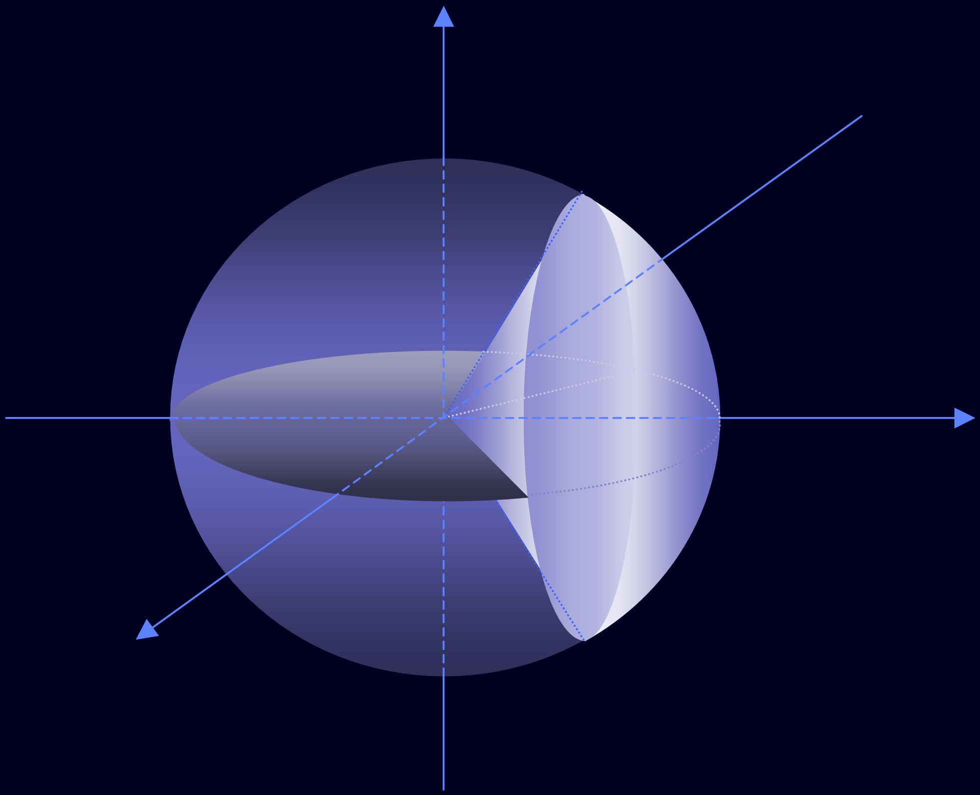
# 거듭제곱과 로그
# Powers and Logarithms



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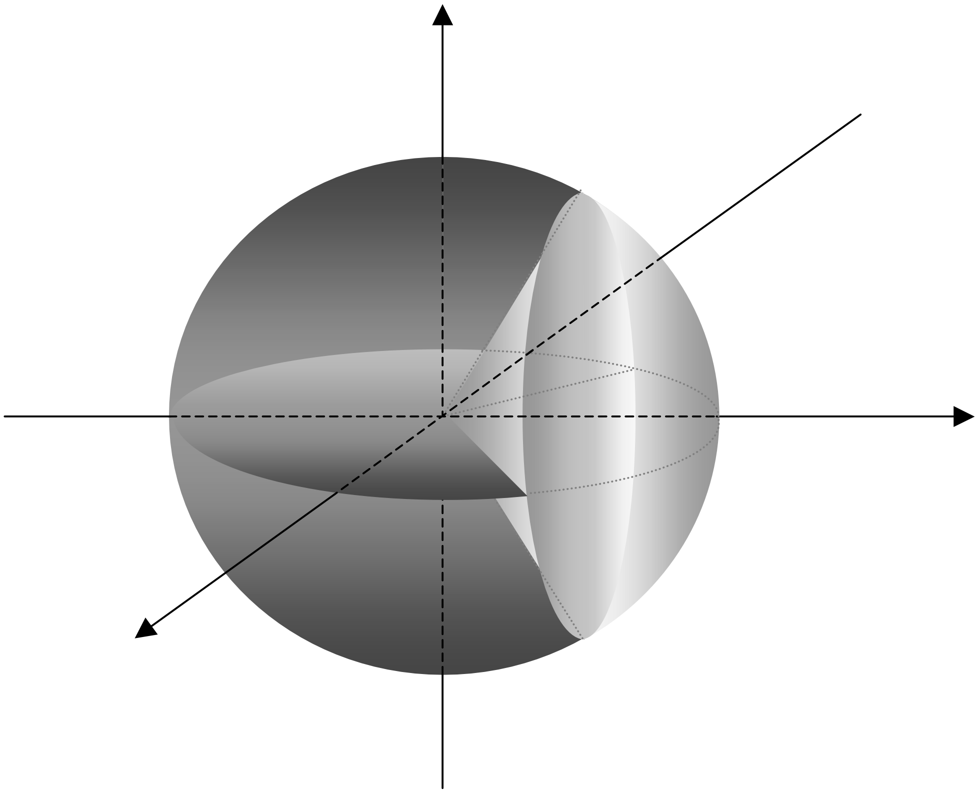

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# Contents

# 0. What is a power?

It is a math tool, and in fact, is a power tool, has quite a bit of power, and gets powered by numbers, together with your reasoning.
Being a power user of such a tool, you can grab and handle readily many values, excessively big as 523347633027360537213511521 or unreasonably small as 0.0000000000000022876792454961.
Also, taking care of ordinary values as 64, 243, 625, 1024, and such, you can effectively handle them, too, using the tool called a power, of course. What then, is the tool?

When adding together many of the same numbers, we don't actually do additions so many times, do we? We just multiply, of course, one of those many by the number of those, and get the product, which is the sum.

So for instance, adding together five **3**s, we don't normally do **3 + 3 + 3 + 3 + 3**, but just do **3·5**, that is, **3 x 5**, and get **15**, which is the sum, as well as the product of 3 and 5.

Next, what if we want to multiply 1 by 3 many times, and specify the product?

If we just want to multiply 1 by 3 twice, we just do 1·3·3, and get 9, which is the product. What if however, we want to multiply 1 by 3 five hundred times, and specify the product?

Let alone specifying such a product, we are probably not going to actually do multiplications so many times, are we?

Suppose however, we actually did do multiplications so many times, and somehow did find the product. Then, we would get a big number, which is a long number.

The number is the product of five hundred **3**s, and has a huge number of digits. In fact, it has a whopping array of 239 digits. So it's an unreasonably long number with a huge value.    What if however, we've got to specify such a huge value anyway?

