Similar Triangles

Click this to start.

Similar Triangles 1

When doing math, particularly doing geometry, we often work with triangles, together with angles and lines. And among triangles, some are similar, and some are not.

A similar triangle sounds easy. Naturally, by similar triangles, we mean those triangles that look alike. When it comes to a problem, though, it's not that easy.

Fig. 1

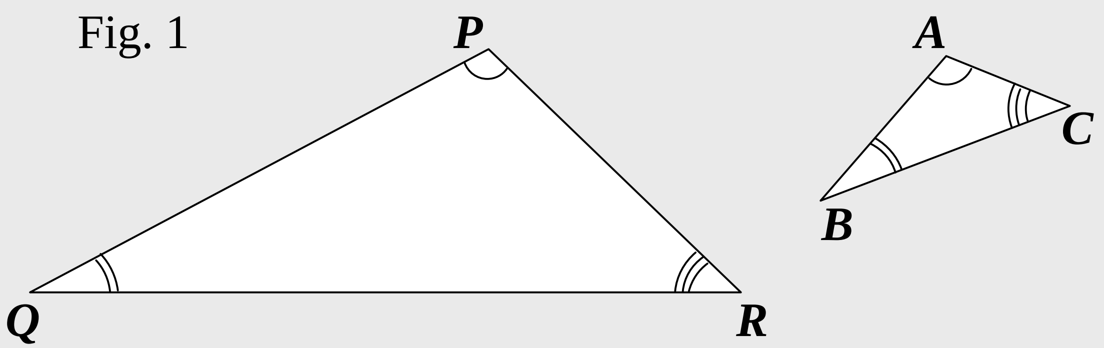
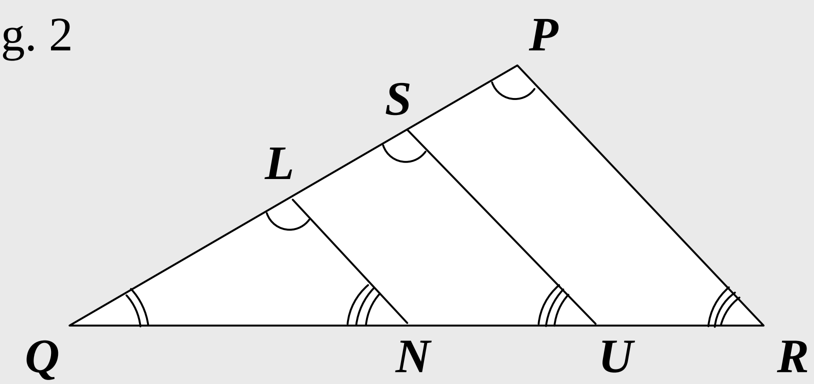


Fig. 2



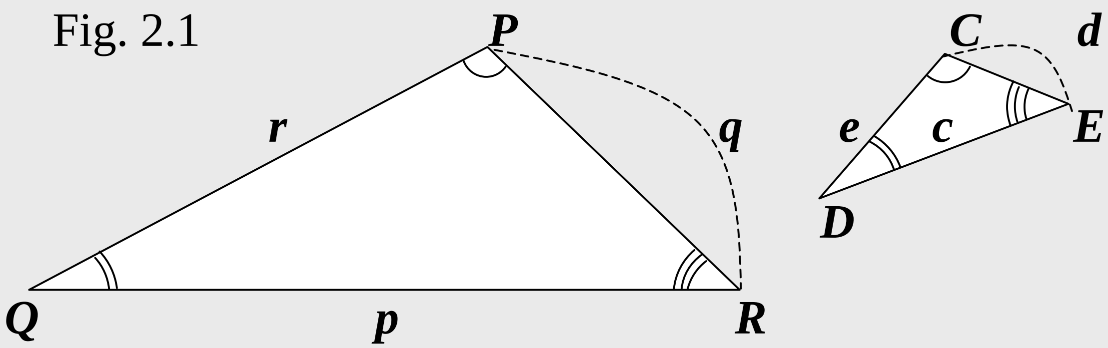
Similar triangles are important and crucial.

They are crucial to solutions to many problems.

So what do we mean by similar triangles?

Getting straight to the point, the bottom line, we can say that the definitions for similar triangles can be as follows.

Fig. 2.1



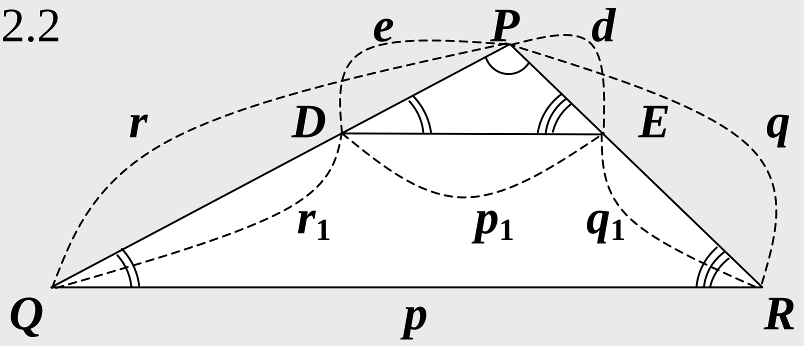
(1) $\Delta PQR \sim \Delta CDE \quad \Leftrightarrow \quad \frac{c}{p} = \frac{d}{q} = \frac{e}{r}$

(2) $\Delta PQR \sim \Delta CDE \quad \Leftrightarrow \quad \angle P = \angle C$, and $\angle Q = \angle D$

(3) $\Delta PQR \sim \Delta CDE \quad \Leftrightarrow \quad \angle P = \angle C$, and $\frac{e}{r} = \frac{d}{q}$

Also. we can say that the definitions for similar triangles can be as follows.

Fig. 2.2



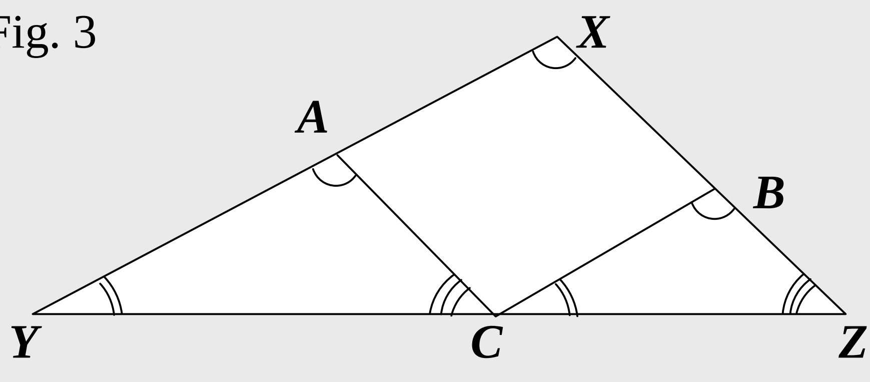
$$\text{(1-1)} \quad \Delta PQR \sim \Delta PDE \quad \Leftrightarrow \quad \frac{d}{q} = \frac{e}{r}$$

$$\text{(2-1)} \quad \Delta PQR \sim \Delta PDE \quad \Leftrightarrow \quad \angle Q = \angle D$$

$$\text{(3-1)} \quad \Delta PQR \sim \Delta PDE \quad \Leftrightarrow \quad \frac{e}{r_1} = \frac{d}{q_1}$$

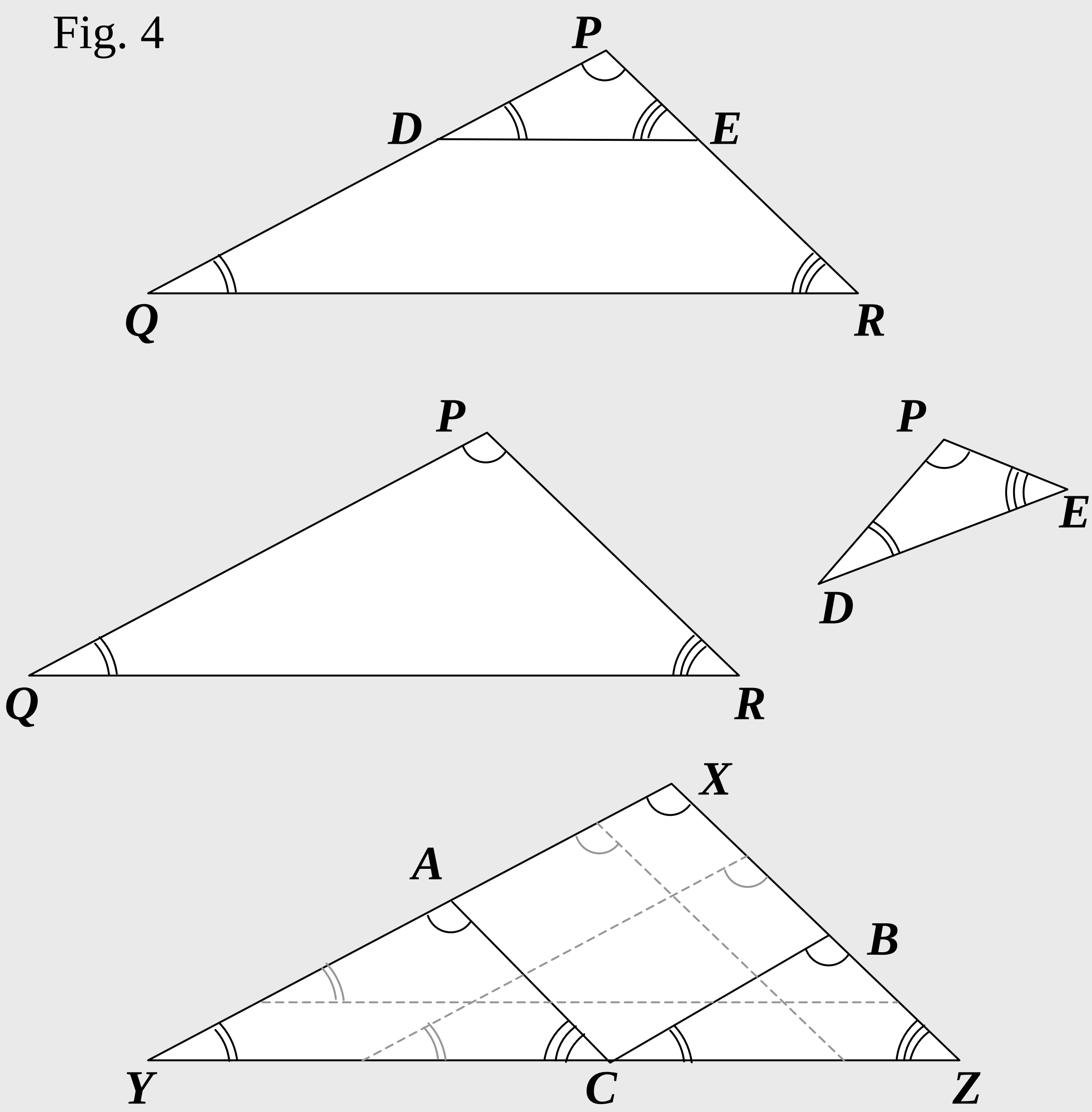
Let's start now, taking the steps to get the idea behind.

Fig. 3



If things resemble each other, we can say that they are similar. And usually, we are talking about their shapes.

