connects s and t.

So next, replacing s and t with x and y to get the new equation, we get
$f(x - a, y - b) = (x - a)^2 + (y - b)^2 - 1 = 0$, which is the new equation.

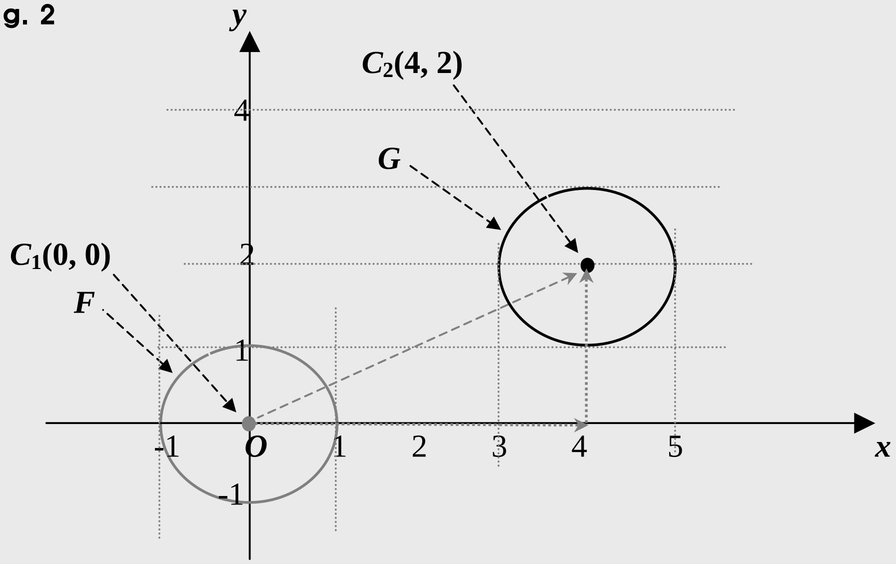
And we have assumed that g is the new equation, that is, $g(x, y) = 0$ is the new equation.
So we get $g(x, y) = f(x - a, y - b) = (x - a)^2 + (y - b)^2 - 1 = 0$.

In short, $g(x, y) = (x - a)^2 + (y - b)^2 - 1 = 0$, which is an equation of a circle of radius 1 centered at (a, b), in the x-y plane, of course.

In fact, the original equation is the equation of a circle of radius 1 centered at the origin.

And if for instance, $a = 4$, and $b = 2$, the new equation is $g(x, y) = f(x - 4, y - 2) = 0$.
More specifically, it is put this way: $g(x, y) = f(x - 4, y - 2) = (x - 4)^2 + (y - 2)^2 - 1 = 0$.
And of course, we can put it this way, too: $(x - 4)^2 + (y - 2)^2 - 1 = 0$.
Or this way: $(x - 4)^2 + (y - 2)^2 = 1$.
And simplifying the equation above, we get $x^2 - 8x + y^2 - 4y + 19 = 0$.

Fig. 2



Let's find this time, a new equation $g = P \bullet f$, where $f(x, y) = (x - 1)^2 + (y - 3)^2 - 1 = 0$, which is the equation of a circle of radius 1 centered at (1, 3).

Take first, (s, t) as the new point, we get $s = x + a \Rightarrow x = s - a$, and $t = y + b \Rightarrow y = t - b$.

Next, put $x = s - a$ and $y = t - b$ into the original equation to get s and t connected.
So putting $x = s - a$ and $y = t - b$ into $f(x, y) = (x - 1)^2 + (y - 3)^2 - 1 = 0$, we get

$$f(s - a, t - b) = \{(s - a) - 1\}^2 + \{(t - b) - 3\}^2 - 1 = (s - a - 1)^2 + (t - b - 3)^2 - 1 = 0.$$

So next, replacing s and t with x and y to get the new equation, we get
$f(x - a, y - b) = (x - a - 1)^2 + (y - b - 3)^2 - 1 = 0$, which is the new equation.

And we have assumed that g is the new equation, that is, $g(x, y) = 0$ is the new equation.
So we get $g(x, y) = f(x - a, y - b) = (x - a - 1)^2 + (y - b - 3)^2 - 1 = 0$, which is the equation of a circle of radius 1 centered at $(a + 1, b + 3)$.

And if for instance, $a = 4$, and $b = -2$, the new equation is $g(x, y) = f(x - 4, y + 2) = 0$.
More specifically, it is put this way: $g(x, y) = f(x - 4, y + 2) = (x - 5)^2 + (y - 1)^2 - 1 = 0$.
And of course, we can put it this way, too: $(x - 5)^2 + (y - 1)^2 - 1 = 0$.
Or this way: $(x - 5)^2 + (y - 1)^2 = 1$, which is a circle of radius 1 centered at (5, 1).
And simplifying the equation above, we get $x^2 - 10x + y^2 - 2y + 25 = 0$.

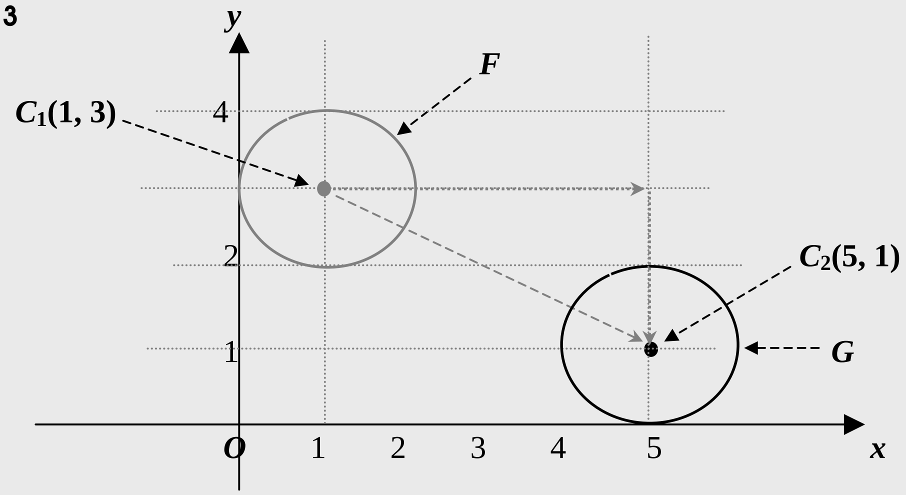
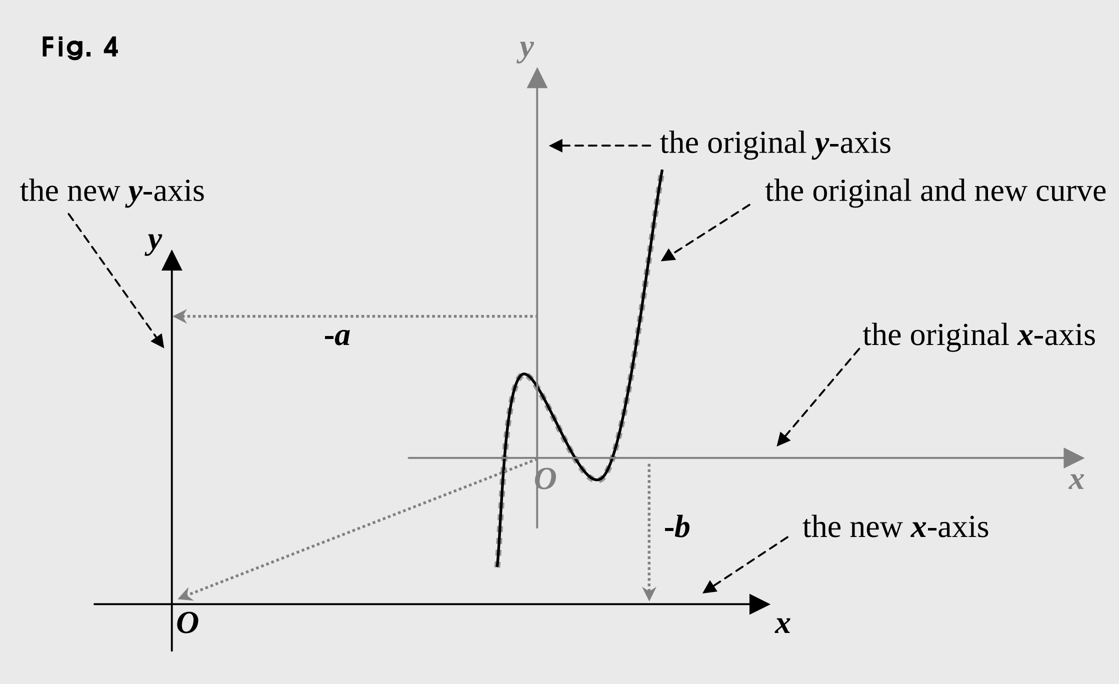