We don't actually specify such a long number digit by digit. We can quickly put such an enormous value in a short and simple manner. Using a special notation, we can effectively specify a humongous value or a number with a gigantic value.

A number indicates a value.
And we can put a value in various forms. We have a variety of ways or methods, where we can indicate the same value. Depending on the purpose, we can show a value in a particular form. For instance, we can put 8 in such many ways as follows.

$1 \times 8$, $8 \cdot 1$, $4 \cdot 2$, $4 \cdot (3 - 1)$, $1 + 2 + 3 + 2$, $3 + 1 + 4$, $1 + 3 - 3 + 7$, $9 - 1$, $\frac{16}{2}$, $\frac{3 \cdot 8}{3}$, etc.

Quite often, we need to show values with unduly large or extremely small magnitudes, but want to express them in a succinct manner. If for instance, we have to put in writing values like 100000000000000000 and 0.000000000001, they not only take up too much space, but are hard to read, too.    What then do we do?

In such a case, we can use a tool, called a *power*. And the tool takes two numbers, one is called an *exponent*, and the other is called a *base*. So we get the tool powered by two numbers called an *exponent* and a *base*.

Suppose for instance, a value is a product we get multiplying 1 by 7 a particular number of times. Then, we can specify the value using a power. Then, the particular number is called the *exponent*, and 7 is called the *base*.

So for instance, multiplying 1 by 9 twenty eight times, and expressing the product using a power, we use 9 as the base, and use 28 as the exponent.

So expressing $1 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$ which is 523347633027360537213511521, we can use a power, where the base is 9, and the exponent is 28.    How then can we specify a power?

Specifying a power, we put it in terms of two numbers, one is called a base, and the other is called an exponent.

A power is therefore, a structured number composed of two parts, one is a base, and the other is an exponent.

And in this book, such a structured number is said to be in *power notation*.
Using power notation, we put an exponent at the upper right hand corner of a base.
In other words, we use an exponent as a superscript.

So for instance, using power notation, we can put 8 in $2^3$, and put 1,000,000,000 in $10^9$. And $2^3$ is called a power, 2 is called the base, and 3 is called the exponent. The same is true for the power $10^9$, too. So 10 is the base of the power $10^9$, and 9 is the exponent.

Putting thus, a value or number in power notation, we make a power, which is a number made of a base and an exponent.

Also, calling a power with a particular base, we call it <u>a power of the particular base</u>.

So for instance, $10^9$ can be called <u>a power of 10</u>, and specifically, it is called the *ninth power* of 10. And also, we can call it differently, too, and will cover shortly how to do so.

And for another instance, we can simply put 523347633027360537213511521 in $9^{28}$, which is a power of 9, and specifically, is called the twenty eighth power of 9. And thus, we can put such a huge value in a succinct manner.

What then about small numbers as 0.0000001?

In such a case, we can use a *negative* number as the *exponent*.

A value can be a reciprocal of a large number, so such a value is very small, and we can get such a number dividing 1 by a particular number many times. For instance, it can be 1/23/23/23/23/23/23, which equals $\frac{1}{148035889}$, which we get dividing 1 by 23 six times.

And expressing a number we get doing divisions by the same number repeatedly, we can use power notation, too. Then, the base is the divisor, which is 23 in the case above, but the exponent is a negative integer, because a division is opposite of a multiplication. So –6 is the exponent, because the division by 23 happens 6 times.

Using thus, power notation, we put $\frac{1}{148035889}$ in $23^{-6}$, which is a power of 23, and specifically, is called the *negative sixth* power of 23.

And we put 0.0000001 in $10^{-7}$, because 0.0000001 = 1/10/10/10/10/10/10/10. And $10^{-7}$ is a power of 10, and is called the negative seventh power of 10.

And multiplying for instance, 1 by $b$ twice, and specifying the product, we can do this: $1 \cdot b \cdot b$, and then, can put the product this way: $bb$. Normally though, we put it this way: $b^2$. And dividing 1 by $b$ twice, we put the result this way: $b^{-2}$. What if however, we want to indicate a value we get multiplying (or dividing) 1 by $b$ an arbitrary number of times?

We can indicate the value by $\underline{b^n \text{ where } n \text{ is an integer}}$. So the superscript $n$ indicates the number of the multiplications made. And the same is true for divisions, too, of course.

So in the power $b^n$ where $n$ is an integer, $n$ indicates the number of times we multiply or divide 1 by $b$. For instance, $1 \cdot b \cdot b \cdot b \cdot b \cdot b = 1bbbbb = b^5$, and $1/b/b/b/b = b^{-4}$. And of course, in cases of divisions, the base $b$ is not 0.

What do we get though, if we put 0 into $n$ in the power $b^n$?

Then, we do neither multiply nor divide 1 by the base $b$. That is, we do nothing to 1. In other words, we leave 1 alone. So we get $b^0 = 1$, and for instance, $2^0 = 3^0 = 4^0 = 1$.

What if $n$ is 1 in $b^n$?

Then, we multiply 1 by $b$ once, and therefore, we get a $b$. So we get $b^1 = b$.

Thus, we can put a value in a form of a power, which takes a form of $b^n$, where $b$ is called a base, and $n$ is called an exponent. How can we read a power though?

We can read $b^n$ as ‘$b$ raised to the $n^{\text{th}}$ power’ or ‘the $n^{\text{th}}$ power of $b$’. And briefly, we can read it as ‘$b$ to the $n^{\text{th}}$’. Even more briefly, we can just read it as ‘$b$ to the $n$’.

For instance, we can read $b^7$ as ‘$b$ raised to the seventh power’, ‘the seventh power of $b$’, or ‘$b$ to the seventh’. And more briefly, we can read it as ‘$b$ to the seven’, too.

Usually though, we read $\boldsymbol{b}^2$ differently, and often read it as ‘$\boldsymbol{b}$ squared’, instead of reading it as ‘$\boldsymbol{b}$ raised to the second power’, ‘the second power of $\boldsymbol{b}$’, or ‘$\boldsymbol{b}$ to the second’.

And the same is true for $\boldsymbol{b}^3$, too. So $\boldsymbol{b}^3$ is often read as ‘$\boldsymbol{b}$ cubed’. And of course, we can read it as ‘$\boldsymbol{b}$ to the third’, too. How then about $\boldsymbol{b}^{-3}$?

It can be read as ‘$\boldsymbol{b}$ to the negative (minus) third’, ‘the negative third power of $\boldsymbol{b}$’, or just can be read briefly as ‘$\boldsymbol{b}$ to the negative three’.
And in fact, we can use any real number as an exponent if the base is positive.