Fig. 4



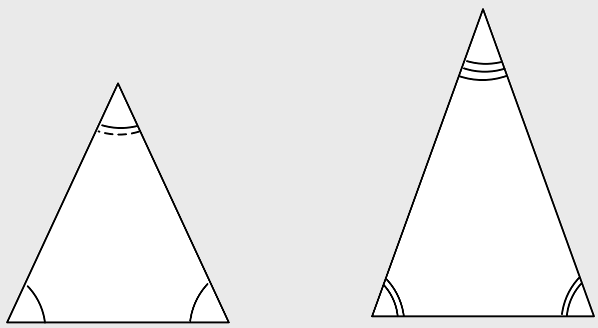
Doing math though, saying ***similar triangles***, we are not really talking about their shapes or looks.

And saying ***similar*** when doing math, we mean ***similar in math***, that is, ***similar mathematically***.

Of course, ***mathematically similar*** triangles can resemble each other or even look quite alike.

What then, about the triangles below?

Fig. 5

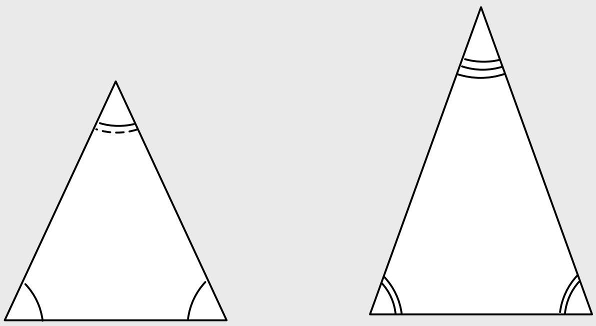


They resemble each other, don’t they? Aren’t they similar?

Though they look like each other, they are not similar.

Not similar mathematically, of course.

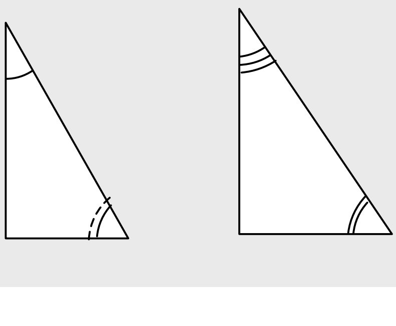
Fig. 6



The two triangles above are not similar.

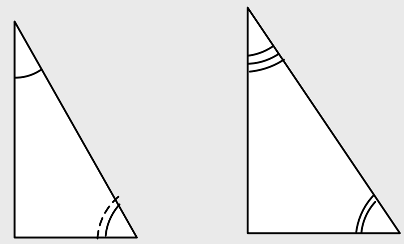
What about the two below?

Fig. 7



They look pretty much alike, but are not similar.

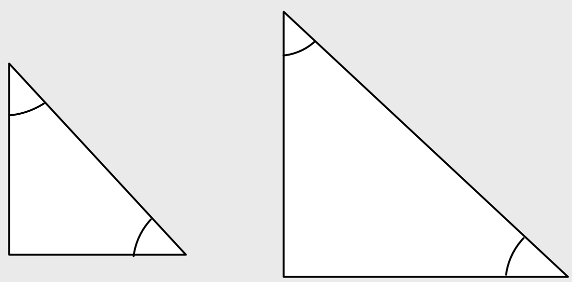
Fig. 7



They are not similar mathematically.

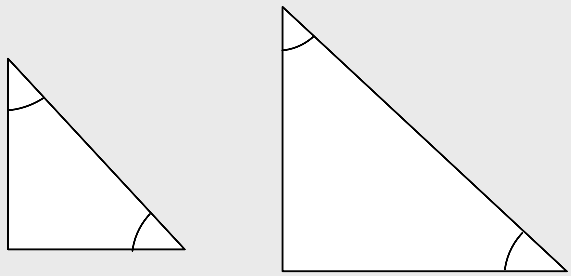
How bout these two below?

Fig. 8



Both not only look alike, but are similar, too.

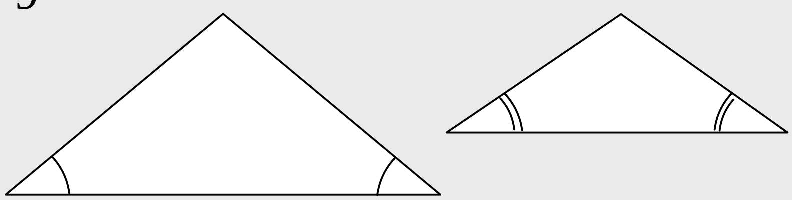
Fig. 8



They are similar mathematically, of course. ☺

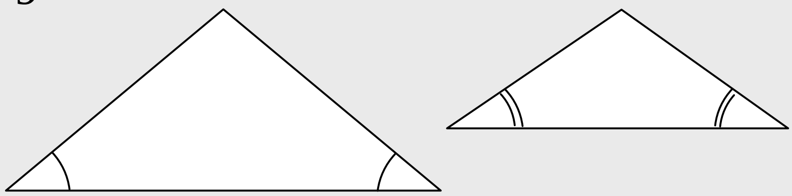
What then about the two below?

Fig. 9



They seem quite similar, but are not.

Fig. 9



They only look alike; they are not similar.

Not similar in math, of course.

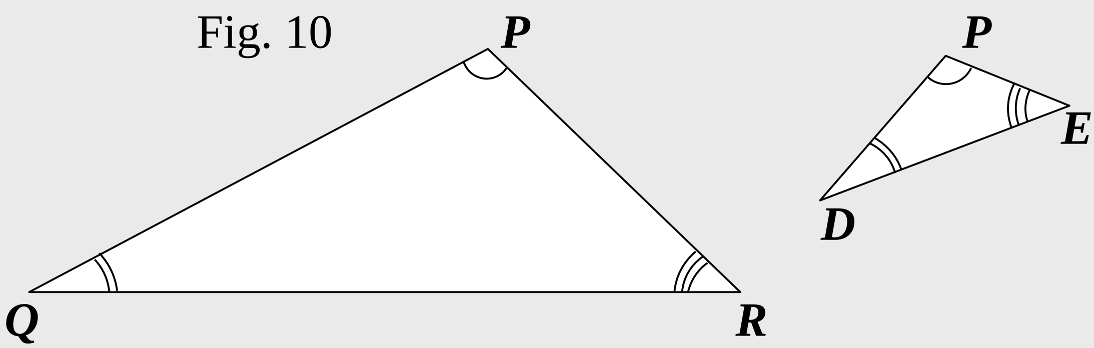
Looking alike but not similar? Not similar in math! ? ???

What then do we mean by similar triangles in math? 😐 ☺

Again, saying similar triangles doing math, we are talking about mathematical properties rather than their shapes.

In the figure below, when we do math, the two triangles ΔPQR and ΔPDE are said to be ***similar***.

Fig. 10



They are similar *mathematically*, of course.

Note that '~' is a math symbol meaning '***is similar to***'.

So indicating the two triangles above are similar, we can put it this way: $\Delta PQR \sim \Delta PDE$, which is therefore, read as

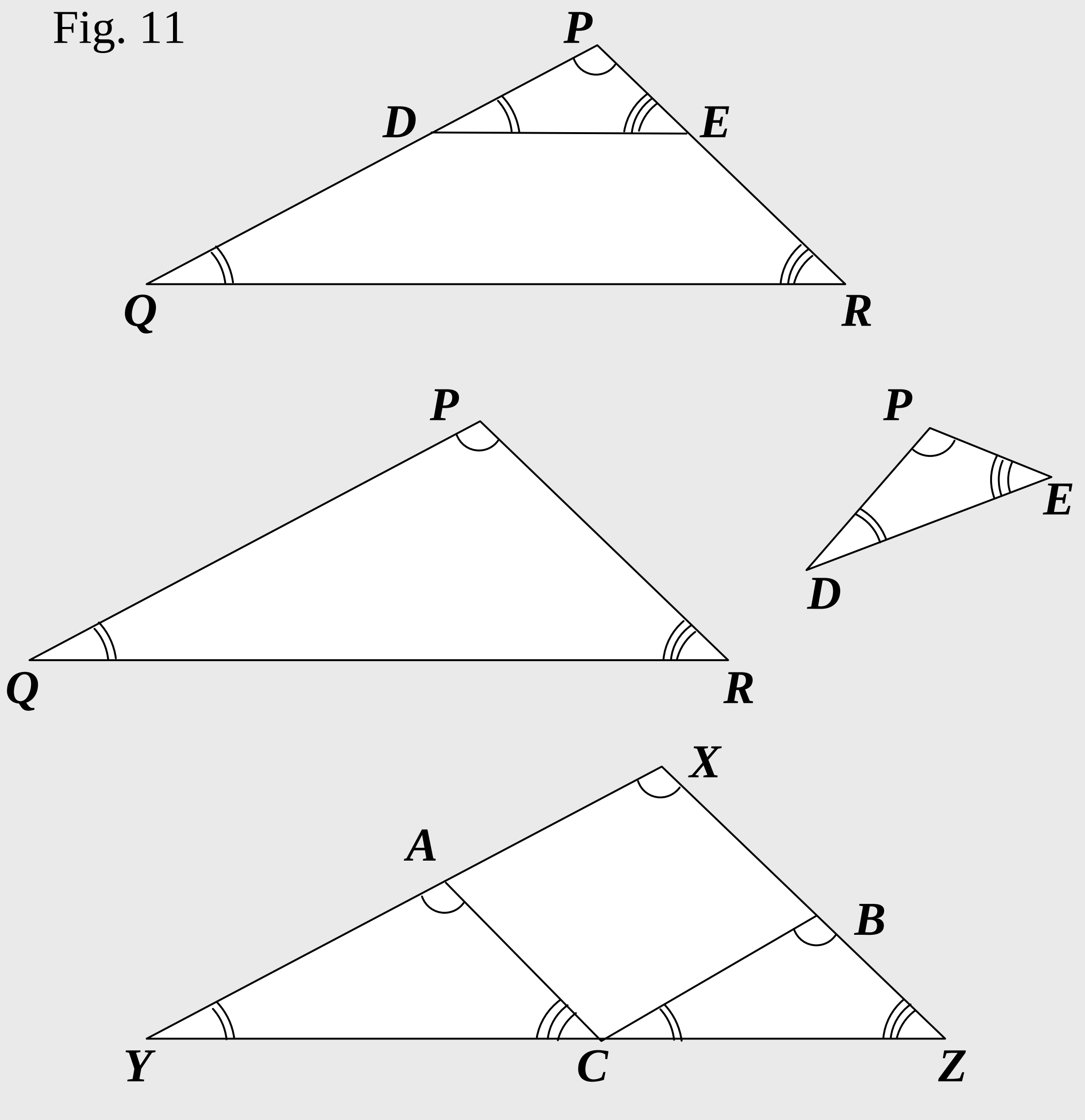
"The triangle PQR is similar to the triangle PDE." or

"ΔPQR and ΔPDE are similar to each other."

What if more than two triangles are similar?

Similar triangles can be ***nested*** together ***or*** can be ***apart*** from each other as shown below.

Fig. 11



In Fig. 11 above, we can see five similar triangles, which are these: ΔPQR, ΔPDE, ΔXYZ, ΔAYC, and ΔBCZ.