Fig. 3

And we can put a parallel transformation this way, too: applying a parallel transformation to a curve, we copy the curve, and then, paste it with no rotation.

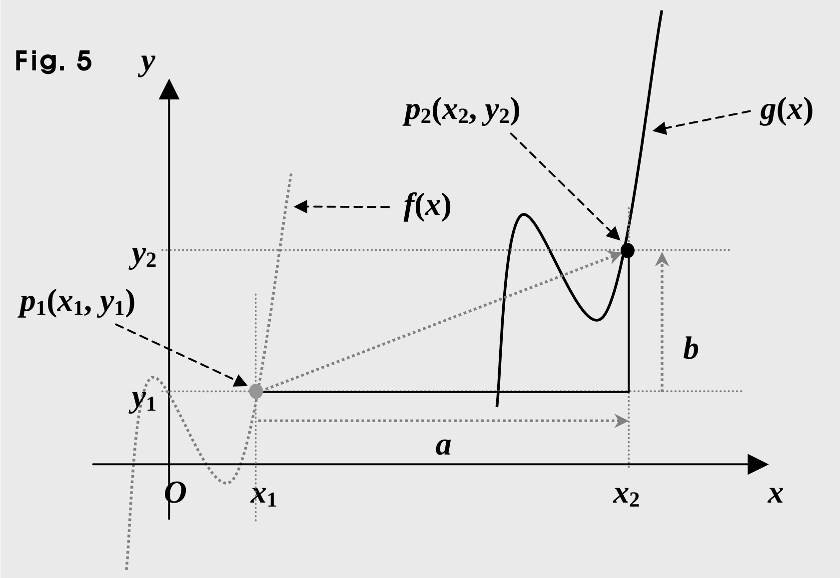
Also, there are two ways we can interpret a parallel transformation.
In one, moving the axes, that is, moving the origin keeping the curve intact, we get

Fig. 4



And in the other, moving the curve, we get

Fig. 5



So transforming a curve by a parallel transformation, we translate the curve in a particular direction and amount, and then, we get a new curve.

That is, applying a parallel transformation to a curve, we shift the curve up or down, sideways, or diagonally.

Shifting the curve up or down, we set $a = 0$, shifting the curve sideways, we set $b = 0$, and shifting the curve diagonally, we set for instance, $a = -1$, and $b = 2$.

- And note that in this book, some important terminologies are used the way as follows.

The original curve: the curve of the function or equation a transformation is applied to
The new curve: the curve of the function or equation made by a transformation

The original function or equation: the function or equation a transformation is applied to
The new function or equation: the function or equation made by a transformation

The original arbitrary point or the original point: the arbitrary point in the original curve.
The new arbitrary point or the new point: the arbitrary point in the new curve.

2.0. Symmetric Transformations

In the case of a symmetric transformation, the new curve is symmetric to the original curve. So the two curves, the original and the new are symmetric to each other.
And thus, the two are the same in shape. That is, both have the same look.

It is often the case however, they are not identical, that is, their equations are different.
What do we mean by a symmetry though, between two curves?

Suppose for instance, the original and the new are parabolas symmetric to each other in the ***x*-*y*** plane. Then, first, the two curves share the same shape.

It is often the case though, the two don't share the same equation.
In other words, the original and the new are usually taken as two different curves.

That is because it is normally the case the new curve is at a different position.

It doesn't mean however, the new curve can be anywhere in the ***x*-*y*** plane.
The new one has to be positioned in a particular manner. What particular manner?

In the case of a symmetric transformation, what matters is <u>how</u> the new and the original curves are symmetric to each other.
More specifically, besides the fact that both share the same shape, what really matters is <u>about what</u> the two are symmetric. So what can the two be symmetric about?

If the two curves are in a plane, the two can be symmetric <u>about a point or a line</u>.
So for instance, the two can be symmetric about a point $\mathbf{(1, 2)}$ or a line $\boldsymbol{y = 2x + 1}$.

And thus, applying a transformation symmetric about a point **(1, 2)**, we get a new curve, and the new curve is symmetric to the original curve about the point **(1, 2)**.

And also, if a transformation is symmetric about a line $y = 2x + 1$, the new curve is symmetric to the original curve about the line $y = 2x + 1$.
So if we fold the plane along the line $y = 2x + 1$, the two curves match exactly.

That is, the two coincide. It is usually the case though, the two are different, because they have different equations. Is there any case then, the two curves are the same? That is, is there any case, both share the same equation?

If we apply to a parabola a transformation symmetric about the axis of symmetry of the parabola, the new curve is the very parabola to which we apply the transformation.
In that case, we say that both are the same or identical, because their equations are the same. So depending on the symmetry we apply, the two can share the same equation.

What do we mean by though, curves symmetric about a point?

Transforming a curve, we don't actually move or change the curve as a whole.
What then do we actually move or change?

It is each and every point of the curve. And in a curve, its arbitrary point represents all the points in the curve. So transforming a curve, we actually apply a transformation to the arbitrary point of the curve.

In the case of a transformation symmetric about a point therefore, each and every point of the original curve gets moved symmetrically about the point.

And the same is true for a transformation symmetric about a line, too.
So in the case of a transformation symmetric about a line, each and every point of the curve gets moved symmetrically about the line.