So for instance, we can read

$\boldsymbol{b}^{0.3}$ as ‘$\boldsymbol{b}$ to the 0.3’, $\boldsymbol{b}^{\pi}$ as ‘$\boldsymbol{b}$ to the pi’, and $\boldsymbol{b}^{\sqrt{2}}$ as ‘$\boldsymbol{b}$ to the square root of 2’.

$\mathbf{0.3}^3$ as 0.3 cubed, and of course, we can read it as 0.3 to the third, too.

$\boldsymbol{b}^{\frac{1}{n}}$ as ‘$\boldsymbol{b}$ to the 1 over $\boldsymbol{n}$’, $\boldsymbol{b}^{-\frac{y}{x}}$ as ‘$\boldsymbol{b}$ to the negative $\boldsymbol{y}$ over $\boldsymbol{x}$’, and $\boldsymbol{b}^{\frac{3}{\pi}}$ as ‘$\boldsymbol{b}$ to the 3 over pi’.

$\boldsymbol{b}^{-\frac{2}{3}}$ as ‘$\boldsymbol{b}$ to the negative 2 over 3’, and also, as ‘$\boldsymbol{b}$ to the negative two thirds’.

$\boldsymbol{b}^{f(2)}$ as ‘$\boldsymbol{b}$ to the $\boldsymbol{f}(\mathbf{1})$’, and $\boldsymbol{b}^{g(x)}$ as ‘$\boldsymbol{b}$ to the $\boldsymbol{g}(\boldsymbol{x})$’.

$\boldsymbol{b}^{x^2}$ as ‘$\boldsymbol{b}$ to the $\boldsymbol{x}$ squared’ or ‘$\boldsymbol{b}$ to the $\boldsymbol{x}$ to the second’, and $\boldsymbol{b}^{x^{\frac{1}{3}}}$ as ‘$\boldsymbol{b}$ to the $\boldsymbol{x}$ to the third’.

# 1. Exponential Identities 1

Doing math or taking courses in math, we can't do much without doing algebra. And doing algebra, we get to work with many tools as formulas, identities, theorems, etc. Or rather, whatever math we do, we have to do algebra, and always need to use such tools.

And also, doing algebra, we often get to work with powers. Or rather, we always have to work with powers. And doing algebra with powers, we do exponential algebra, and get to work with exponents and bases, of which the powers are made.

And doing exponential algebra, we can hardly do much without using some essential tools, called exponential identities, which are therefore, crucial. And thus, we want to know them very well. What do we mean by though, knowing them very well?

Knowing them very well, we know how they work, and more importantly, know how to work with them. So let's see now how they work and how we can work with them.

To begin with, keep in mind that all the letters used in the math expressions here take real numbers unless specified otherwise. Also, note in particular that we do not rule out cases where the letters can be 0 or negative unless specified otherwise.

Now, suppose that both $\boldsymbol{a}$ and $\boldsymbol{b} \neq \mathbf{0}$, and that both $\boldsymbol{m}$ and $\boldsymbol{n}$ are integers.
Then, the tools below are called exponential identities, and *always* work.

**0.** $a^m a^n = a^{m+n}$

**1.** $a^m / a^n = \dfrac{a^m}{a^n} = a^{m-n}$

**2.** $(a^m)^n = a^{mn}$

**3.** $(ab)^n = a^n b^n$

**4.** $(a/b)^n = \left(\dfrac{a}{b}\right)^n = a^n / b^n = \dfrac{a^n}{b^n}$

Let's see now, what's going on in each of the identities listed above.

**0.** $a^m a^n = a^{m+n}$, where $a \neq 0$, and both $m$ and $n$ are integers.

To begin with, we know multiplying 1 by $a$, $(m + n)$ times, we get a power $a^{m+n}$.

What then, do we get if multiplying a power $a^m$ by another power $a^n$?

We get a new power $a^{m+n}$. It's because $a^m$ is the product of $m$ of $a$s, and $a^n$ is the product of $n$ of $a$s, so $a^m a^n$ is the product of $(m + n)$ of $a$s.

Taking thus, the product of powers sharing the same base, we get a new power, where the base remains the same, and the exponent is the sum of the exponents. For instance,

$3^4 3^5 = (3\cdot3\cdot3\cdot3)(3\cdot3\cdot3\cdot3\cdot3) = 3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3 = 3^9$, so we get $3^4 3^5 = 3^{4+5} = 3^9$.

$3^3 3^4 3^2 = (3\cdot3\cdot3)(3\cdot3\cdot3\cdot3)(3\cdot3) = 3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3 = 3^9$, so we get $3^3 3^4 3^2 = 3^{3+4+2} = 3^9$.

$(-3)^2(-3)^3 = \{(-3)\cdot(-3)\}\{(-3)\cdot(-3)\cdot(-3)\} = (-3)\cdot(-3)\cdot(-3)\cdot(-3)\cdot(-3) = (-3)^5$, so we get

$(-3)^2(-3)^3 = (-3)^{2+3} = (-3)^5$, which equals $-3^5$, of course.

What if $m$ and $n$ are negative?

Then, we have $m < 0$, and $n < 0$, so we get $-(m + n) > 0$. That is, $-(m + n)$ is a positive integer. So dividing 1 by $a$, $-(m + n)$ times, we get $\frac{1}{a^{-(m+n)}}$.

And the denominator is a power, which is the product of $-(m + n)$ of $a$s.
And thus, putting $\frac{1}{a^{-(m+n)}}$ in the power notation, we get $a^{m+n}$, where $m + n < 0$.

So for instance, dividing 1 by $a$, 3 times, we get $1/a/a/a = 1/(aaa) = \frac{1}{aaa} = \frac{1}{a^3} = a^{-3}$.

What then do we get if multiplying a power $a^m$ by another power $a^n$ where $m$ and $n$ are negative?

We have $-m > 0$, and $-n > 0$. So we can set $a^m = \frac{1}{a^{-m}}$, and $a^n = \frac{1}{a^{-n}}$, where $a^{-m}$ is the product of $-m$ of $a$s, and $a^{-n}$ is the product of $-n$ of $a$s.

Thus, we get $\boldsymbol{a^m a^n} = \dfrac{\mathbf{1}}{\boldsymbol{a^{-m}}} \cdot \dfrac{\mathbf{1}}{\boldsymbol{a^{-n}}} = \dfrac{\mathbf{1}}{\boldsymbol{a^{-(m+n)}}} = \boldsymbol{a^{m+n}}$.

For instance, we can get $\boldsymbol{a^{-2}a^{-3}} = \dfrac{\mathbf{1}}{\boldsymbol{a^2}} \cdot \dfrac{\mathbf{1}}{\boldsymbol{a^3}} = \dfrac{\mathbf{1}}{\boldsymbol{a^{2+3}}} = \boldsymbol{a^{-(2+3)}} = \boldsymbol{a^{-5}}$.