And indicating that the five above are similar to each other using the math symbol '~', we can put it the way as follows.

$$\Delta PQR \sim \Delta PDE \sim \Delta XYZ \sim \Delta AYC \sim \Delta BCZ.$$

The expression above is saying therefore,

"ΔPQR is similar to ΔPDE, which is similar to ΔXYZ, which is similar to ΔAYC, which is similar to ΔBCZ, "

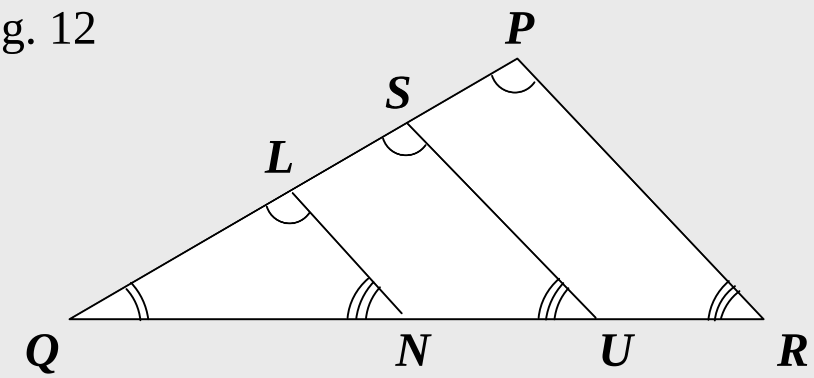
And of course, it is saying the statement below, too:

"ΔPQR, ΔPDE, ΔXYZ, ΔAYC, and ΔBCZ are all similar to each other"

They are similar mathematically, of course.

In the figure below, you can find three triangles that are similar. Find the three, and indicate that they are similar using the math symbol '~'.

Fig. 12



Also, in the figure below, you can find three similar triangles, so find the three, and indicate that they are similar using the math symbol.

Fig. 13

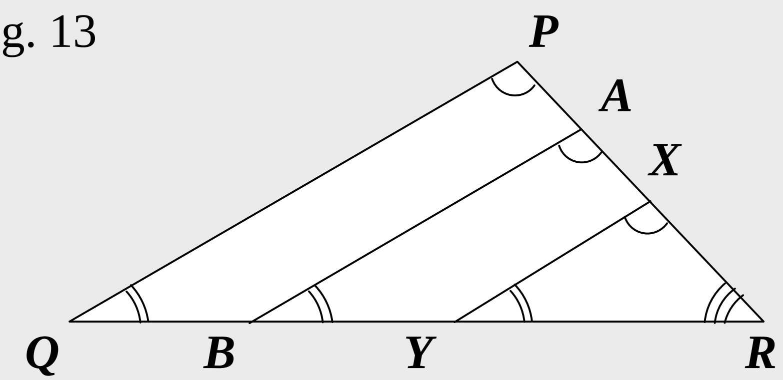
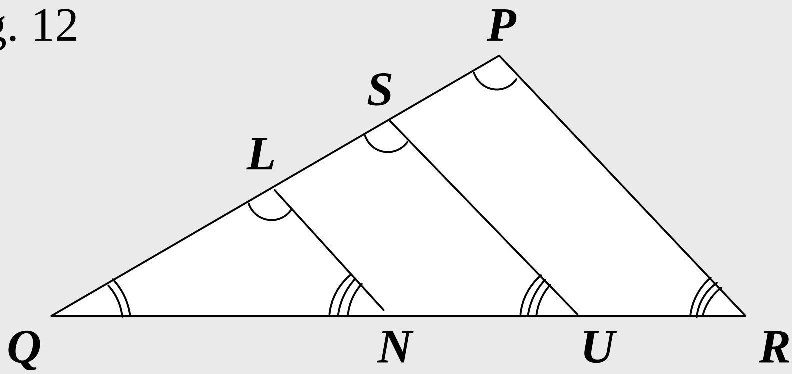
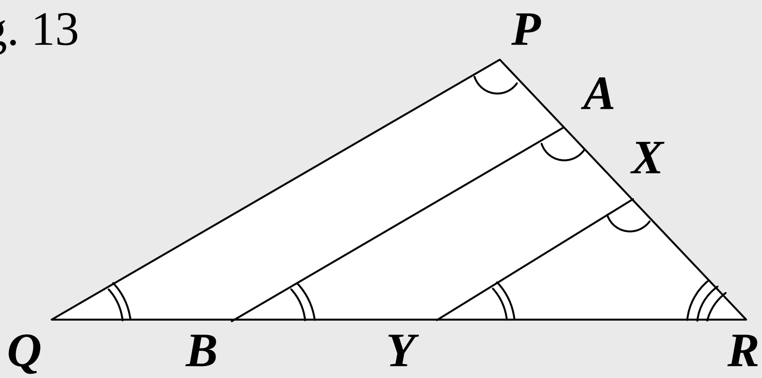


Fig. 12



$\Delta PQR \sim \Delta SQU \sim \Delta LQN$

Fig. 13



$\Delta PQR \sim \Delta ABR \sim \Delta XYR$

And putting together the two facts above, we get this:

$\Delta PQR \sim \Delta SQU \sim \Delta LQN \sim \Delta SQU \sim \Delta LQN$.

So all the five triangles are similar to each other.

What triangles are then similar mathematically?

Though they are often said to be ***the triangles that share the same shape but have different sizes***, it seems kind of foggy to some students. And in fact, many don't just get it.

They can decipher the statement above describing what similar triangles are only after they got the math idea behind. Getting the idea, they can understand it. They seem to start getting the idea getting and thinking about the answer to a question.

And the question is as follows.

What do we mean by the same shape & the different sizes?

The same shape?
What do we mean by a size of a triangle?

Well, a triangle is a shape. It is, of course.

So is a quadrangle, and the same is true of a circle. Too.

So do all triangles share the same shape, and are they all similar?

Of course not. Many are similar, yet many are not.
In math, the similarity doesn't have much to do with shapes, but has a lot to do with some math properties.

So it's about ***not*** appearances or looks but math properties.

Math properties often used, very often.

What math properties, though?

They are exactly what you gonna get from this lesson.
And you'll often use them solving problems.

They are ***easy***, ***wieldy*** when you get the ***concept*** and the ***fluency***, which you get doing examples with the help here.

And they are ready to be used in your math tool box.

So they are pretty handy tools ready for use when you take tests, exams, and of course, when you do your homework.

The math properties will be yours as you understand, make sure of, and fully grab the triangle basics covered here. So let's now, continue with this lesson to get the concepts of the math properties.

Now, before moving on, though, you may want to quickly go over some important math properties on the same angles.

First, vertical angles are two same angles.

If two lines meet at a point, we get two different pairs of vertical angles, which are two same angles.

Next, angles with parallel lines are the same.

The angles made by parallel lines and the transversal are the same.

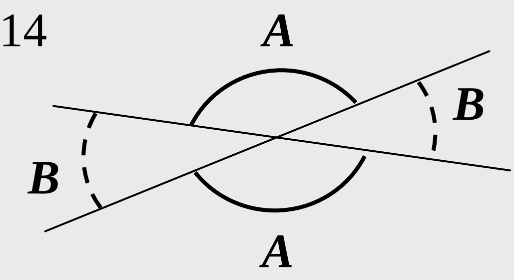
And next, with parallel lines, corresponding angles are equal, and so are alternate angles.

If the lines crossed by the transversal are parallel, corresponding angles are equal, and so are alternate angles.

And we can put the math properties above in figures the way as follows.

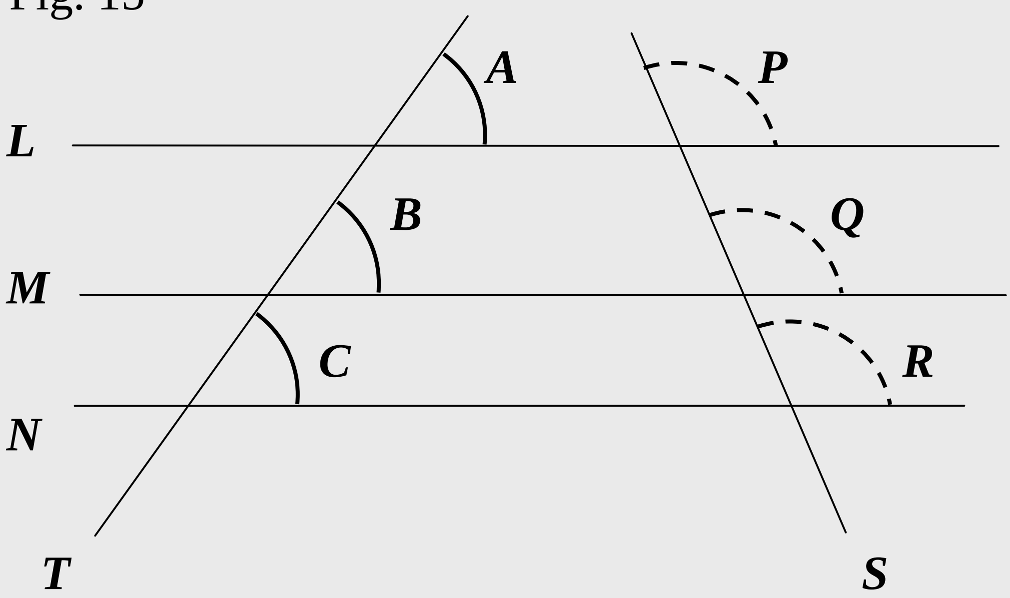
First off, vertical angles are two same angles, and are made by two lines crossing at a point as shown below.

Fig. 14



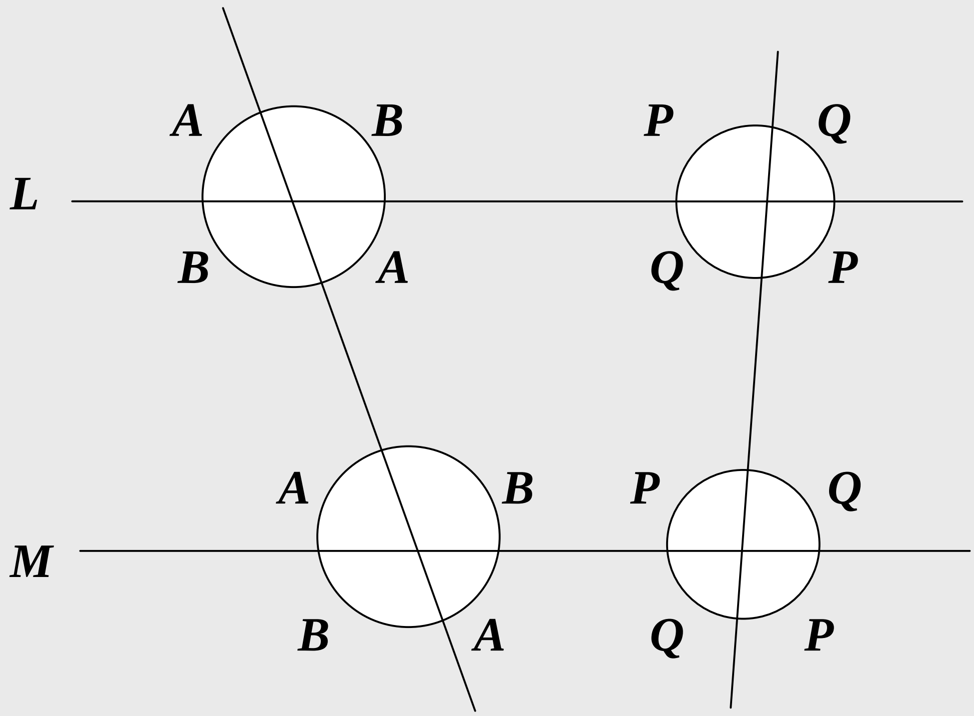
Next, in the figure below, if we have this: $L \parallel M \parallel N$, we get these: $\angle A = \angle B = \angle C$, and $\angle P = \angle Q = \angle R$.