Thus, applying a symmetric transformation, we actually apply the transformation to the arbitrary point of the original curve.

So for instance, applying a transformation symmetric about a point **(1, 2)**, we get each and every point to be moved symmetrically about a point **(1, 2)**. And then, all the points moved form a new curve, which is said to be symmetric about the point **(1, 2)**.

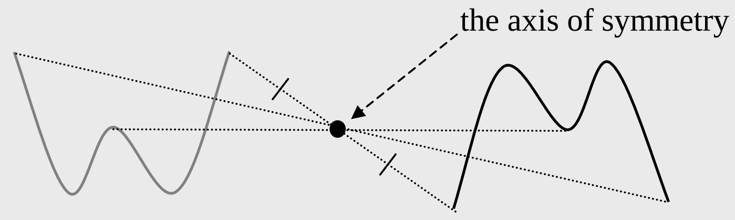
That is, the new curve is said to be symmetric to the original curve about the point **(1, 2)**.

Thus, saying two curves are symmetric to each other, we mean in both curves, each and every pair of corresponding points are symmetric to each other.
What do we mean by then, two points symmetric about a point **(1, 2)**?

If two points are symmetric about a point **(1, 2)**, the point **(1, 2)** is the midpoint between the two points. And the two points and the point **(1, 2)** are in a line.

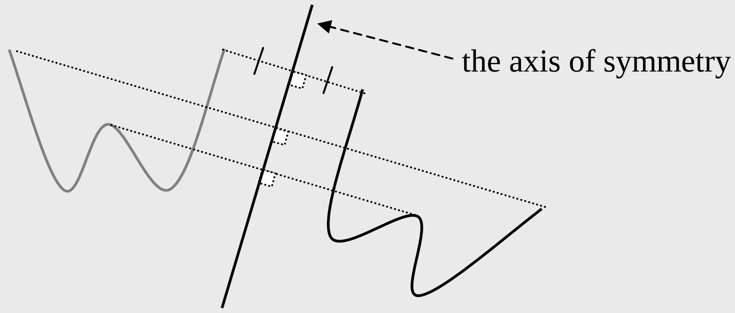
And the point **(1, 2)** can be called the axis of symmetry.
So if a point is the axis of symmetry, the point is the midpoint between every pair of two corresponding points in the two curves, the new and the original.

Fig. 0



And the same is true, also, for two points symmetric about a line $y = 2x + 1$.
So in that case, we call the line the axis of symmetry.

Fig. 1



So if a line is the axis of symmetry, the line is perpendicular to the line that connects the two points, and passes through the midpoint between the two points.

And thus, saying two curves are symmetric to each other, we mean in both curves, each and every pair of corresponding points are equal distance away from a particular point or line called the axis of symmetry.

And of course, the axis of symmetry is right in the middle of the two points in each of the pairs.

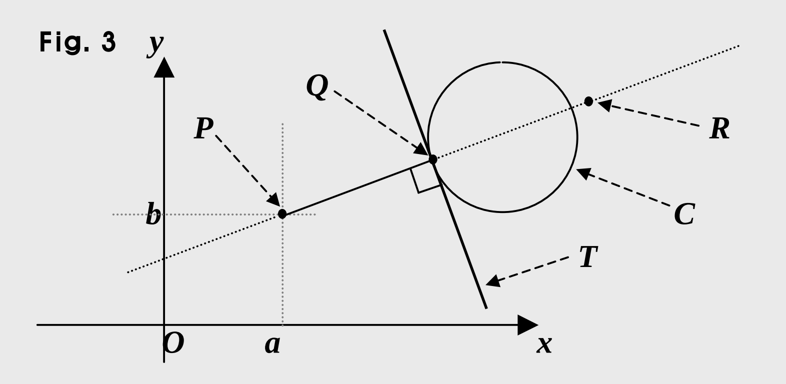
What do we mean by though, a distance in math?

Saying a distance in math, we mean the shortest distance, called the perpendicular distance, too.

What do we mean by the perpendicular distance though?

Suppose ***D*** is the distance from a point ***P*(*a*, *b*)** to a circle ***C*** as shown below.

Fig. 3



Suppose also, ***T*** is a line tangent to the circle ***C*** at a point ***Q*** as shown above.

Then, if ***D*** is the length of the line segment between ***P*** and ***Q***, the line segment is perpendicular to the tangent line ***T***.

So taking the perpendicular distance from ***P*** to the circle, we take ***D*** as the distance. Also, just saying a distance, we take a perpendicular distance as a distance.

Besides, the line containing the line segment passes through the center of the circle. That is, the line passing through the three points ***P***, ***Q***, and ***R*** passes though the center of the circle.

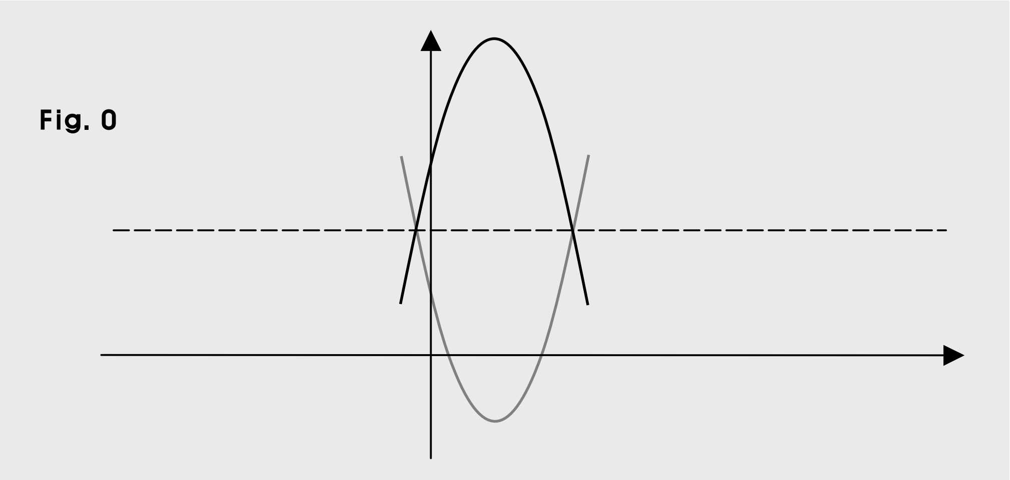
Also, ***D*** is called the distance from the point ***P*** to the line ***T***.

So in the graph above, if the point ***R*** is symmetric to the point ***P*** about the point ***Q*** or the line ***T***, the point ***R*** is in the line containing the line segment connecting ***P*** and ***Q***, and of course, we take ***D*** as the distance from ***R*** to ***Q***, also.

So let's see now how symmetric transformations work. We are going to begin with a simple transformation, which is symmetric about a line ***y* = *b***, which is a horizontal line, which is a line parallel to the ***x***-axis in the ***x-y*** plane.

2.1. Symmetric about a line 1

Fig. 0



• First, note that in this book, some important terminologies are used the way as follows.