And assuming $\boldsymbol{m = n = 0}$, we get $\boldsymbol{a^m a^n = a^0 a^0 = 1 \cdot 1 = 1}$. And we have $\boldsymbol{a^{0+0} = a^0 = 1}$.

So we get $\boldsymbol{a^m a^n = a^{m+n}}$ where $\boldsymbol{a \neq 0}$, and both $\boldsymbol{m}$ and $\boldsymbol{n}$ are integers. What if $\boldsymbol{a = 0}$?

Then, we need to have $\boldsymbol{m > 0}$, and $\boldsymbol{n > 0}$.

For instance, assuming $\boldsymbol{m = -1}$, we get $\boldsymbol{a^m = 0^{-1}} = \dfrac{1}{0}$, which is not possible.

What if $\boldsymbol{m = 0}$, though?

Then, we get $\boldsymbol{a^m = 0^0 = 1}$. That's because multiplying 1 by 0, no times, we get 1.

Thus, putting threads together, we get $\boldsymbol{a^m a^n = a^{m+n}}$, where $\boldsymbol{a \neq 0}$, and both $\boldsymbol{m}$ and $\boldsymbol{n}$ are integers.

**1.** $\boldsymbol{a^m / a^n} = \dfrac{\boldsymbol{a^m}}{\boldsymbol{a^n}} = \boldsymbol{a^{m-n}}$, where $\boldsymbol{a \neq 0}$, and both $\boldsymbol{m}$ and $\boldsymbol{n}$ are integers.

We are going to look at two cases. In one, a power $\boldsymbol{a^m}$ gets divided by $\boldsymbol{a}$, $\boldsymbol{n}$ times. And in the other, the power $\boldsymbol{a^m}$ gets divided by another power $\boldsymbol{a^n}$.

So let's first, divide the power $\boldsymbol{a^m}$ by $\boldsymbol{a}$, $\boldsymbol{n}$ times.

Then, assuming first, $\boldsymbol{m \geq n}$, and dividing the power $\boldsymbol{a^m}$ by $\boldsymbol{a}$, $\boldsymbol{n}$ times, we get a new power $\boldsymbol{a^{m-n}}$, where $\boldsymbol{m - n \geq 0}$. How?

In the power $\boldsymbol{a^m}$, $\boldsymbol{n}$ of $\boldsymbol{a}$s get canceled due to $\boldsymbol{n}$ divisions by $\boldsymbol{a}$. So for instance, doing 3 divisions by $\boldsymbol{a}$ to a power $\boldsymbol{a^5}$, we get $\dfrac{\boldsymbol{a^5}}{\boldsymbol{a^3}} = \dfrac{\boldsymbol{a \cdot a \cdot a \cdot a \cdot a}}{\boldsymbol{a \cdot a \cdot a}} = \boldsymbol{a \cdot a} = \boldsymbol{a^2} = \boldsymbol{a^{5-3}}$

- Suppose next, $\boldsymbol{m < n}$.

Then, though dividing the power $\boldsymbol{a^m}$ by $\boldsymbol{a}$, $\boldsymbol{n}$ times, we still get the power $\boldsymbol{a^{m-n}}$, in which however, $\boldsymbol{m - n < 0}$. How?

First, if $\boldsymbol{m < n}$, we can say that $\boldsymbol{a^n}$ has $(\boldsymbol{n - m})$ more $\boldsymbol{a}$s than $\boldsymbol{a^m}$ has.

For instance, since 2 < 5, $\boldsymbol{a^5}$ has ($\mathbf{5 - 2 = 3}$) more $\boldsymbol{a}$s than $\boldsymbol{a^2}$ has.

So if $\boldsymbol{m < n}$, and the power $\boldsymbol{a^m}$ gets divided by $\boldsymbol{a}$, $\boldsymbol{n}$ times, what happens first is that all of the $\boldsymbol{a}$s in the power $\boldsymbol{a^m}$ get canceled due to the first $\boldsymbol{m}$ divisions by $\boldsymbol{a}$, and we get 1, and then, 1 gets divided by $\boldsymbol{a}$, $(\boldsymbol{n - m})$ more times.

Thus, we get $\dfrac{\mathbf{1}}{\boldsymbol{a^{n-m}}}$. So we get $\boldsymbol{a^{m-n}}$.

So for instance, if a power $\boldsymbol{a^2}$ gets divided by $\boldsymbol{a}$, 5 times, what happens first is that all of the two $\boldsymbol{a}$s in the power $\boldsymbol{a^2}$ get canceled due to the first two divisions by $\boldsymbol{a}$, and we get 1, and then, 1 gets divided by $\boldsymbol{a}$, (5 – 2 = 3) more times.

Thus, we get $\dfrac{\mathbf{1}}{\boldsymbol{a^{5-2}}}$. So we get $\boldsymbol{a^{5-2}}$.

Thus, dividing 1 by the same number, many times, we get a negative exponent, and take the same number as the base in the power we get.

So assuming $\boldsymbol{m < n}$, and dividing $\boldsymbol{a^m}$ by $\boldsymbol{a}$, $\boldsymbol{n}$ times, we get $\boldsymbol{a^{m-n}}$, too. Thus, for instance, dividing $\boldsymbol{a^3}$ by $\boldsymbol{a}$, five times, we get $\dfrac{\boldsymbol{a}^3}{\boldsymbol{a}^5} = \dfrac{\boldsymbol{a \cdot a \cdot a}}{\boldsymbol{a \cdot a \cdot a \cdot a \cdot a}} = \dfrac{\mathbf{1}}{\boldsymbol{a \cdot a}} = \boldsymbol{a}^{-2} = \boldsymbol{a}^{3-5}$.

So either way, dividing $\boldsymbol{a^m}$ by $\boldsymbol{a}$, $\boldsymbol{n}$ times, we get the power, $\boldsymbol{a^{m-n}}$.

- Suppose this time, we divide the power $\boldsymbol{a^m}$ by another power $\boldsymbol{a^n}$. Then, we get $\dfrac{\boldsymbol{a^m}}{\boldsymbol{a^n}}$.

So assuming first, $\boldsymbol{m \geq n}$, we get a new power, $\boldsymbol{a^{m-n}}$. How?

Initially, the numerator $\boldsymbol{a^m}$ has $\boldsymbol{m}$ of $\boldsymbol{a}$s, and the denominator $\boldsymbol{a^n}$ has $\boldsymbol{n}$ of $\boldsymbol{a}$s.