Fig. 15



And assuming ***L* // *M*,** we have this:

Fig. 16



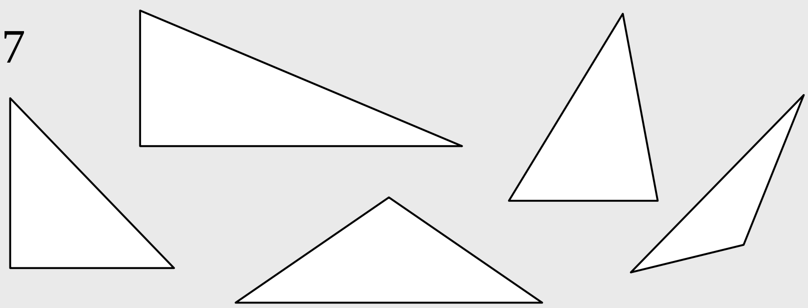
So if two lines crossed by the transversal are parallel, not only vertical angles but corresponding and alternate angles are the same, too.

If not quite sure of those math properties above, you may want to stop for a moment, and move to the lessons called ***Angles and Lines*** that can help get familiar with those properties. Then, continue with these lessons.

Similar Triangles 2

Now, back to the discussions on similar triangles.

Fig. 17



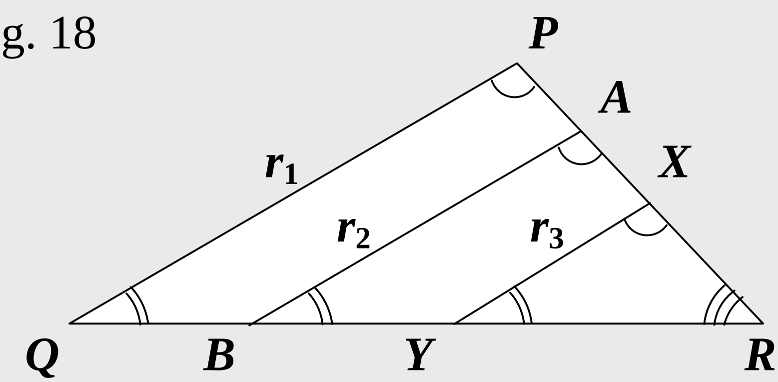
The triangles above don't share the same shape.

So in math, they are said to have different shapes, and are not similar. And indeed, they all look quite different.

Some do share the same shape; thus, they are similar.

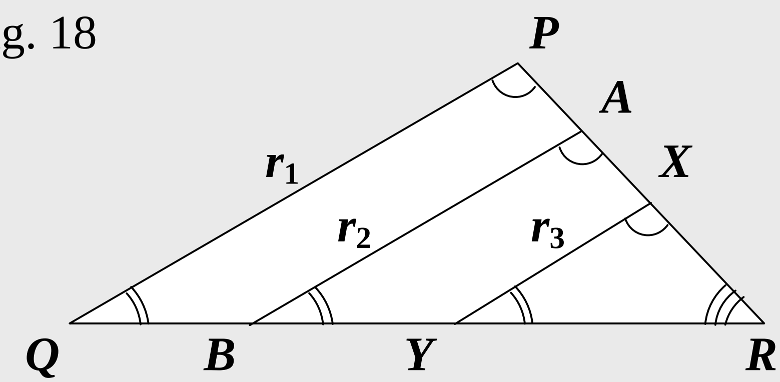
And some of the examples are shown below.

Fig. 18



In the figure below, we can say that the sides r_1, r_2, and r_3 are all ***parallel*** to each other.

Fig. 18



Also, we can say that the three triangles ΔPQR, ΔABR, and ΔXYR are all similar to each other.

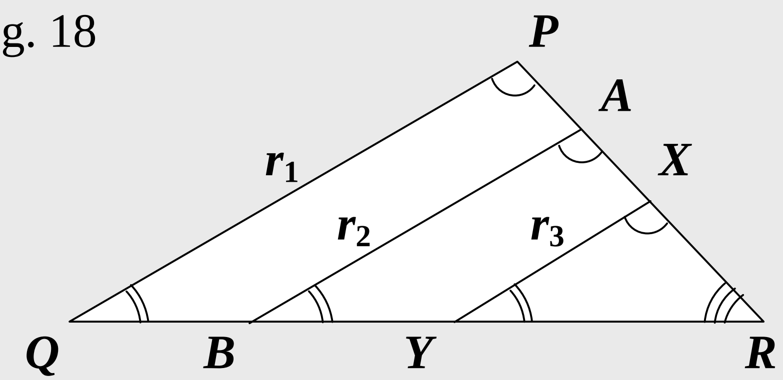
And using the math symbol, we can put them this way:
$\Delta PQR \sim \Delta ABR \sim \Delta XYR$.

Note that the sides r_1, r_2, and r_3 are all ***parallel***.

Why are they parallel?

it's because we have this: $\angle PQR = \angle ABR = \angle XYR$.

Fig. 18



Or equivalently, we have this: $\angle QPR = \angle BAR = \angle YXR$.

And vice versa, so if the sides are parallel, the angles are equal.

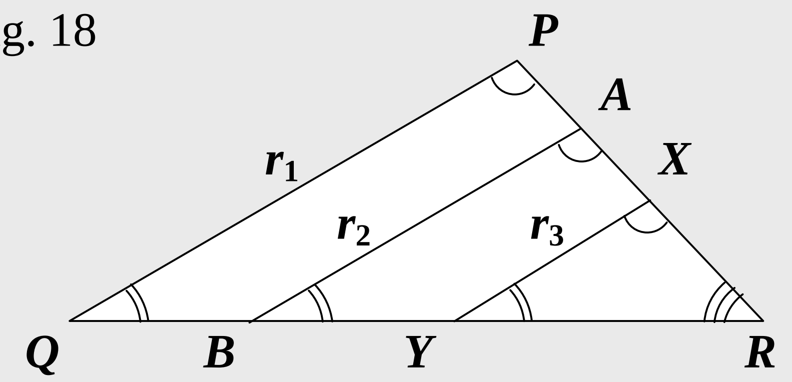
Thus, in short,

> ***angles with parallel lines*** are the same, and
> ***lines with the same angles*** are parallel.

Why are the three angles, though, all the same?

It's because the three triangles ΔPQR, ΔABR, and ΔXYR are all ***similar*** to each other.

Fig. 18



Since ΔPQR, ΔABR, and ΔXYR are ***similar***, we get these:

$\angle PQR = \angle ABR = \angle XYR$, and $\angle QPR = \angle BAR = \angle YXR$.

And since we have this: $\angle PQR = \angle ABR = \angle XYR$, the sides r_1, r_2, and r_3 are all ***parallel***.

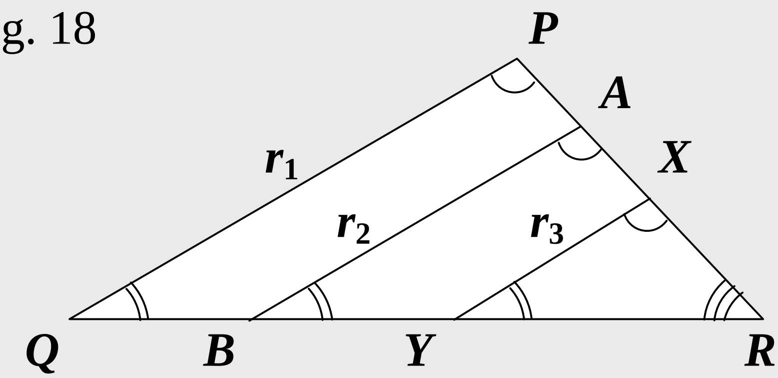
So since ΔPQR, ΔABR, and ΔXYR are ***similar***, the sides r_1, r_2, and r_3 are all ***parallel***.

And more importantly, all the reverses of the three statements above are true. And the same is true of this, too: $\angle QPR = \angle BAR = \angle YXR$.

We know that since ΔPQR, ΔABR, and ΔXYR are ***similar***, we get these:

$\angle PQR = \angle ABR = \angle XYR$, and $\angle QPR = \angle BAR = \angle YXR$.

Fig. 18



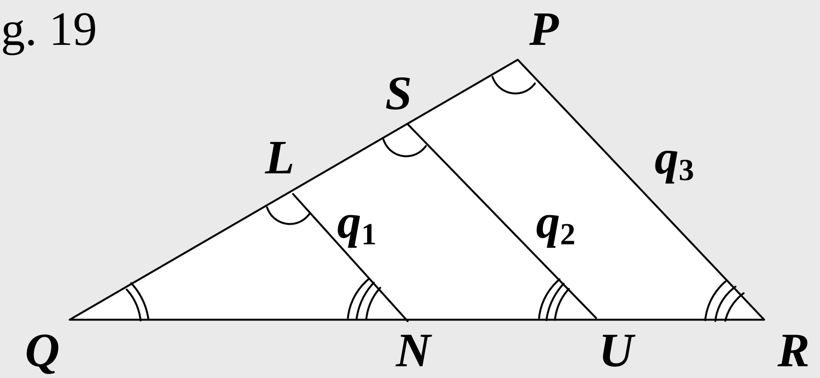
And since we have this: $\angle QPR = \angle BAR = \angle YXR$, the sides r_1, r_2, and r_3 are all ***parallel***.

So since ΔPQR, ΔABR, and ΔXYR are ***similar***, the sides r_1, r_2, and r_3 are all ***parallel***.

And again, the reverses of all the statements above are true. So for instance, since the sides r_1, r_2, and r_3 are all ***parallel***, we have this: $\angle QPR = \angle BAR = \angle YXR$, and this, too: $\angle PQR = \angle ABR = \angle XYR$.

Next, in Fig. 19 below. we can also, say that the three sides q_1, q_2, and q_3 are all ***parallel*** to each other,

Fig. 19



And we can say that the three triangles ΔPQR, ΔSQU, and ΔLQN are all similar to each other.

Also, using the math symbol, we can put them this way:

$\Delta PQR \sim \Delta SQU \sim \Delta LQN$.

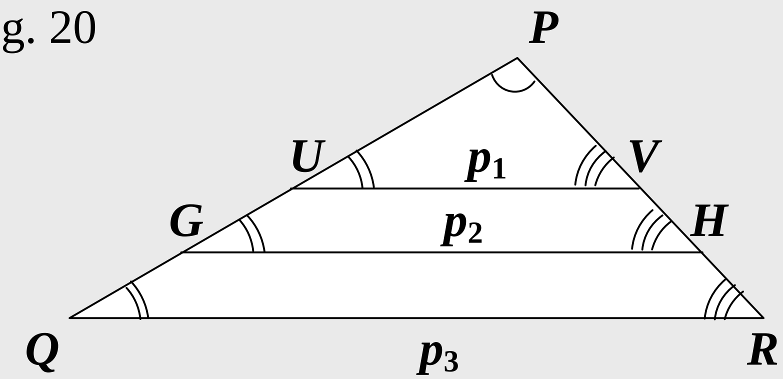
And as in the case of Fig. 18, the sides q_1, q_2, and q_3 are all ***parallel*** to each other, because we have this:

$\angle QNL = \angle QUS = \angle QRP$, or equally, we have this:

$\angle QLN = \angle QSU = \angle QPR$.   And, of course, vice versa, so if the sides are parallel, the angles are the same.

And next, in Fig. 20 below, we can again, say that the sides $\boldsymbol{p_1}$, $\boldsymbol{p_2}$, and $\boldsymbol{p_3}$ are all ***parallel***.