The original curve: the curve of the function or equation a transformation is applied to
The new curve: the curve of the function or equation made by a transformation

The original function or equation: the function or equation a transformation is applied to
The new function or equation: the function or equation made by a transformation

The original arbitrary point or the original point: the arbitrary point in the original curve.
The new arbitrary point or the new point: the arbitrary point in the new curve.

Applying a transformation symmetric about a line, we get a new curve, which is symmetric to the original curve about the line, which is called therefore, the axis of symmetry. How then is the new curve made?

In the case of a transformation symmetric about a line, every point in the original curve gets moved symmetrically about the line, and then, all the points moved form the new curve, which is thus, said to be symmetric to the original curve about the line.

What do we mean by then, transforming a function symmetrically about a line?

Transforming a function symmetrically about a line, we get a new curve, which is symmetric about the line to the curve of the function.   So do we get a new function?

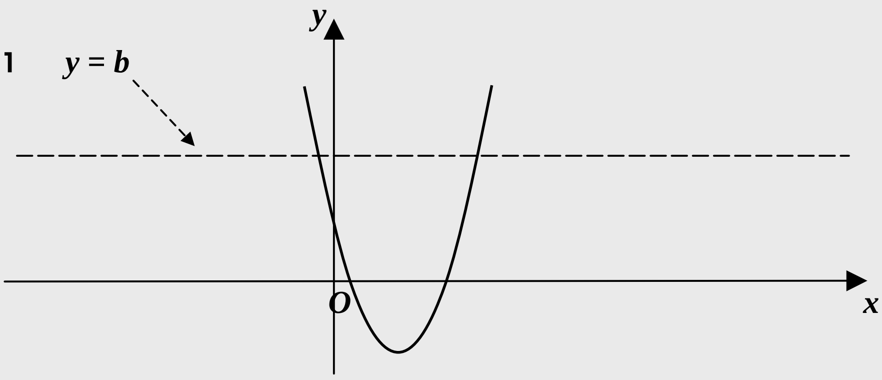
Maybe. It depends on the original function and the line taken as the axis of symmetry. If the line is however, horizontal or vertical, that is, parallel or perpendicular to a coordinate axis as the $\boldsymbol{x}$-axis, we get a new function.

And what's covered in this section is a case where the axis of symmetry is a horizontal line. So covered here is a transformation symmetric about a line parallel to the $\boldsymbol{x}$-axis.

Thus, the two curves, the original and the new in such a transformation are symmetric about a line $\boldsymbol{y = b}$, where $\boldsymbol{b}$ is constant. And if $\boldsymbol{b = 0}$, the axis of symmetry is the $\boldsymbol{x}$-axis.

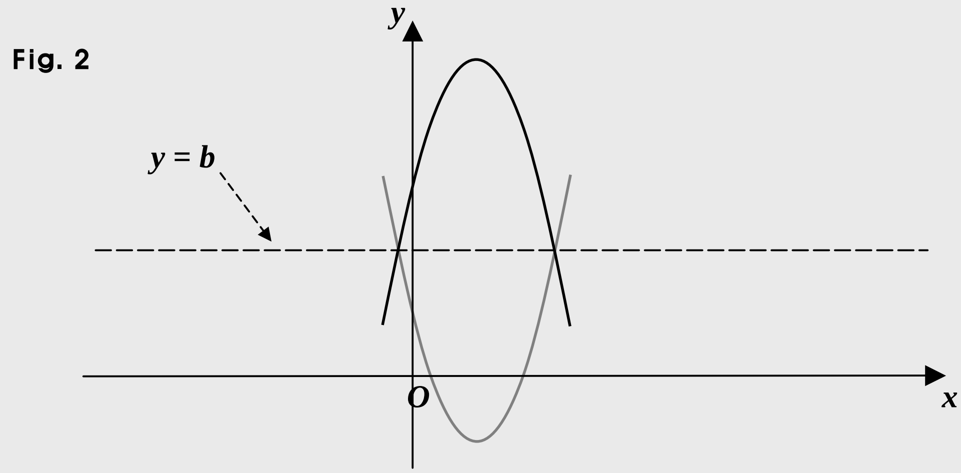
Suppose now that the original curve is the curve of a function $\boldsymbol{y = f(x)}$, and is put in a graph the way as follows.

Fig. 1



Then, applying to the curve of the function $\boldsymbol{f}$ the symmetric transformation stated above, we get a new curve, which is of course, symmetric to the curve of the function $\boldsymbol{f}$ about the line $\boldsymbol{y = b}$. Thus, the new curve is the curve in black in the graph as follows.

Fig. 2



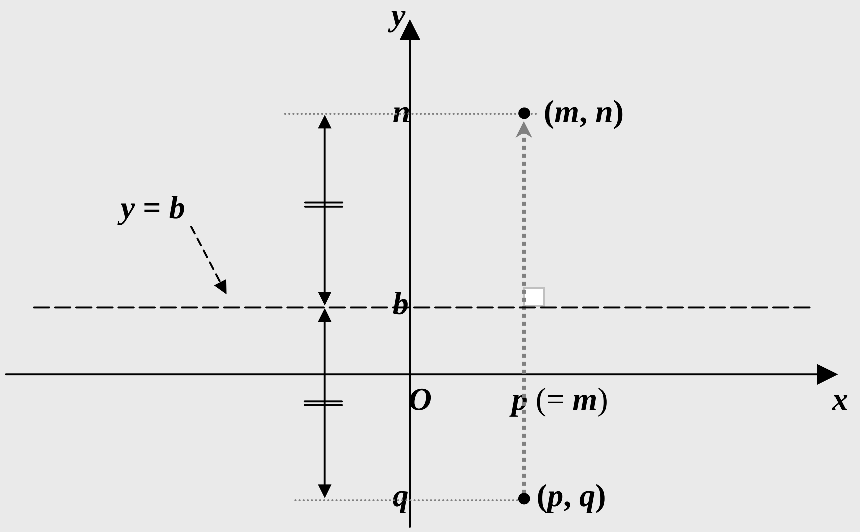
How then can we get the new curve, that is, the new equation?

A transformation definition shows specifically how every point gets changed in the original curve. So it specifies the way the original curve changes, that is, the way the new curve gets made. Thus, we want to get the definition first.

To begin with, if two points are symmetric about a line, the two are equal distance away from the line, and both points are in a line perpendicular to the line.

So for instance, assuming two points **(p, q)** and **(m, n)** are symmetric about the line **$y = b$**, we can put the two points in a graph the way as follows.

Fig. 3



Then first, since the two points (p, q) and (m, n) are symmetric about the line $y = b$, their x-coordinates are the same, that is, we get first, $p = m$.

And next, the distance from (p, q) to the line equals the distance from (m, n) to the line. In other words, b is right in the middle of n and q. That is, b is the average of n and q.

So we get $b = \frac{n+q}{2} \Rightarrow 2b = n + q \Rightarrow n = 2b - q$. How though, is it the average?

Since b is at the center between n and q, adding to b the distance from b to q, we get n.