During the course of the division of $\boldsymbol{a^m}$ by $\boldsymbol{a^n}$, $\boldsymbol{n}$ of $\boldsymbol{a}$s each in both the numerator and denominator get canceled, so the resultant numerator is left with ($\boldsymbol{m - n}$) of $\boldsymbol{a}$s, and the resultant denominator is 1.

Thus, we get $\frac{a^{m-n}}{1}$, which is $\boldsymbol{a^{m-n}}$.

So for instance, we get $\frac{3^5}{3^3} = \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3} = \frac{3 \cdot 3}{1} = 3^2 \Rightarrow \frac{3^5}{3^3} = 3^{5-3} = 3^2$.

And we get $\frac{(\frac{1}{3})^5}{(\frac{1}{3})^3} = \frac{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}}{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}} = \frac{\frac{1}{3} \cdot \frac{1}{3}}{1} = (\frac{1}{3})^2 \Rightarrow \frac{(\frac{1}{3})^5}{(\frac{1}{3})^3} = (\frac{1}{3})^{5-3} = (\frac{1}{3})^2$.

And assuming next, $\boldsymbol{m < n}$, we still get the power indicated by $\boldsymbol{a^{m-n}}$.
This time though, we get $\boldsymbol{m - n < 0}$. How?

Initially, the numerator $\boldsymbol{a^m}$ has $\boldsymbol{m}$ of $\boldsymbol{a}$s, and the denominator $\boldsymbol{a^n}$ has $\boldsymbol{n}$ of $\boldsymbol{a}$s.

During the course of the division, $\boldsymbol{m}$ of $\boldsymbol{a}$s each in both the numerator and denominator get canceled, so the numerator ends up with 1, and the denominator is left with ($\boldsymbol{n - m}$) of $\boldsymbol{a}$s.

Thus, we get $\frac{1}{a^{n-m}}$, which is the result we get if 1 gets divided by $\boldsymbol{a}$, ($\boldsymbol{n - m}$) times.

And we know $\frac{1}{a^{n-m}} = a^{-(n-m)} = a^{m-n}$.

Thus either way, dividing a power $\boldsymbol{a^m}$ by another power $\boldsymbol{a^n}$, we get a new power $\boldsymbol{a^{m-n}}$.

For instance, we get $\frac{3^3}{3^5} = \frac{3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = \frac{1}{3 \cdot 3} = \frac{1}{3^2} = 3^{-2} \Rightarrow \frac{3^3}{3^5} = 3^{3-5} = 3^{-2} = \frac{1}{3^2}$.

And we get $\frac{(\frac{1}{3})^3}{(\frac{1}{3})^5} = \frac{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}}{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}} = \frac{1}{\frac{1}{3} \cdot \frac{1}{3}} = \frac{1}{(\frac{1}{3})^2} = (\frac{1}{3})^{-2} \Rightarrow \frac{(\frac{1}{3})^3}{(\frac{1}{3})^5} = (\frac{1}{3})^{3-5} = (\frac{1}{3})^{-2}$.

So doing a division with two powers with the same base $\boldsymbol{a}$, we get a new power with the same base $\boldsymbol{a}$, and the magnitude of the new exponent is the difference between the two exponents of the two powers.

Thus, we get $a^m/a^n = \dfrac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$, and both $m$ and $n$ are integers.

**2.** $(a^m)^n = a^{mn}$, where $a \neq 0$, and both $m$ and $n$ are integers.

Multiplying 1 by $a$, $m$ times, we get a power $a^m$.

So multiplying 1 by the power $a^m$, $n$ times, we get a new power $(a^m)^n$.

And let's see now what each multiplication can produce.

First, multiplying 1 by the power $a^m$, we just get $a^m$, which is the product of $m$ of $a$s.

Next, multiplying $a^m$ by $a^m$, that is, multiplying 1 by $a^m$, 2 times, we get $a^m \cdot a^m = (a^m)^2$.

We know $a^m$ is the product of $m$ of $a$s. So $a^m \cdot a^m$ is the product of $2m$ of $a$s.

Thus, $(a^m)^2$ is the product of $2m$ of $a$s, too. So we get $(a^m)^2 = a^{2m}$.

Next, multiplying $a^{2m}$ by $a^m$, that is, multiplying 1 by $a^m$, 3 times, we get $a^m \cdot a^m \cdot a^m = (a^m)^3$.

Also, we know $a^m$ is the product of $m$ of $a$s. So $a^m \cdot a^m \cdot a^m$ is the product of $3m$ of $a$s.

Thus, $(a^m)^3$ is the product of $3m$ of $a$s, too. So we get $(a^m)^3 = a^{3m}$.

Next, multiplying $a^{2m}$ by $a^{2m}$, that is, multiplying 1 by $a^m$, 4 times, we get $a^{2m} \cdot a^{2m} = (a^{2m})^2$.

We know $a^{2m}$ is the product of $2m$ of $a$s. So $(a^{2m} \cdot a^{2m})$ is the product of $4m$ of $a$s.

Thus, $(a^{2m})^2$ is the product of $4m$ of $a$s, too. So we get $(a^{2m})^2 = a^{4m}$.

And we have $a^m a^n = a^m \cdot a^n = a^{m+n}$.  So we get $a^{4m} = a^m \cdot a^m \cdot a^m \cdot a^m = (a^m)^4$.

Thus, we get $a^{4m} = (a^m)^4$.  Now, what then do we get multiplying 1 by $a^m$, $n$ times?

We get $(a^m)^n = a^{mn}$.  So for instance, $(5^3)^2 = (5\cdot5\cdot5)\cdot(5\cdot5\cdot5) = 5\cdot5\cdot5\cdot5\cdot5\cdot5 = 5^{3\cdot2} = 5^6$.

Also, dividing 1 by $a$ twice, we get $a^{-2}$, which is $\frac{1}{a^2}$, which is $1/a/a$, too.

So dividing 1 by $a^3$ twice, we can get $(a^3)^{-2} = 1/(a^3)/(a^3) = 1/(a^3 \cdot a^3) = 1/a^6 = a^{-6} = a^{3(-2)}$.

And multiplying 1 by $a^{-3}$ twice, we can put it this way: $(a^{-3})^2$, and we get
$(a^{-3})^2 = (1/a^3)(1/a^3) = 1/(a^3 \cdot a^3) = 1/a^6 = a^{-6} = a^{3(-2)}$, too.  So we get $(a^3)^{-2} = (a^{-3})^2 = a^{-6}$.

And dividing 1 by $a^{-3}$ twice, we get $(a^{-3})^{-2} = 1/(a^{-3})/(a^{-3}) = 1/(a^{-3} \cdot a^{-3}) = 1/a^{-6} = a^6$.
And we know $a^6 = a^{(-3)(-2)}$.