Fig. 20



Also, we can, of course, say that the three triangles $\boldsymbol{\Delta PUV}$, $\boldsymbol{\Delta PGH}$, and $\boldsymbol{\Delta PQR}$ are all similar, and can indicate the fact the way as follows: $\boldsymbol{\Delta PUV \sim \Delta PGH \sim \Delta PQR}$.

And many people will probably say that the triangles above look pretty much alike with some differences in size.

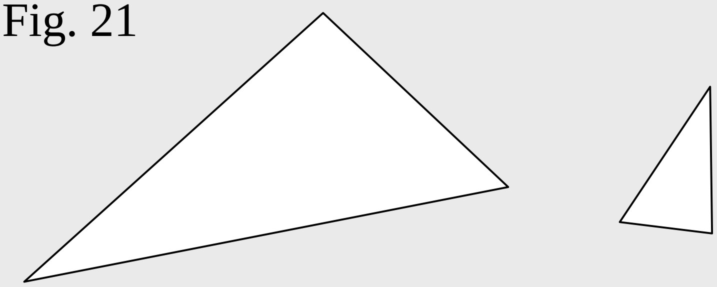
Indeed, it was stated earlier that similar triangles share the same shape but have different sizes, which is not however, the main idea in the concept of the similarity.

The similarity is not much about shapes or looks triangles have, but is a lot about ***the math properties similar triangles have to share between themselves***.

Similar triangles can look alike, and can even share exactly the same appearance.
And yet they can have different looks, also.

In fact, they can even look very different from each other, because, for instance, as shown below, one of two similar triangles can be way much bigger, and both can have different orientations respectively.

Fig. 21



The two triangles above are actually similar to each other.

So the two share the same shape.

They don’t look though, like each other.

And they can even look quite different.

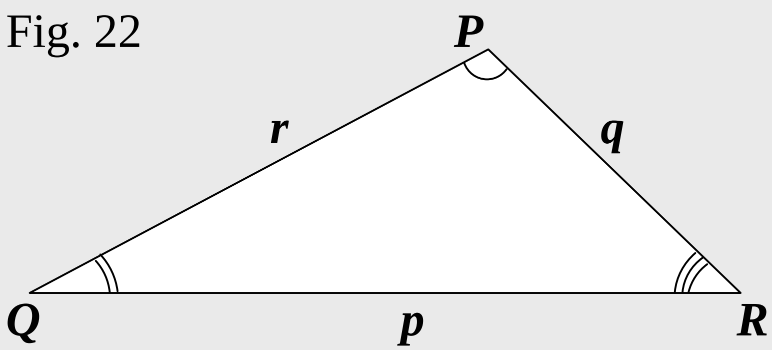
What then, do we mean by the same shape?!? 😐 ☺

It's not just a shape. In the math on triangles, the same shape has to do with not only shapes, i.e., appearances or looks, but also the *components making a triangle*.

The shapes don't really matter; the components do.

What are the components? What makes a triangle?

Fig. 22



What is the triangle above made of?

It’s not just made of sides but angles, too.

So a triangle is made of angles and sides.

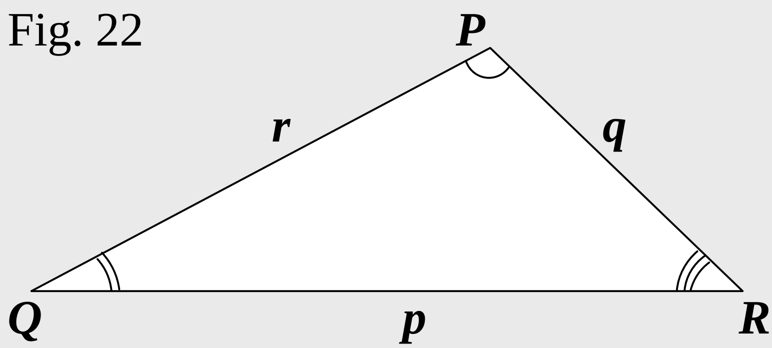


Fig. 22

In Fig. 22, ΔPQR is made of three angles, which are $\angle P$, $\angle Q$, and $\angle R$, along with three sides, which are p, q, and r.

By the way, saying ***sides*** talking about triangles, rectangles, etc., we often mean ***their lengths***, as well as ***line segments***.

So, we can say that a triangle is made of two groups, one is a group of three ***angles***, and the other is a group of three *line segments*, usually just called ***sides***.

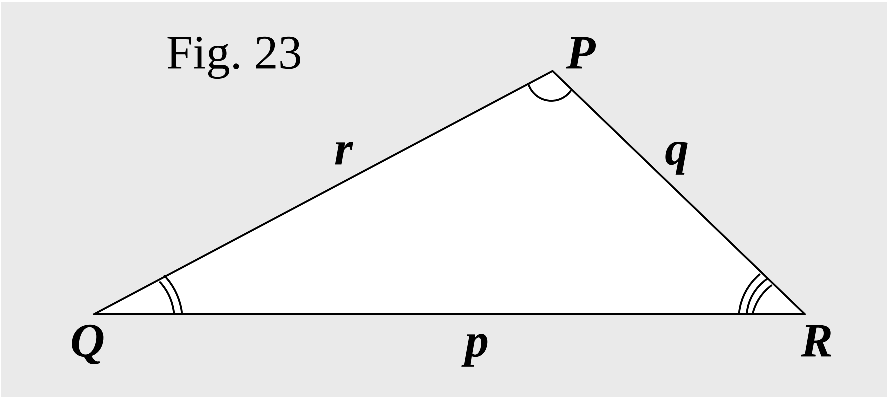
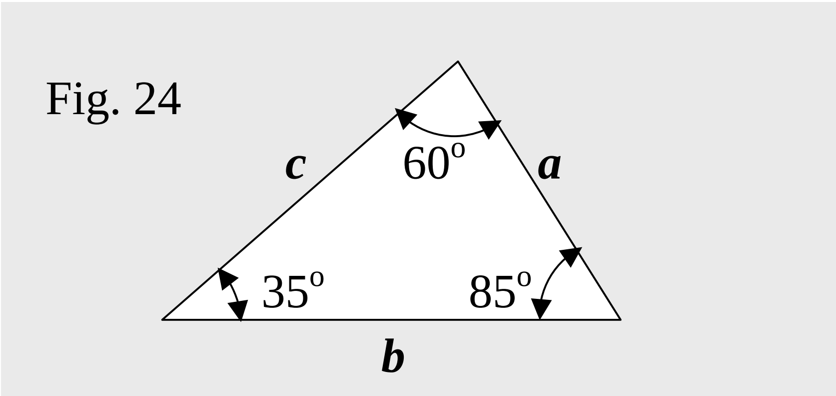


Fig. 23

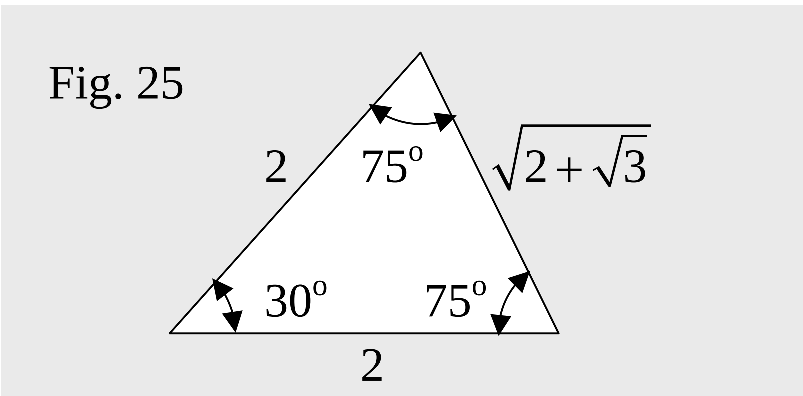
The triangle above is called **the triangle PQR**, and using a symbol, Δ, we can put it this way: ΔPQR, where the angle group is ($\angle P$, $\angle Q$, $\angle R$) and the side group is (p, q, r).

So what are the components making a triangle?

They are three angles and three sides. And more examples are as follows.



The triangle above is made of three sides, ***a***, ***b***, and ***c***, and the three angles, 35^{o}, 60^{o}, and 85^{o}.

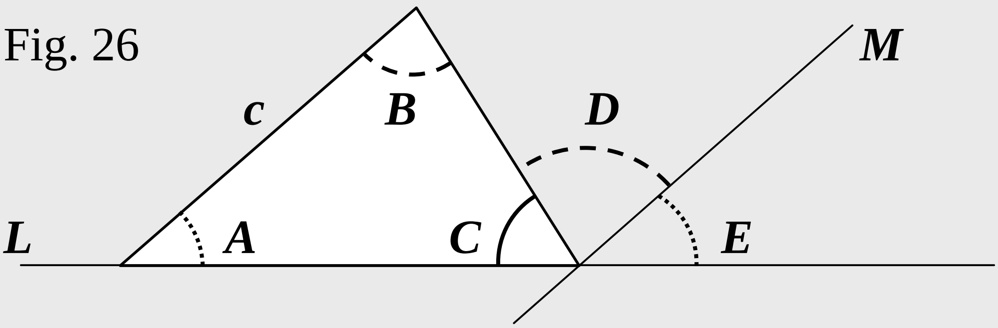


The triangle above is made of three sides, 2, $\sqrt{2+\sqrt{3}}$, and 2, and the three angles, 30^{o}, 75^{o}, and 75^{o}.

Notice that the sum of the three angles is 180^o, which is true no matter what the triangle it may be. Why, though?

To begin with, let's now, put a triangle on a line ***L*** as shown in the figure below.

Fig. 26



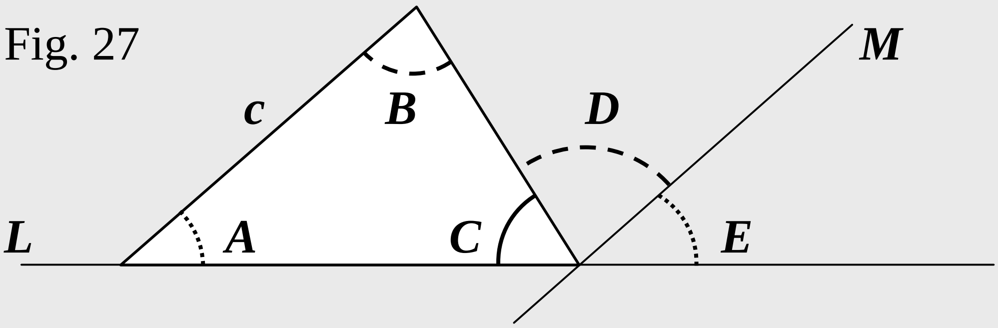
Suppose next, the side ***c*** is parallel to the line ***M*** shown above.

Then, the line ***L*** is the transversal, and we can say that the two angles $\angle A$ and $\angle E$ are corresponding angles that are equal, and the two angles $\angle B$ and $\angle D$ are alternate angles that are equal. That is, we get $\angle A = \angle E$, and $\angle B = \angle D$.

So next, what can we get?

So we can get this: $\angle A + \angle B + \angle C = \angle E + \angle D + \angle C$, which is a straight angle, which is 180^{o}.

Fig. 27



That is to say that in a triangle, the sum of all the three (internal) angles is 180^{o}. Besides, notice that in a triangle, the sum of two internal angles is equal to an external angle, which is supplement to the other internal angle.   Why?

First, we have this: $\angle A + \angle B + \angle C = \angle E + \angle D + \angle C$, which is 180^{o}, so we have this: $\angle A + \angle B = \angle E + \angle D$, which is the sum of two internal angles.