That is, we get $n = b + (b - q) \Rightarrow 2b = n + q \Rightarrow b = \frac{n+q}{2}$, which is the average.

So we get $m = p$, and $n = 2b - q$.

Thus, assuming B is the transformation symmetric about the line $y = b$, and applying B to the point (p, q), we get $(m, n) = (p, 2b - q)$, which shows therefore, the correlation between (p, q) and (m, n) in the symmetry about the line $y = b$.

What then is the correlation between the original arbitrary point and the new arbitrary point in the symmetric transformation?

We know that the arbitrary point in a curve represents all the points in the curve, which means that every point in the curve acts the way the arbitrary point does, and vice versa.

So assuming (p, q) is in the original curve, and (x, y) is the original arbitrary point, we can say that (p, q) acts the way (x, y) does, and (x, y) acts the way (p, q) does.

Also, assuming (m, n) is in the new curve, and (s, t) is the new arbitrary point, we can say that (m, n) acts the way (s, t) does, and (s, t) acts the way (m, n) does.

So we get a correlation as follows. $(s, t) = (x, 2b - y)$.

Thus, we can put the transformation B the way as follows.

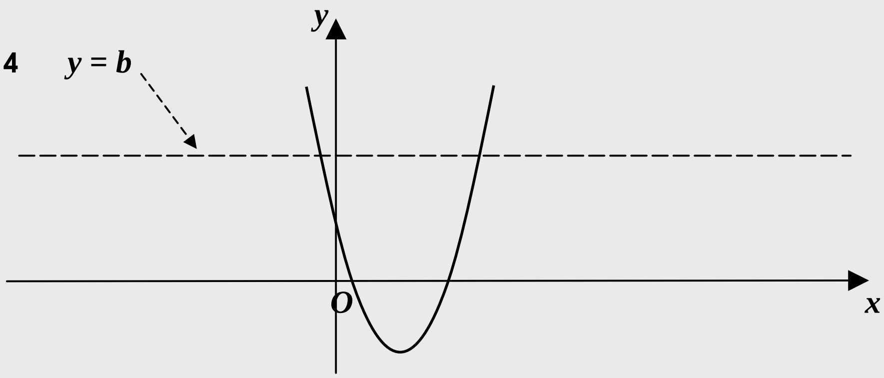
$B: (x, y) \longrightarrow (s, t) = (x, 2b - y)$, where b is constant.

And briefly, we can put it this way, too: $B: (x, y) \longrightarrow (x, 2b - y)$.

So we can call the expression above the definition of the transformation symmetric about a line ***y* = *b***. And the core of the transformation definition is **(x, 2*b* – *y*)**, which is the new arbitrary point, so we set **(*s*, *t*) = (*x*, 2*b* – *y*)**. And it shows how each and every point in the original curve gets changed, that is, the way every point in the new curve gets made. Is the new curve then a curve of a function?

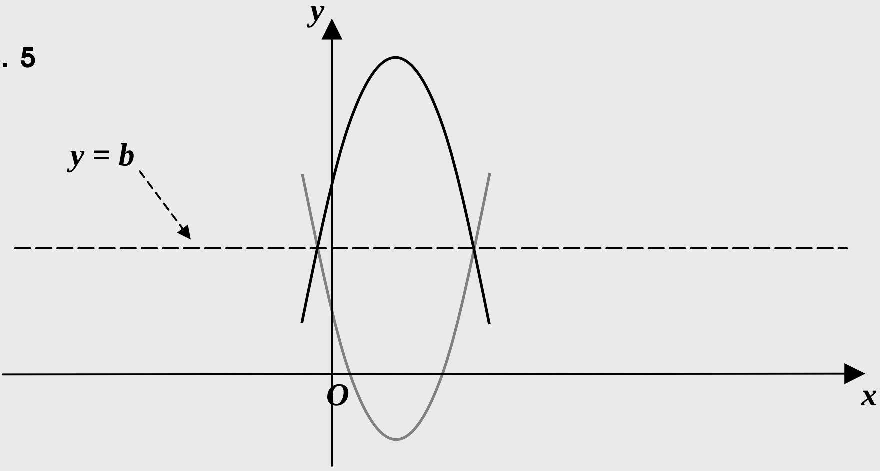
Yes. Applying to a function a transformation symmetric about a horizontal line ***y* = *b***, we get another function. That is, if the original curve is a curve of a function, the new curve is a curve of a function, too. In this example, the original curve is as follows.

Fig. 4



Applying the transformation ***B*** to the original curve above, we get a new curve, which is the curve in black below, and can be a curve of a function, which is a new function.

Fig. 5



And the transformation of a function by ***B*** is a function, too.
How then do we get the new curve, that is, the equation of the new curve?

We can get it getting the equation connecting the coordinates of the new arbitrary point (s, t), that is, the connective equation between s and t. And we can get the connective equation mixing the new arbitrary point (s, t) into the original curve. And we can get them mixed using $(x, 2b - y)$. And using it, we begin with this: $(s, t) = (x, 2b - y)$.

Then, we can set $s = x$, and $t = 2b - y$, which is the system of equations stated above. So solving it for x and y, we get $x = s$, and $y = 2b - t$.

Thus, we can now mix the new arbitrary point (s, t) into the original curve. That is, we can mix now, (s, t) into the original equation $y = f(x)$ to get the connective equation.

And getting the mixture, that is, the connective equation, putting $x = s$ and $y = 2b - t$ into the original equation. That is, in $y = f(x)$, we replace x with s, and y with $2b - t$.

So doing the substitutions, we get $y = f(x) \Rightarrow 2b - t = f(s)$, which is the connective equation between s and t, and thus, is the new equation indicating the new curve.

And we know that the new curve is in the x-y plane, and that s is another name for the variable x, and t is another name for the variable y.

So replacing s with x, and t with y, we get the new equation in the x-y system. That is, we get $2b - t = f(s) \Rightarrow 2b - y = f(x)$, which is the new equation.

So in sum, transforming an equation $y = f(x)$ symmetrically about a line $y = b$, we get a new equation, and the new equation is $2b - y = f(x)$.

And we can put it the way below, too.

Transforming an equation $h(x, y) = 0$ symmetrically about a line $y = b$, and assuming k is the new equation, we get $k(x, y) = h(x, 2b - y) = 0$. Why?

We have $(s, t) = (x, 2b - y)$.
So we can get $s = x$, and $t = 2b - y$, and thus, we get $x = s$, and $y = 2b - t$.

Next, mixing (s, t) into the original equation $h(x, y) = 0$, we get the connective equation.

And getting the mixture, that is, the connective equation, putting $x = s$ and $y = 2b - t$ into the original equation. That is, in $h(x, y) = 0$, we replace x with s, and y with $2b - t$.

So doing the substitutions, we get $h(x, y) = 0 \Rightarrow h(s, 2b - t) = 0$, which is the connective equation between s and t, and thus, is the new equation.

And we know that the new curve is in the x-y plane, and that s is another name for the variable x, and t is another name for the variable y.