- So we get $(a^m)^n = a^{mn}$, where $a \neq 0$, and $m$ and $n$ are integers.

So if a power gets raised to another power, keep the base, and take the product of the exponents.

**3.** $(ab)^n = a^n b^n$, where both $a$ and $b \neq 0$, and both $m$ and $n$ are integers.

Multiplying 1 by $a$, $n$ times, we get a power $a^n$.

So multiplying 1 by a product $ab$, $n$ times, we get a power $(ab)^n$.

And let's see now what each multiplication can produce.

First, multiplying 1 by the product $ab$, we just get $ab$, which is the product of 1 of $a$s and 1 of $b$s.
Next, multiplying $ab$ by $ab$, that is, multiplying 1 by $ab$, 2 times, we get
$(ab)^2 = ab \cdot ab = aabb$, because multiplications are commutative, that is, $xy = yx$.

So we get $(ab)^2 = a^2b^2$.

Next, multiplying 1 by $ab$, 3 times, we get

$(ab)^3 = ab \cdot ab \cdot ab = aaabbb$, since multiplications are commutative.

So we get $(ab)^3 = a^3b^3$.

Now, what then do we get multiplying 1 by $ab$, $n$ times?

We get $(ab)^n = a^nb^n$. So for instance, $(3\cdot5)^3 = (3\cdot5)\cdot(3\cdot5)(3\cdot5) = 3\cdot5\cdot3\cdot5\cdot3\cdot5 = 3\cdot3\cdot3\cdot5\cdot5\cdot5 = 3^35^3$, and $(12\cdot7\cdot5)^2 = 12^27^25^2$.

Also, dividing 1 by $ab$ twice, we get $(ab)^{-2}$, which is $1/(ab)/(ab)$, which is $1/(ab)^2$, too.

So we get $(ab)^{-2} = 1/(ab)^2 = 1/(a^2b^2) = (1/a^2)(1/b^2) = a^{-2}b^{-2}$.
Thus, we get $(ab)^{-2} = a^{-2}b^{-2}$.

Besides, we have $(ab)^0 = 1$, since multiplying 1 by $ab$ no times, we just get 1.

And we have $a^0 = 1$, and $b^0 = 1$, so we get $a^0b^0 = 1$. Thus, we get $(ab)^0 = a^0b^0$.

So we get $(ab)^n = a^nb^n$, where both $a$ and $b \neq 0$, and both $m$ and $n$ are integers.

Thus, if the base is a product, take a product of powers.

**4.** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where both $a$ and $b \neq 0$, and both $m$ and $n$ are integers.

Multiplying 1 by $a$, $n$ times, we get a power $a^n$.

So multiplying 1 by a fraction $\frac{a}{b}$, $n$ times, we get a power $\left(\frac{a}{b}\right)^n$.

And let's see now what each multiplication can produce.

First, multiplying 1 by the fraction $\frac{a}{b}$, we just get $\frac{a}{b}$.

Next, multiplying $\frac{a}{b}$ by $\frac{a}{b}$, that is, multiplying 1 by $\frac{a}{b}$, 2 times, we get

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b}\cdot\frac{a}{b} = \frac{aa}{bb}. \quad \text{So we get } \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}.$$

Next, multiplying 1 by $\frac{a}{b}$, 3 times, we get

$$\left(\frac{a}{b}\right)^3 = \frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b} = \frac{aaa}{bbb}. \quad \text{So we get } \left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}.$$

Now, what then do we get multiplying 1 by $\frac{a}{b}$, $n$ times?

We get $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$. So for instance, $\mathbf{(3/5)^3 = (3/5)(3/5)(3/5) = 3\cdot 3\cdot 3/(5\cdot 5\cdot 5) = 3^3/5^3}$, and $\mathbf{(12/7/5)^2 = 12^2/7^2/5^2}$. By the way, we have $12/7/5 = 12/(7\cdot 5) = 12/35$.

Also, dividing 1 by $\frac{a}{b}$ twice, we get $\mathbf{(a/b)^{-2}}$, which is $\mathbf{1/(a/b)/(a/b)}$, which is $\mathbf{1/(a/b)^2}$, too.

So we get $\mathbf{(a/b)^{-2} = 1/(a/b)^2 = 1/(a^2/b^2) = b^2/a^2}$.

That is, we get $\left(\frac{a}{b}\right)^{-2} = \frac{1}{\left(\frac{a}{b}\right)^2} = \frac{1}{\frac{a^2}{b^2}}$.

And multiplying by $b^2$, the numerator and the denominator above, that is, multiplying the numerator 1 by $b^2$, and multiplying the denominator $\frac{a^2}{b^2}$ by $b^2$, we get $\frac{b^2}{a^2}$.

So we get $\left(\frac{a}{b}\right)^{-2} = \frac{1}{\left(\frac{a}{b}\right)^{2}} = \frac{1}{\frac{a^2}{b^2}} = \frac{b^2}{a^2}$. That is, we get $\left(\frac{a}{b}\right)^{-2} = \frac{b^2}{a^2}$.

And we have $b^2 = \frac{1}{b^{-2}}$, because $\frac{1}{b^{-2}} = b^{-(-2)} = b^2$.

Also, we have $\frac{1}{a^2} = a^{-2}$.

So we get $\frac{b^2}{a^2} = b^2\left(\frac{1}{a^2}\right) = \frac{1}{b^{-2}} \cdot a^{-2} = \frac{a^{-2}}{b^{-2}}$. Thus, we get $\frac{b^2}{a^2} = \frac{a^{-2}}{b^{-2}}$.

And we have $\left(\frac{a}{b}\right)^{-2} = \frac{b^2}{a^2}$, too.

Besides, we have $\left(\frac{a}{b}\right)^{0} = 1$, since multiplying 1 by $\frac{a}{b}$ no times, we just get 1.

And we have $a^0 = 1$, and $b^0 = 1$, so we get $a^0/b^0 = 1$. Thus, we get $(a/b)^0 = a^0/b^0$.

So in sum, we get $(a/b)^n = a^n/b^n$, where both $a$ and $b \neq 0$, and both $m$ and $n$ are integers.

For instance, we can get $\left(\frac{3}{5}\right)^{3} = \left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right) = \frac{3 \cdot 3 \cdot 3}{5 \cdot 5 \cdot 5} = \frac{3^3}{5^3}$.