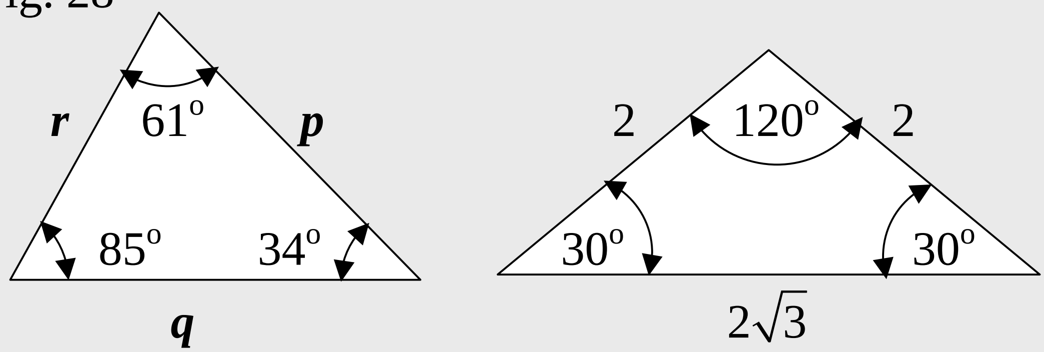
And next, assuming $\angle X = \angle E + \angle D$, we can say that $\angle X$ is the sum of two internal angles, and is supplement to $\angle C$, which is the other internal angle.

And of course, when solving many problems, we often use the two facts above, ***the sum of the three*** is $\mathbf{180^o}$, and ***the sum of the two internal is the one external***. So you may want to keep those in mind. How to keep them in mind?

Use them often doing many examples.

Now, in the figure below, the triangle on the left is made of three sides, ***p***, ***q***, and ***r***, along with the three angles, 85^o, 61^o, and 34^o.

Fig. 28



And the triangle on the right is made of three sides, 2, 2, and $2\sqrt{3}$, and the three angles, 30^o, 120^o, and 30^o.

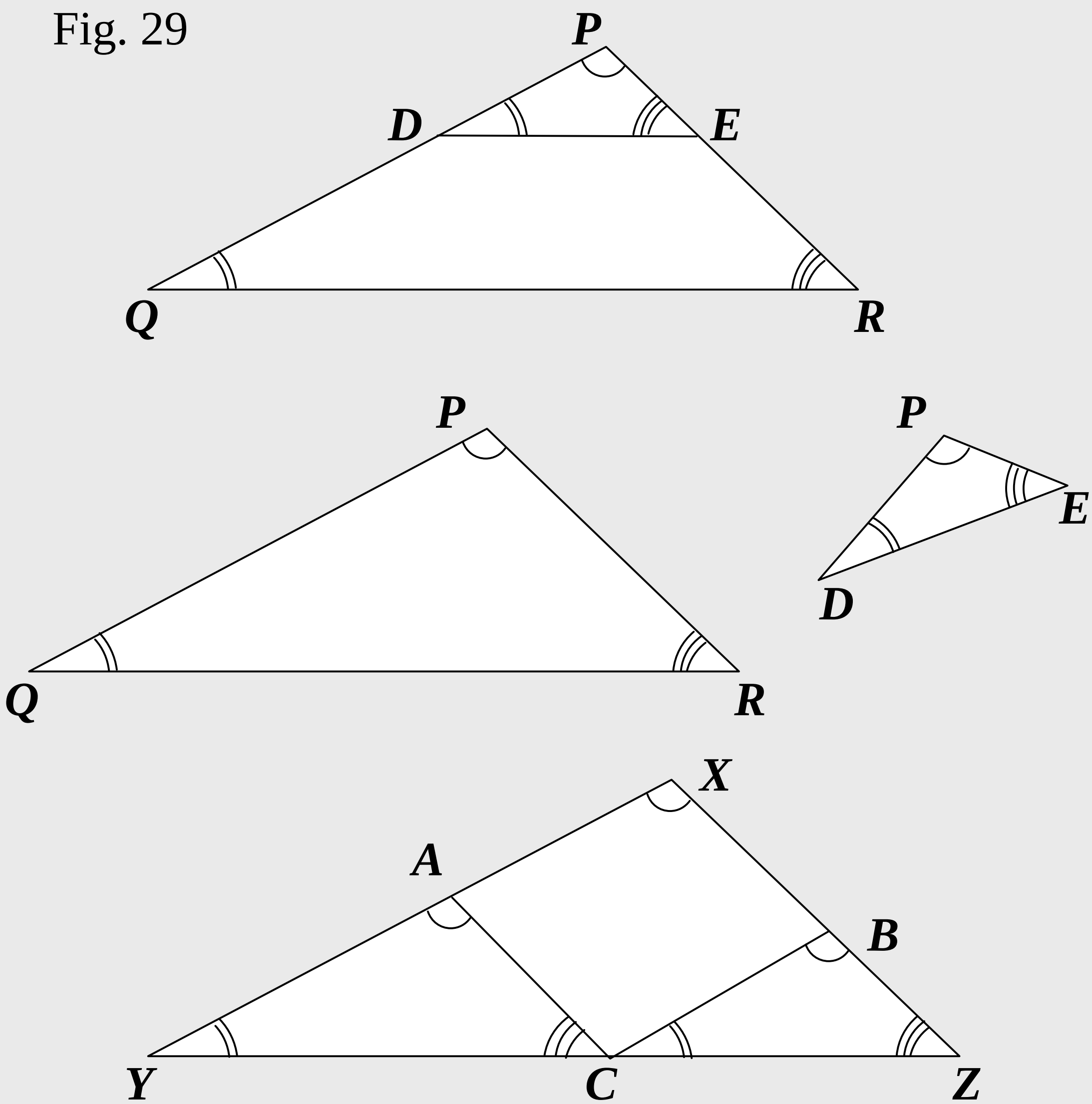
Thus, the components making a triangle are three angles and three sides. So?

So the same shape has much to do with angles and sides.

That is, the similarity has a lot to do with the components called angles and sides.

How then, does it have to do with those components?

Fig. 29



So we have several questions to think about.

How does the similarity have to do with those components?

What angles make the same shape?

What sides make a similar triangle?

Not all triangles are similar.

So not all angles and sides do, of course; some do.

Some angles and sides can make it.

So if the angles and the sides are eligible, the same shape happens, that is, the triangle is similar.

What then, is the eligibility? ☺

In math, we define similar triangles.

And in the definition, the eligibility is specified.

So you want to see the definition.

The definition is in three versions.

One is about ***angles***.

Another is about ***sides***.

And the other is about ***the combination*** of angles and sides.

We'll start talking about the definition in the next lesson.

Similar Triangles 3

Let's now start discussions on the definition for similar triangles. We'll talk about the eligibility for similar triangles.

In this lesson, it'll be about angles, so we'll cover the eligibility regarding angles, which is the first of the three versions in the definition.
So beginning with the first version now, we can say,

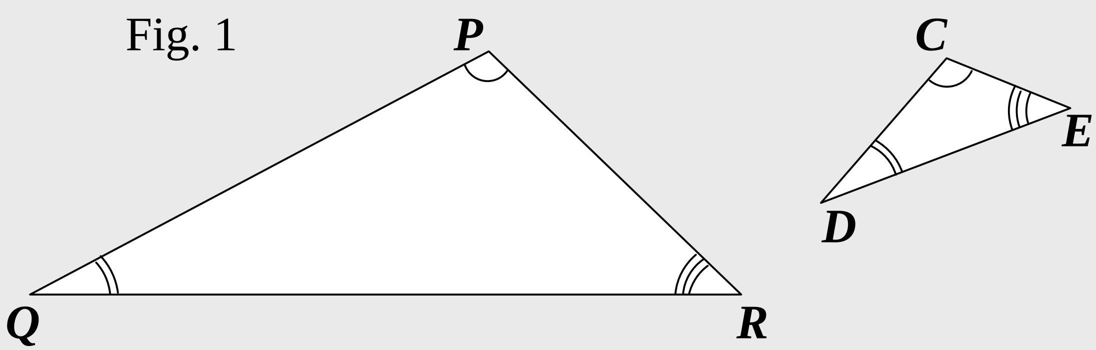
> ***By definition***, similar triangles share three angles.

Triangles in the same shape, that is, similar triangles can be different in size, but their angle groups are the same.

So similar triangles have the same angle groups.

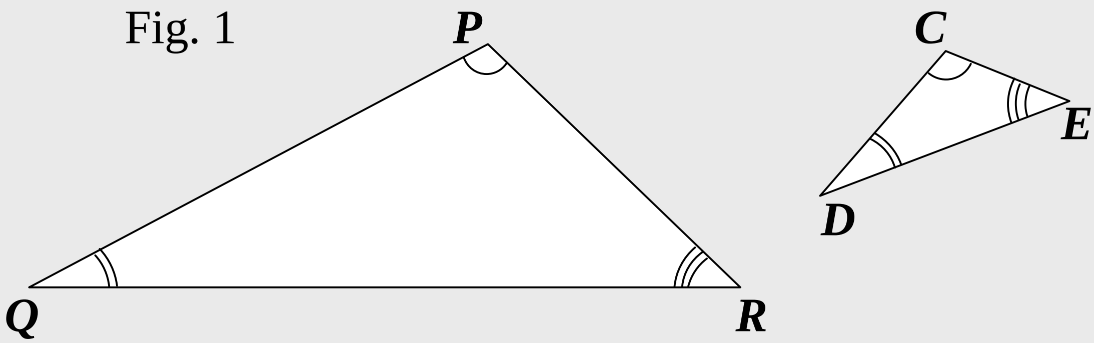
That is, they have the same groups of three angles.

Fig. 1



So the two triangles below are similar, that is, we have this: $\Delta PQR \sim \Delta CDE$.

Fig. 1



And of the three angles, two or all the three can be equal.

For instance, similar triangles can share this angle group: $(40^o, 60^o, 80^o)$ or this group: $(30^o, 30^o, 120^o)$, or even this: $(60^o, 60^o, 60^o)$.

So either way, in each similar triangle, the angle group is the same.

Similar triangles, therefore, share a group of three angles whether the three are the same or not. A triangle is made of three angles and three sides.

So in short, similar triangles share three angles same or not.

And the reverse is true, too.

> So now, reversely, ***by definition***, those triangles that share three angles are similar.

Triangles sharing the same angle group can be different in size, but they share the same shape, that is, they are similar.

So triangles with the same group of three angles are similar.

We now have two definitions equivalent, one is above, and the other is as follows.

> ***By definition***, similar triangles share three angles.

And putting the two definitions together, we can make one, and simply, can put it this way:

> By definition, similar triangles mean the same angle groups, which mean also, similar triangles.

Not simple enough?

We can make it simpler using a math symbol.

Many mathematicians are not happy with many words. So they came up with symbols and signs, and love to use them to replace words.