So replacing s with x, and t with y, we get the new equation in the x-y system.
That is, we get $h(s, 2b - t) = 0 \Rightarrow h(x, 2b - y) = 0$, which is the new equation.

Thus, assuming k is the new equation, we get $k(x, y) = h(x, 2b - y) = 0$.

And next, we know applying to a function such a symmetric transformation, we get a new function. How then can we get the new function?

Expressing a function, we put the output variable in terms of the input variable.
That is, we set y equal to an expression in terms of x, and the expression is the function.
In this case, we have $2b - y = f(x)$. So we can set $y = 2b - f(x)$.
Assuming thus, g is the new function, we can set $y = g(x) = 2b - f(x)$.

Suppose for a concrete case, the original function is $y = f(x) = 2x^2 - 4x + 1$, and it gets transformed symmetrically about a line $y = 2$, that is, $b = 2$ in the axis of symmetry $y = b$.

Then, we can get the new equation, the way as follows.

First, we have $(x, y) \longrightarrow (x, 2b - y)$. So we have $(x, y) \longrightarrow (x, 4 - y)$.

So next, we can set $(s, t) = (x, 4 - y)$. Thus, we get $s = x$, and $t = 4 - y \Rightarrow y = 4 - t$.

So we get $y = f(x) \Rightarrow 4 - t = f(s) \Rightarrow 4 - t = 2s^2 - 4s + 1 \Rightarrow t = -2s^2 + 4s + 3$.

So we now have the equation connecting s and t. Replacing thus, s with x, and t with y, we get $y = -2x^2 + 4x + 3$, which is the new equation.

Next, assuming $\boldsymbol{g}$ is the new function, we get $\boldsymbol{y = g(x) = -2x^2 + 4x + 3}$.

And we can put the example above the way below, too.

We can have $\boldsymbol{y = f(x) = 2x^2 - 4x + 1 \Rightarrow y = 2x^2 - 4x + 1 \Rightarrow 2x^2 - 4x + 1 - y = 0}$.
And we can put it this way: $\boldsymbol{h(x, y) = 2x^2 - 4x + 1 - y = 0}$.

So suppose this time, the original equation is $\boldsymbol{h(x, y) = 2x^2 - 4x + 1 - y = 0}$, and it gets transformed symmetrically about the line $\boldsymbol{y = 2}$. Then, we can get the new equation the way as follows.

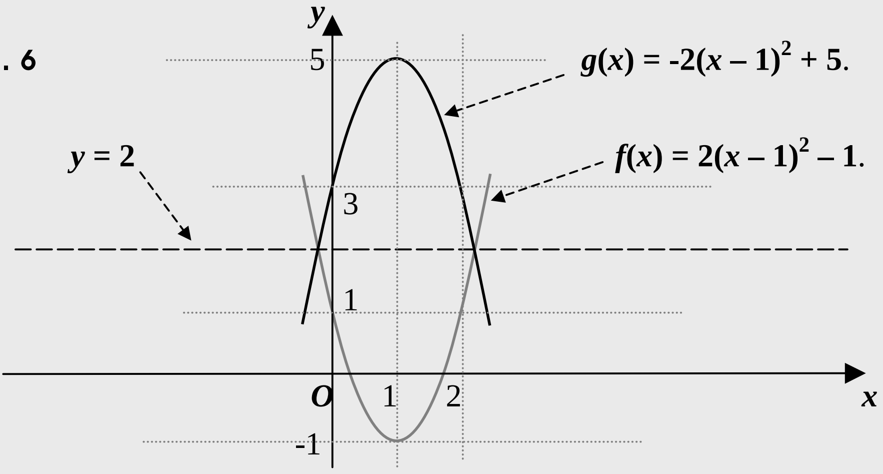
Since we apply the same transformation as the one above, we get $\boldsymbol{x = s}$, and $\boldsymbol{y = 4 - t}$.

So next, $\boldsymbol{h(x, y) = 0 \Rightarrow h(s, 4 - t) = 0 \Rightarrow 2s^2 - 4s + 1 - (4 - t) = 0 \Rightarrow t = -2s^2 + 4s + 3}$, which is the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$. Replacing thus, $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we get $\boldsymbol{y = -2x^2 + 4x + 3}$, which is the new equation.

And assuming $\boldsymbol{k(x, y) = 0}$ is the new equation above, we can set
$\boldsymbol{k(x, y) = 2x^2 - 4x - 3 + y = 0}$.

And next, assuming $\boldsymbol{g}$ is the new function, we get $\boldsymbol{y = g(x) = -2x^2 + 4x + 3}$.

Fig. 6



So in sum, transforming $y = f(x)$ symmetrically about a line $y = b$, we can begin with setting $(s, t) = (x, 2b - y)$, where (s, t) is the arbitrary point in the new curve, that is, the new arbitrary point, and (x, y) is the arbitrary point in the original curve, the curve of f, and thus, is the original arbitrary point.

Next, using $(s, t) = (x, 2b - y)$, along with the original equation $y = f(x)$, we can get the equation connecting the coordinates of the new point, that is, s and t. And next, in the connective equation, replacing s with x, and t with y, we get the new curve.

Thus, getting the new equation, we begin with setting $s = x$, and $t = 2b - y$.
So next, we get $s = x$, and $y = 2b - t$. Thus, we get $y = f(x) \Rightarrow 2b - t = f(s)$, which is the equation connecting the coordinates of (s, t), which is the arbitrary point in the new curve, which is in the x-y plane.

So replacing s with x, and t with y, we get $2b - t = f(s) \Rightarrow 2b - y = f(x)$, which is the new equation, which indicates the curve of the new function. So assuming $y = g(x)$ is the new function, we get $2b - y = f(x) \Rightarrow y = 2b - f(x) \Rightarrow y = g(x) = 2b - f(x)$.

Thus, assuming for instance, the original function is $y = f(x) = x^3 - 2x^2 - x + 2$,
we get $y = g(x) = 2b - f(x) = 2b - (x^3 - 2x^2 - x + 2) = 2b - x^3 + 2x^2 + x - 2$.

So the new function is $y = g(x) = 2b - x^3 + 2x^2 + x - 2$.

And we can put the same the way below, too.

Assuming (x_f, y_f) is the original arbitrary point, (x_g, y_g) is the new arbitrary point, and B is the transformation symmetric about a line $y = b$, we can put B the way as follows.

$$B: (x_f, y_f) \longrightarrow (x_g, y_g) = (x_f, 2b - y_f).$$

That is, f is the original function, and g is the new.

So we get $x_g = x_f$, and $y_g = 2b - y_f$, and thus, we get $x_f = x_g$, and $y_f = 2b - y_g$.