And we can get $\left(\frac{3}{5}\right)^{-3} = \frac{1}{\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)} = \frac{1}{\frac{3 \cdot 3 \cdot 3}{5 \cdot 5 \cdot 5}} = \frac{1}{\frac{3^3}{5^3}} = \frac{5^3}{3^3} = 5^3 \cdot \frac{1}{3^3} = \frac{1}{5^{-3}} \cdot 3^{-3} = \frac{3^{-3}}{5^{-3}}$.

Too easy? So boring? Not quite?

Right after the next section, we will get the extended versions of the identities **2**, **3**, and **4** listed above. Thus, we will get some more identities, which can give us more power when we do exponential algebra.

They are basically the same though, as the ones listed above.
The exponents $m$ and $n$ used in the identities above are integers, but the exponents in the extended versions are rational numbers. So the actual extensions are done on exponents.

(In fact, the exponents can be irrational, too, and thus, can be real. It is not the case though, the exponents can be any real numbers for any bases. And we will see how it is not the case in the section, **Problem Exponents**.)

Prior to the extended version, that is, in the next section, we will get familiar with another important tool, called an $n^{\text{th}}$ root, which is a solution to an equation of degree $n$, and is in fact, another form of a power.
And we are going to use the idea of such a root constructing the version extended.

# 2. What is an N$^{th}$ Root?

Multiplying 1 by a number a particular number of times, we get a power made of a base and an exponent. For instance, multiplying 1 by 5, seven times, we get $5^7$, where 5 is the base, and 7 is the exponent. In general, multiplying 1 by $b$, $n$ times, we get $b^n$, where $b$ is the base, and $n$ is the exponent. And the same idea applies to divisions, too.

Dividing thus, 1 by a number a particular number of times, we get a power, too. For instance, dividing 1 by 3, eight times, we get $3^{-8}$, where 3 is the base, and -8 is the exponent. In general, dividing 1 by $b$, $n$ times, we get $b^{-n}$, where $b$ is the base, and $-n$ is the exponent. So we get a power, also, doing divisions. That is because a division can be taken for a multiplication by the reciprocal.

What if however, we want to express a value, which has the converse nature of a power? In other words, assuming we get a value if multiplying 1 by a certain value a particular number of times, we need to find the certain value.

- That is, given the value of a power and an exponent, we have to find the base.

Suppose for instance, multiplying 1 by $x$ seven times, we get 9.
What then is $x$? That is, by what value do we have to multiply 1, seven times to get 9?
So we want to find $x$, by which we multiply 1 seven times to get 9.

We call it a seventh root. What do we mean by though, such a root?

Saying a root in math, we can mean the solution to an equation. So finding the root, we can set up the equation describing the situation we have. And the situation is as follows.

- Multiplying 1 by an unknown value, 7 times, we get 9.

What equation then can we set up?

Assuming $x$ is the unknown, we can set $x^7 = 9$. Then, the value of $x$, that is, the solution to the equation above is called a 7th root. What then is an $n$th root?

It is the solution to an equation describing a situation as follows.

- Multiplying 1 by an unknown value, $n$ times, we get a value called $A$.

So assuming again, $x$ is the unknown, we can set up the equation the way as follows.

$x^n = A$ where $A > 0$, and $n$ is a ***positive integer***.

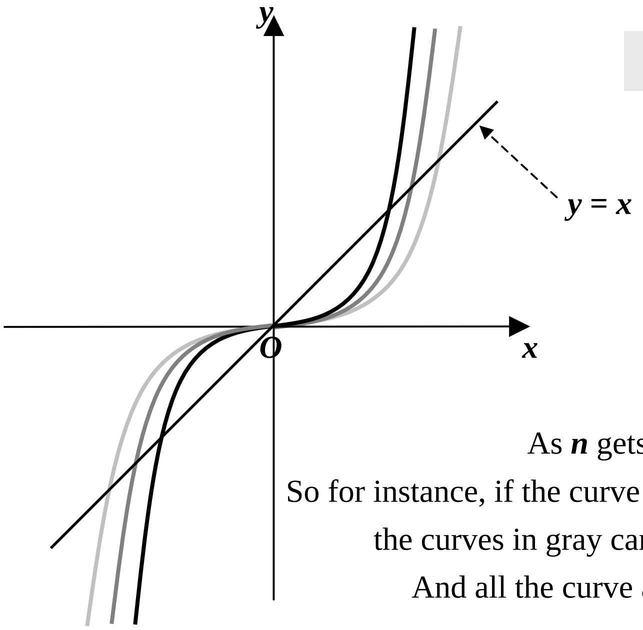
And we call the solution an $n$th root. How then can we get the solution, the $n$th root?

Let's first, take a look at some curves of an equation $y = x^n$ for some values of $n \geq 1$.

If $n = 1$, we just get a line passing through the origin and making 45° against the $x$-axis.

Next, if $n$ is odd, and $\geq 3$, we can put in the $x$-$y$ plane, the curves the way as follows.

**Fig. 2.0**

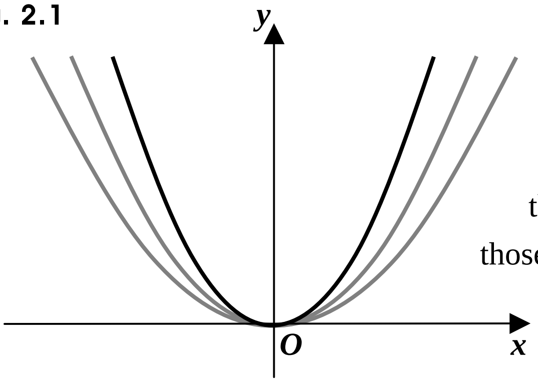


$y = x^n$ for $n$ odd, and $\geq 1$

As $n$ gets larger, the curve gets steeper.
So for instance, if the curve in black is the curve of $y = x^7$,
the curves in gray can be those of $y = x^5$ and $y = x^3$.
And all the curve are passing through the origin.

And next, if $n$ is even, and $\geq 2$, we can put in the $x$-$y$ plane, the curves the way below.