So now, using a math symbol that looks like this: ⇔, read as ***if and only if***, we can put the definition above the way as follows:

Similar Triangles ⇔ The Same Angle Groups

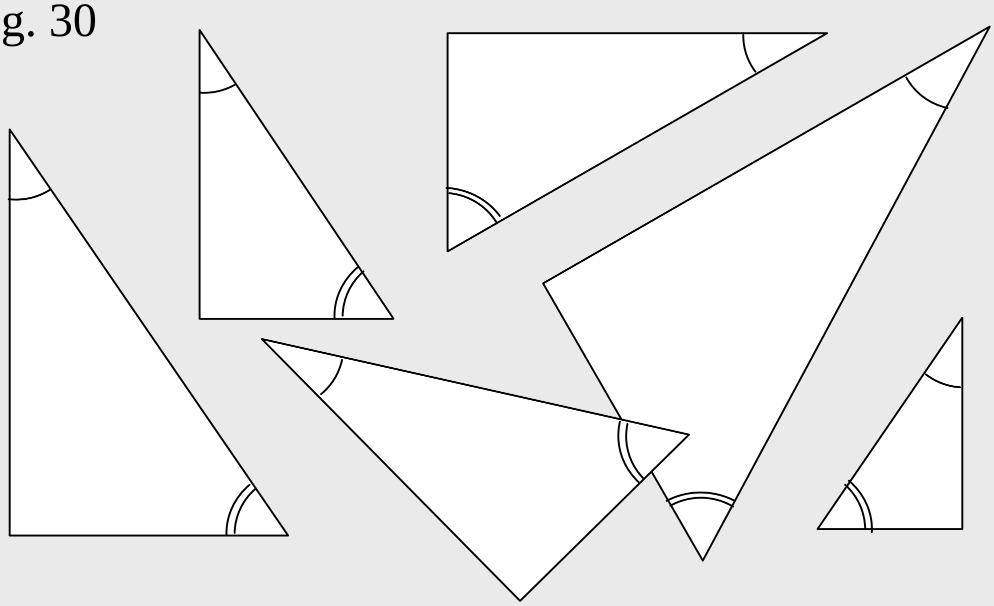
And it’s saying that if similar, the triangles have the same angle groups, that is, they share three angles, and also, if triangles have the same angle groups, they are similar.

In short in plane language, similar triangles share three angles, and triangles sharing three angles are similar.

For instance, if the angles in a triangle are 20^o, 70^o, and 90^o, and the angles in another triangle are 20^o, 70^o, and 90^o, too, both triangles have the same angle groups, and are similar.

Assuming all the triangles below are similar, we can say that they all have the same angle groups, that is, they share three angles. For instance, they share (30^o, 60^o, 90^o).

Fig. 30



And reversely, assuming each of all the triangles above has the same angle group, which is, for instance, (30^o, 60^o, 90^o), we can say that they all are similar to each other.

So, we can say, "*By definition*, similar triangles have the same angle groups, that is, they share three angles, which means also, similar triangles."

Well then, how do we know if two triangles are similar?

In other words, how do we check to see if two are similar?

We don’t need to check all the three angles each triangle has.　Why not?

We have only to check two angles in each triangle.
There are two cases where we can see if two triangles are similar. And one case is as follows.

We know a triangle has three angles, and have already covered the fact that no matter what the triangle may be, the sum of the three angles is a straight angle, 180^{o}.　So?

So getting to see that two triangles share two angles, we can say that the two triangles are similar.　Why, though?

It’s because the two triangles have to share the other angle, too, since the sum of the three angles every triangle has needs to be equal. And the sum is a straight angle, 180^{o}.
So?

So for instance, if in each triangle, two angles are 30^{o} and 60^{o}, the other angle has to be 90^{o}, which has to be shared by both triangles.

And thus, the two triangles share three angles.

That is, both have the same angle groups.

What then, is the other case, the other way we can see if two triangles are similar?

If knowing two angles in each triangle, we can see if two triangles are similar. How do we know?

We can again, use the fact that the sum of the three angles in every triangle is a straight angle, 180^{o}.

How then can we use it?

For instance, if we know that in one triangle, two angles are 50^o and 70^o, and in the other triangle, two angles are 60^o and 50^o, we can see that the two triangles are similar using the fact stated above. How do we know if they are similar?

In each triangle, since the sum of the three angles is 180^o, in one triangle where two angles are 50^o and 70^o, the other angle has to be 60^o, and in the other triangle where two angles are 60^o and 50^o, the other angle has to be 70^o.

So both triangles share this angle group: (50^o, 60^o, 70^o), that is, both have the same angle groups.

And for another example, what if we know, in one triangle, two angles are 40^o and 80^o, and in the other triangle, two angles are 60^o and 50^o?

Then, they are not similar. Why not?

In each triangle, since again, the sum of the three angles is 180^o, in one triangle where two angles are 40^o and 80^o, the other angle has to be 60^o, and in the other triangle where two angles are 60^o and 50^o, the other angle has to be 70^o.

So one triangle has this angle group: (40^o, 80^o, 60^o), and the other triangle has this angle group: (60^o, 50^o, 70^o).

That is to say that the two triangles don't share three angles, that is, the two have a different angle group each.

And thus, we have only to check two angles in each triangle.

It's because of the fact that the sum of the three angles in every triangle is a straight angle, 180^o, which means, the sum is the same in every triangle.

So now, we can put the definition the way below, too.

Similar triangles share two angles.

Or this way:

Sharing two angles, the triangles are similar.

And either way, it means the same group of three angles, which can be called for short, the same angle group. So we can still put the definition the way below.

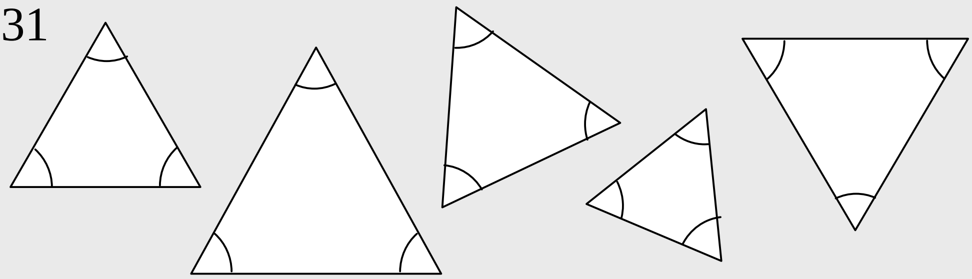
Similar Triangles ⇔ The Same Angle Groups

What then, about regular (equilateral) triangles?

All regular (equilateral) triangles are similar, because they all share the same group of three angles, in which, the three angles are all the same, too: (60^o, 60^o, 60^o).

So, in the case of three equal angles, it is a regular triangle, as well as in the case of three equal sides.

Fig. 31



The triangles above are regular, have the same groups of three angles, and therefore, are all similar.

And all regular triangles are similar.

So in the set of all regular triangles, every triangle is similar, to all the other triangles in the set, of course.

That’s not all.

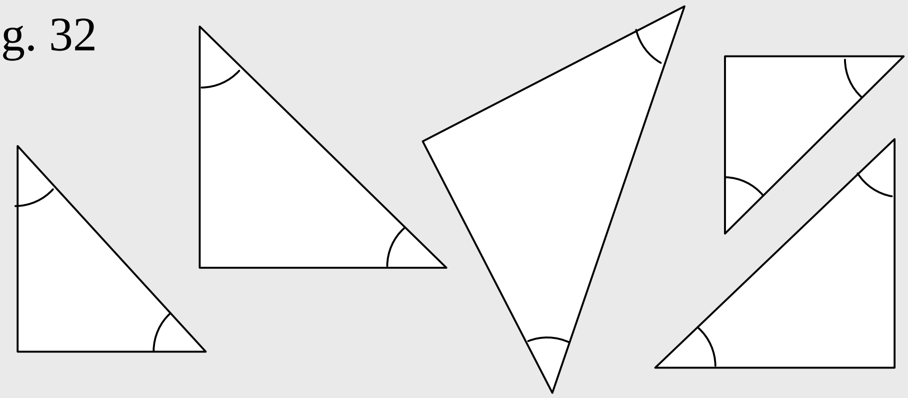
We have another set where all the triangles are similar.

What then, can be the set?

It is a triangle set, where each triangle is isosceles, and has a right angle, that is, 90^o. And we know that in every triangle, the sum of all the three angles is 180^o, and that an isosceles triangle has two equal angles, as wells as two equal sides.

So the angle group is (90^o, 45^o, 45^o), and they are often called ***isosceles right triangles*** or right triangles isosceles.

Fig. 32

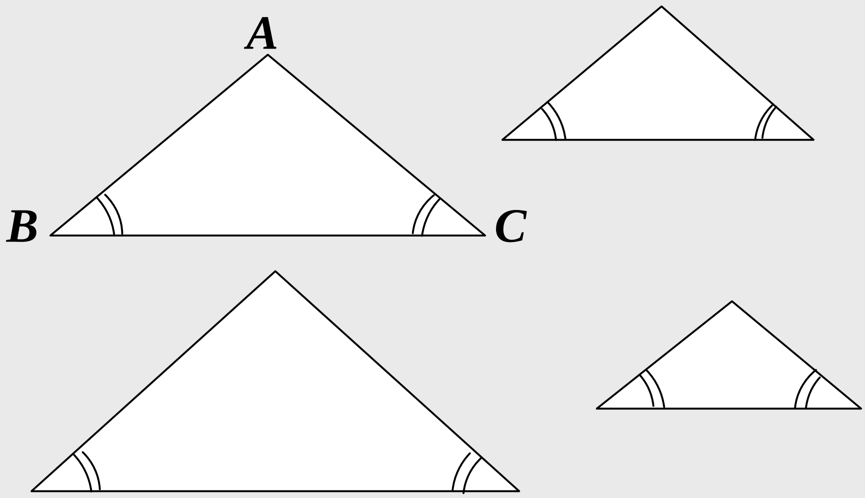


All the triangles above are isosceles right triangles, have the same angle groups, and therefore, are all similar.

Still, that's not it. We have many other sets, in each of which, all the triangles are similar and isosceles. How many?

Infinitely many.
For instance, we can have a set where one angle in each isosceles triangle is 100^o, so the other two vertices have 40^o each. That is to say that in the set, the group of angles all those isosceles triangles share is this: (100^o, 40^o, 40^o).

Fig. 33



So in Fig. 33 above, if $\angle A = 100^o$, and $\angle B = \angle C = 40^o$, we can say that all the four triangles share (100^o, 40^o, 40^o), and therefore, are all similar.