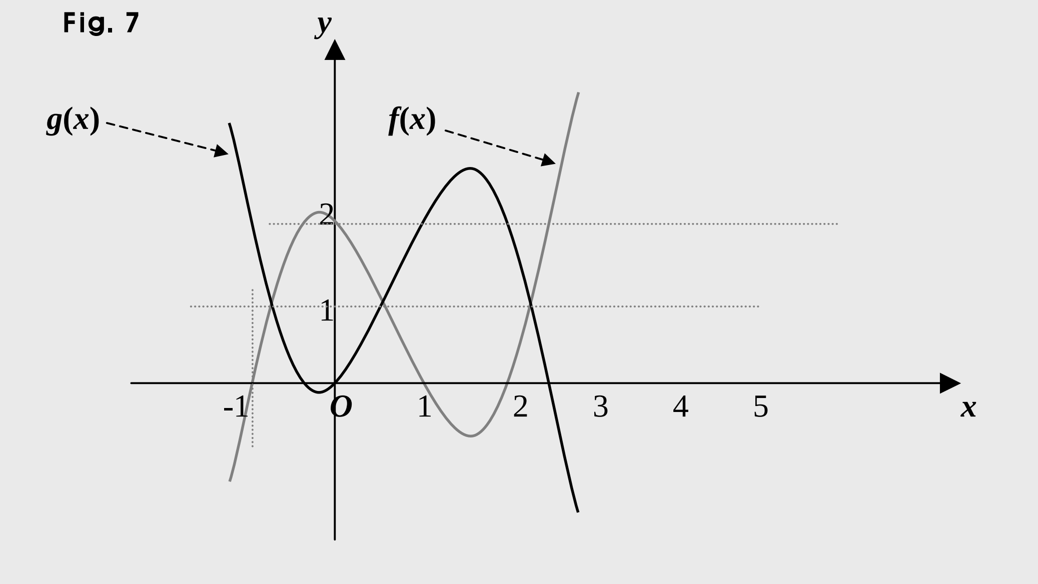
So next, we get $y_f = f(x_f) \Rightarrow 2b - y_g = f(x_g) \Rightarrow y_g = 2b - f(x_g) = g(x_g)$.

Thus, we get $y = g(x) = 2b - f(x) = 2b - x^3 + 2x^2 + x - 2$.

And assuming for instance, the axis of symmetry is $y = 1$, that is, $b = 1$, we get

$y = f(x) = x^3 - 2x^2 - x + 2$, and $y = g(x) = 2 - f(x) = 2 - x^3 + 2x^2 + x - 2 = -x^3 + 2x^2 + x$.

Fig. 7



What if we want the transformation to be symmetric about the x-axis?

Setting $b = 0$ in the equation that $y = b$, we get the transformation symmetric about the x-axis. And we have $B: (x, y) \longrightarrow (x, 2b - y)$. So we want to set $b = 0$ in $2b - y$.

Thus, assuming the transformation is X, we can put it the way below.

$X: (x, y) \longrightarrow (x, -y)$.

So assuming a function h is the transformation of f by X, we can get it the way below.

To begin with, we get $x_h = x_f$, and $y_h = -y_f$, and thus, we get $x_f = x_h$, and $y_f = -y_h$.

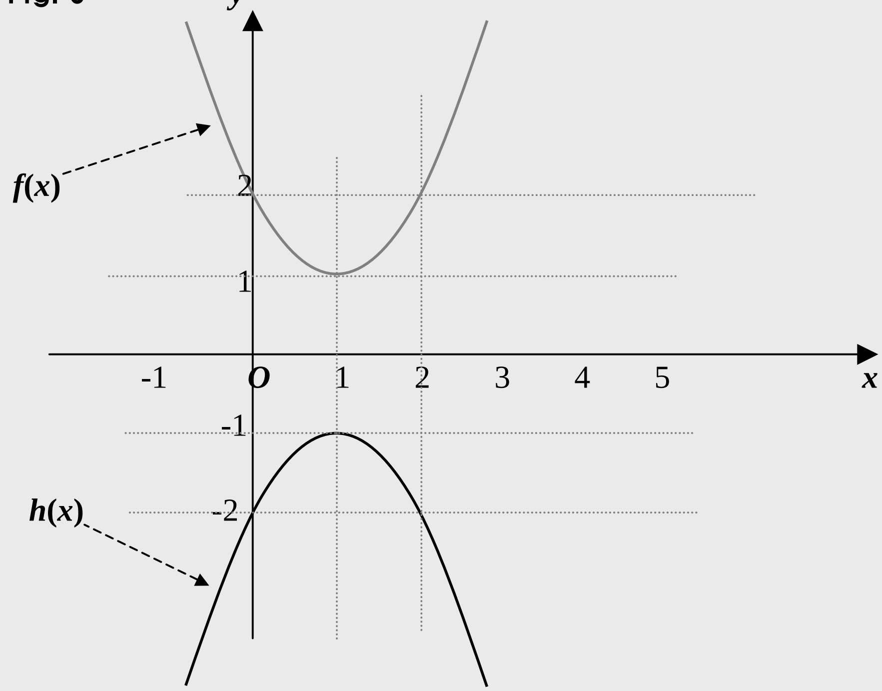
So next, we get $y_f = f(x_f) \Rightarrow -y_h = f(x_h) \Rightarrow y_h = -f(x_h) = h(x_h)$.

Thus, we get $y = h(x) = -f(x)$.

So for instance, assuming $y = f(x) = (x - 1)^2 + 1$, we get

$y = h(x) = -f(x) = -\{(x - 1)^2 + 1\} = -(x - 1)^2 - 1 \Rightarrow y = h(x) = -(x - 1)^2 - 1$.

Fig. 8



Let's next, for another instance, transform an equation $A(x, y) = \frac{y^2}{1 - x} + 2 = 0$ symmetrically about a line $y = 1$.

To begin with, we have $B: (x, y) \longrightarrow (x, 2b - y)$, which is the transformation symmetric about a line $y = b$.

So setting $b = 1$ in $2b - y$, we get the transformation symmetric about the line $y = 1$. Thus, assuming the transformation is K, we can put it the way below.

$K: (x, y) \longrightarrow (x, 2 - y)$.

So assuming an equation $H(x, y) = 0$ is the transformation of A by K, we can get it the way below.

We get first, $x_H = x_A$, and $y_H = 2 - y_A$, and thus, we get $x_A = x_H$, and $y_A = 2 - y_H$.

So next, we get $A(x_A, y_A) = 0 \Rightarrow A(x_H, 2 - y_H) = 0$.

Thus, we get $H(x_H, y_H) = A(x_H, 2 - y_H) = 0$.

And we have $A(x, y) = \dfrac{y^2}{1-x} + 2 = 0$. That is, we have $A(x_A, y_A) = \dfrac{{y_A}^2}{1-x_A} + 2 = 0$.

So we get $A(x_A, y_A) = \dfrac{{y_A}^2}{1-x_A} + 2 = 0 \Rightarrow A(x_H, 2 - y_H) = \dfrac{(2-y_H)^2}{1-x_H} + 2 = 0$.

Thus, we get $H(x_H, y_H) = \dfrac{(2-y_H)^2}{1-x_H} + 2 = 0$.

That is, we get $H(x, y) = \dfrac{(2-y)^2}{1-x} + 2 = 0$.

So we can put $H(x_H, y_H) = A(x_H, 2 - y_H) = 0$ this way, too: $H(x, y) = A(x, 2 - y) = 0$.