**Fig. 2.1**



$y = x^n$ for $n$ even, and $\geq 2$

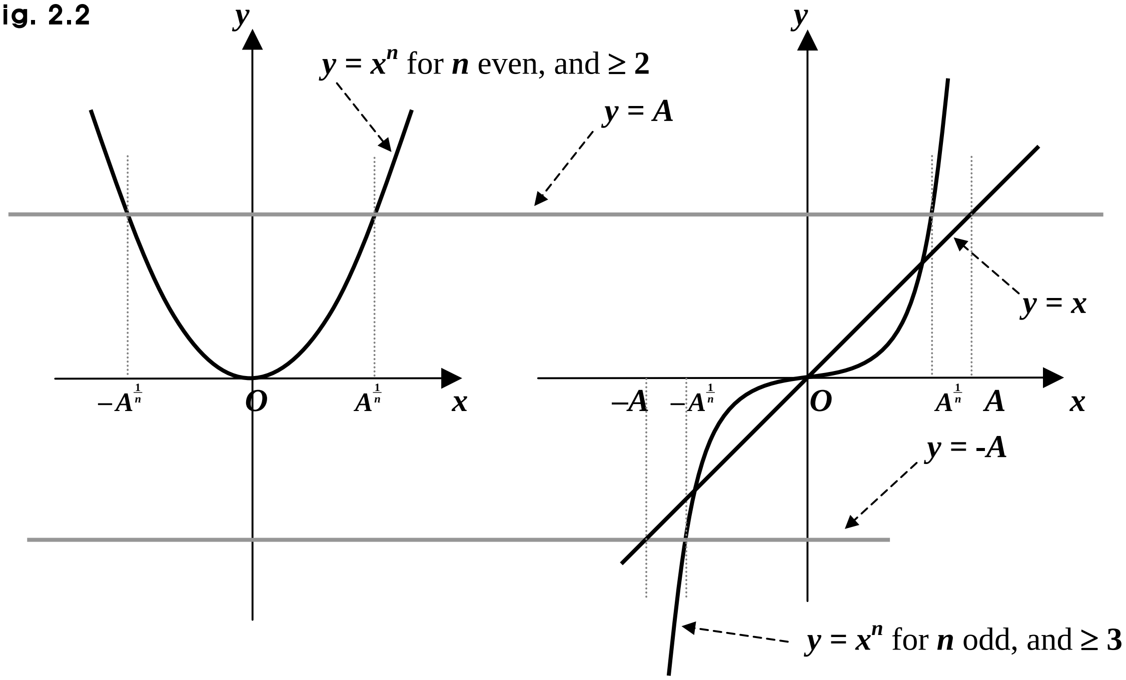
As $n$ gets larger, the curve gets steeper. So for instance, if the curve in black is the curve of $y = x^6$, the curves in gray can be those of $y = x^4$ and $y = x^2$. And all the curves are passing through the origin.

Let's now get back to the equation $x^n = A$ where $A > 0$, and $n$ is a ***positive integer***.

Then, considering the identity $(a^m)^n = a^{mn}$, where $a \neq 0$, and both $m$ and $n$ are integers, we can notice that we can put an $n^{\text{th}}$ root this way: $x = A^{\frac{1}{n}}$.

And putting schematically in the $x$-$y$ plane, the curves of $y = x^n$, $y = A$, and $y = -A$, we can put them the way below.

**Fig. 2.2**



Then, we can see that the solution can be either one $n^{\text{th}}$ root or two $n^{\text{th}}$ roots.

It depends on the value of $n$. So let's see now how it depends.

To begin with, if $n = 1$ or $A = 0$, there isn't much point of calling the solution an $n^{\text{th}}$ root. That's because if $n = 1$, we just get $x = A$, and if $A = 0$, we simply get $x = 0$.

Next, when $n = 2$ or $3$, we normally call the solution a ***square root*** or a ***cube root*** rather than a second root or a third root.

What if $A < 0$, though?

Then, only if $n$ is odd, we can get a solution, which is an $n^{\text{th}}$ root, too, which is negative though, as shown in the graph above. So if $n$ is even, we get no solution.

- Thus, if we want to get an $n^{\text{th}}$ root for ***any*** integer $n > 1$, we need to have $A > 0$.

As we can see in Fig. 2.2 above, if $A > 0$, we can get a solution for any positive integer $n$.

So putting threads together, we can say these:

If $n$ is even and $A > 0$, we get two $n^{\text{th}}$ roots, which are indicated by $A^{\frac{1}{n}}$ and $-A^{\frac{1}{n}}$.
If $n$ is odd, we get one $n^{\text{th}}$ root, which is indicated by $A^{\frac{1}{n}}$.
Either way, if $A > 0$, we get the $n^{\text{th}}$ root $A^{\frac{1}{n}}$, which is positive.

- If however, $A < 0$ and $n$ is even, we get no solution, that is, no root, and thus, we get no $n^{\text{th}}$ root.

So what is an $n^{\text{th}}$ root about?

It is about powers, and in fact, is a power, too.

Getting back to the equation, we have $x^n = A$, which is saying that multiplying 1 by an unknown value, ***n*** times, we get a value called ***A***.

And the solution is $x = A^{\frac{1}{n}}$, which is called an ***n***th root.

How then can we call ***A*** and $\frac{1}{n}$?

We can call ***A*** the **base**, and can call $\frac{1}{n}$ the exponent.

So the ***n***th root $A^{\frac{1}{n}}$ is a power, too, and thus, is about powers.

Besides, the equation $x^n = A$ is originally from the situation below.

- Multiplying 1 by a certain value, ***n*** times, we get a value called ***A***.

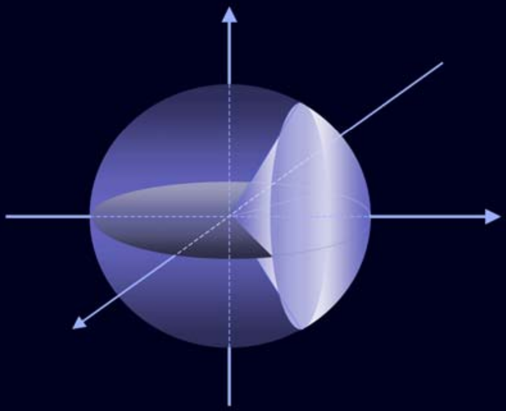
And we know now that the certain value is the ***n***th root, which is $A^{\frac{1}{n}}$.

So multiplying 1 by the ***n***th root $A^{\frac{1}{n}}$, ***n*** times, we get ***A***. And multiplying 1 by the ***n***th root $A^{\frac{1}{n}}$, ***n*** times, we get a power $(A^{\frac{1}{n}})^n$. Thus, we get $(A^{\frac{1}{n}})^n = A$.

That is, the ***n*th** root $A^{\frac{1}{n}}$ is the certain value we want, and multiplying 1 by it, ***n*** times, we get ***A***, which is the base in the power called the ***n*th** root, which is $A^{\frac{1}{n}}$.

Therefore, for instance, multiplying 1 by $9^{\frac{1}{7}}$, 7 times, we get 9, that is, we get $(9^{\frac{1}{7}})^7 = 9$.

So we can notice now that there can be exponential identities where fractions are used as exponents. And in fact, we are going to see what such identities are, and get those identities, so we’ll get to see how they work, and how to use them in the next section.



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## Powers and Logarithms

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