Besides, using the transformation operator, the dot •, we can put it this way, too:

$H = K \bullet A = A(x, 2 - y) = \dfrac{(2-y)^2}{1-x} + 2 = 0.$ And Let’s now put the curves in a graph.

To begin with, converting the two equations, we get

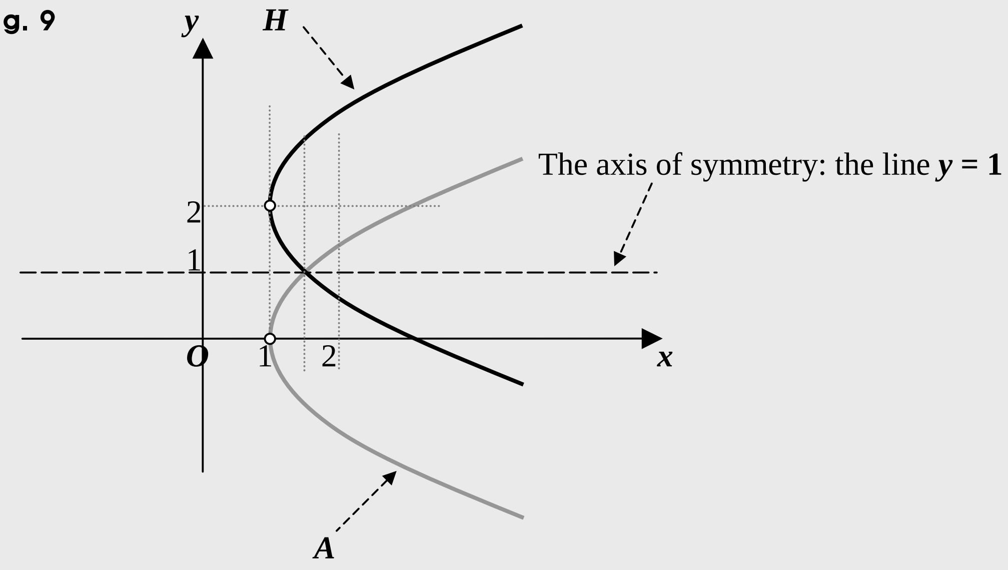
$$A(x, y) = \frac{y^2}{1-x} + 2 = 0 \Rightarrow y^2 + 2(1-x) = 0 \Rightarrow x = \frac{y^2}{2} + 1.$$

$$H(x, y) = \frac{(2-y)^2}{1-x} + 2 = 0 \Rightarrow (2-y)^2 + 2(1-x) = 0 \Rightarrow x = \frac{(2-y)^2}{2} + 1.$$

We want to note however, that the actual equations are $\dfrac{y^2}{1-x} + 2 = 0$ and $\dfrac{(2-y)^2}{1-x} + 2 = 0$.

So we can’t get $x = 1$, since a denominator cannot be 0.

Fig. 9



And let's next, for another instance, transform the function below symmetrically about a line $\boldsymbol{y = 1}$.

$$\boldsymbol{y = f(x) = 0.2x^5 - 2.25x^4 + 9x^3 - 15.5x^2 + 12x - 2}.$$

Then, we can get the new equation, the way as follows.

First, we have $\boldsymbol{(x, y) \longrightarrow (x, 2b - y)}$. So we have $\boldsymbol{(x, y) \longrightarrow (x, 2 - y)}$.

So next, we can set $\boldsymbol{(s, t) = (x, 2 - y)}$.

Thus, we get $\boldsymbol{s = x}$, and $\boldsymbol{t = 2 - y}$, so we get $\boldsymbol{x = s}$, and $\boldsymbol{y = 2 - t}$.

Thus, we get $\boldsymbol{y = f(x) \Rightarrow 2 - t = f(s)}$

$$\boldsymbol{\Rightarrow 2 - t = 0.2s^5 - 2.25s^4 + 9s^3 - 15.5s^2 + 12s - 2}$$

$$\boldsymbol{\Rightarrow t = -(0.2s^5 - 2.25s^4 + 9s^3 - 15.5s^2 + 12s) + 4}$$

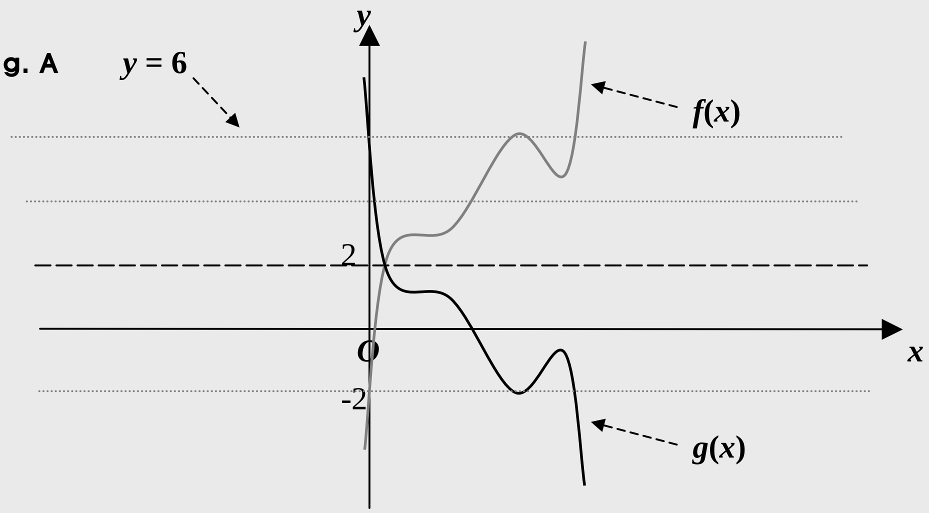
$$\boldsymbol{\Rightarrow t = 4 - 0.2s^5 + 2.25s^4 - 9s^3 + 15.5s^2 - 12s}.$$

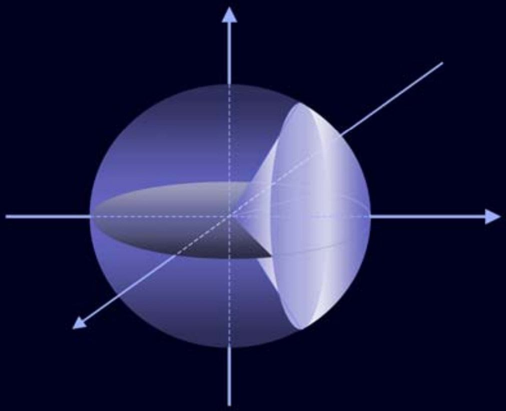
So we now have the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$. Replacing thus, $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we get $\boldsymbol{y = 4 - 0.2x^5 + 2.25x^4 - 9x^3 + 15.5x^2 - 12x}$, which is the new equation.

And next, assuming ***g*** is the new function, we get

$$y = g(x) = 4 - 0.2x^5 + 2.25x^4 - 9x^3 + 15.5x^2 - 12x.$$

Fig. A





함수의 변환

Function Transformations

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