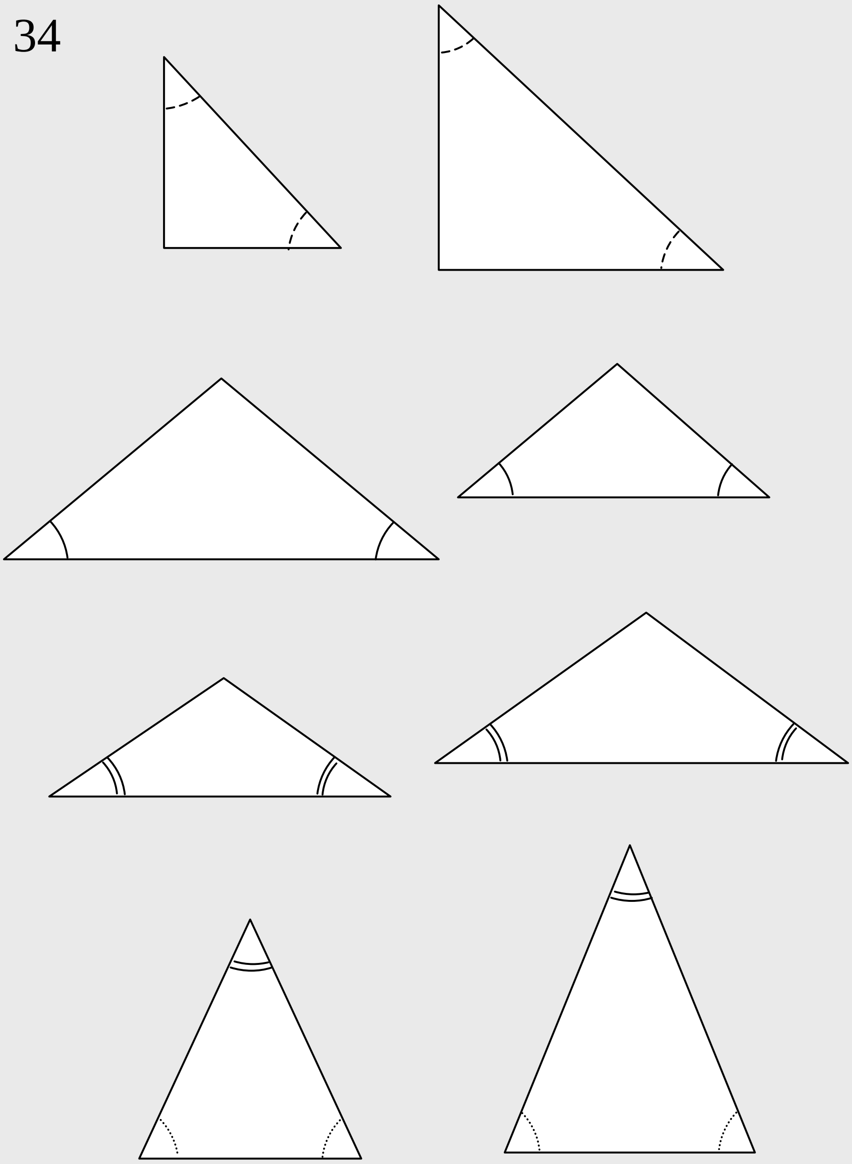
And there can be infinitely many different sets, in each of which, all the triangles are similar.

Is it the case then, all isosceles triangles are similar?

No, it isn’t.   Why not?

We can have these:

Fig. 34



It’s not the case that all the triangles above have the same angle groups, that is, the same groups of three angles.

Some are similar, and some are not.
So a counter example can be as follows.

An isosceles right triangle is not similar to the isosceles triangle where the angle group is (100^o, 40^o, 40^o).

And another counter example can be an isosceles triangle where the angle group is (40^o, 70^o, 70^o).

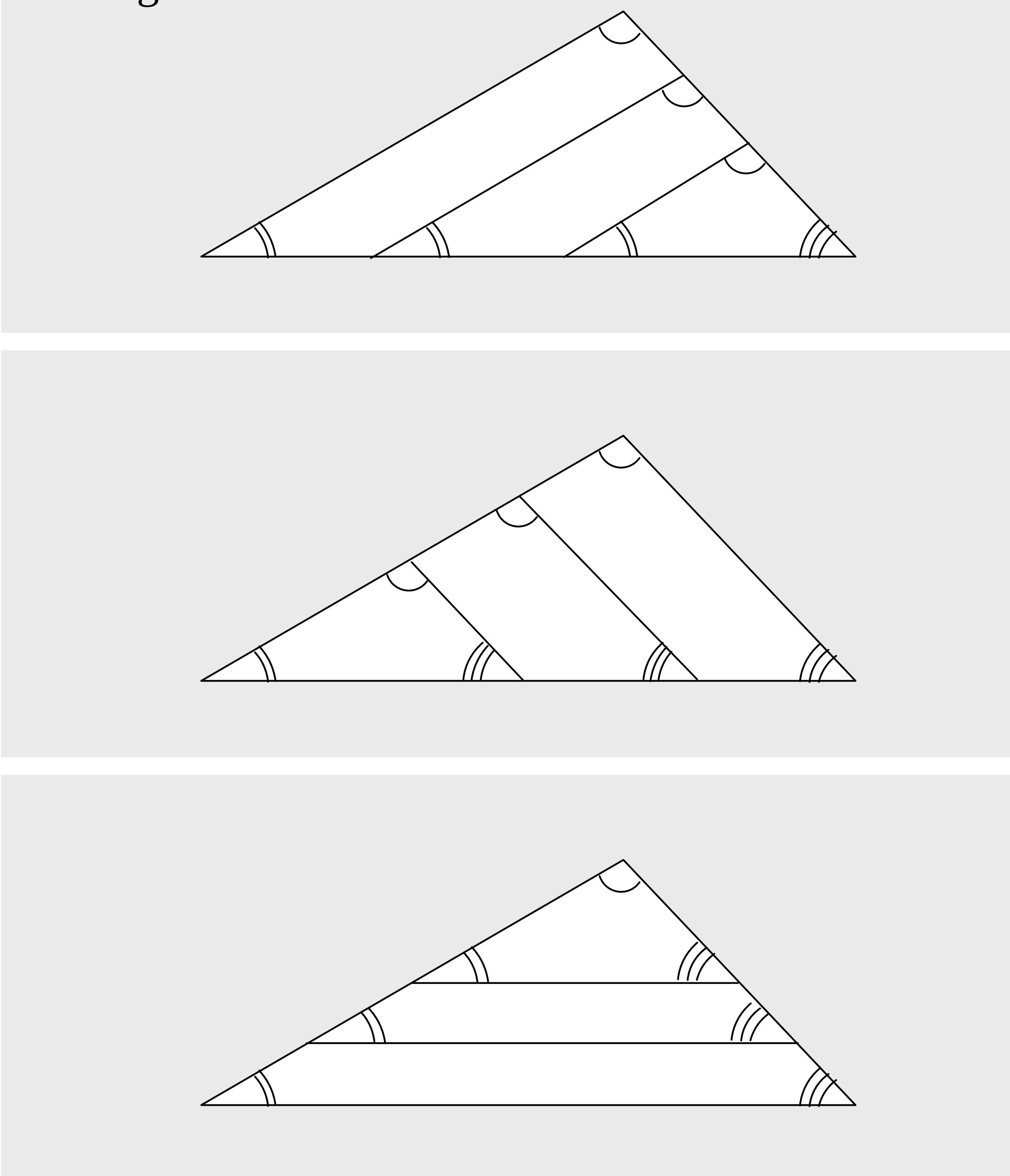
So it is not true that all isosceles triangles are similar.
As shown in Fig. 34 above, we can have groups of isosceles triangles, in each of which ***only***, the triangles are similar.

And with one exception, similar triangles have different sizes.
That is to say that they except one case have different areas.

What is the exception?

It is the case of identical triangles. Why is it?

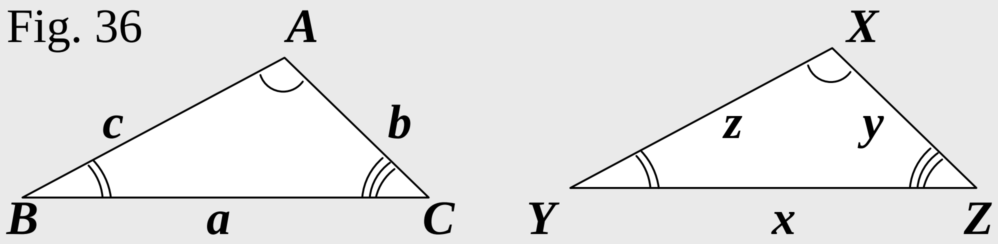
Fig. 35



All the triangles above are similar to each other.

Technically, identical triangles are similar, too, because they share three angles, that is, they have the same angle groups. And of course, they share the same area, too, as well as three sides.

Fig. 36



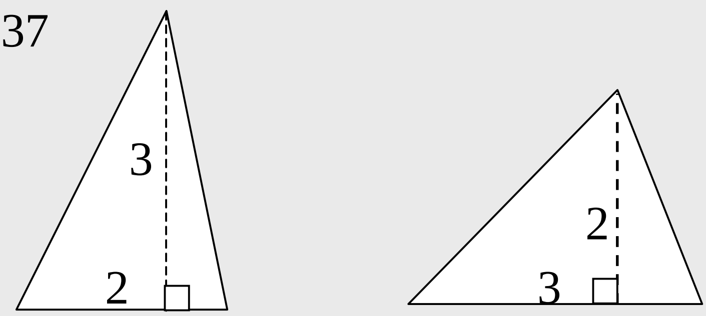
In Fig. 36 above, if the two triangles ΔABC and ΔXYZ are identical to each other, we get these: $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$, $a = x$, $b = y$, and $c = z$.

And using a math symbol, we can put them the way as follows: $\Delta ABC \equiv \Delta XYZ$, which is saying therefore, "ΔABC is identical to ΔXYZ." or is saying that ΔABC and ΔXYZ are identical.

So '≡' is a math symbol meaning '***is identical to***'.

By the way, triangles not identical can share the same area, also. For instance, if the base of a triangle is 2 and the height is 3, and if the base of another triangle is 3 and the height is 2, both are not identical but share the same area, which is 3, the base times the height divided by 2.

Fig. 37



The two triangles above share the same area.

And if different triangles share the same area, they are not similar. Why not?

If similar but not identical, the triangles have to be able to be nested any of the three ways below, so all their areas are different.

Fig. 38

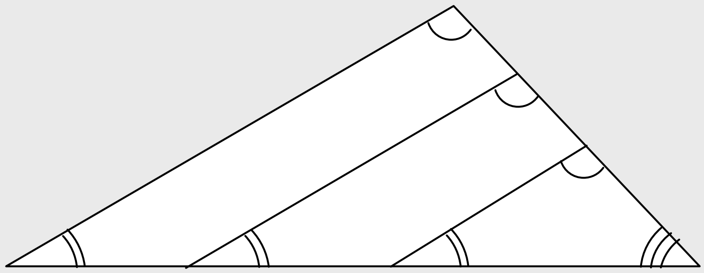


Fig. 39

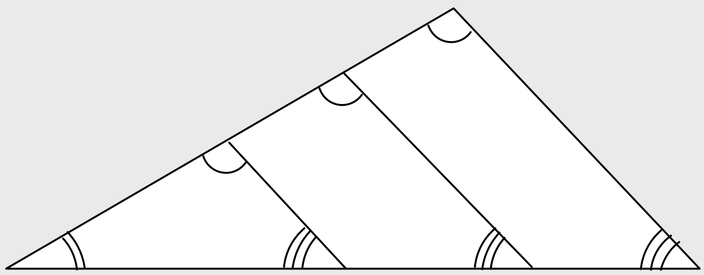
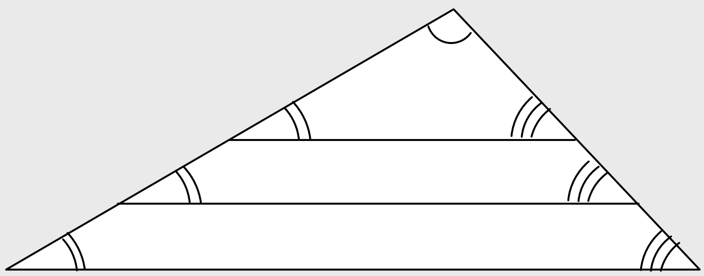


Fig. 40



All the triangles in all the three figures above are similar, and have different areas. So if similar but not identical, all the triangles have to have different areas.

And summing up, for now, we can put the bottom line the way as follows.

Similar Triangles ⇔ The Same Angle Groups

It is saying that if similar, the triangles have the same angle groups, that is, they share three angles, and also, if having the same angle groups, the triangles are similar.

> Equivalently, or rather, practically or efficiently, if ***triangles share two angles, they are similar***.

And the statement above can be taken as a definition for similar triangles, which is not, though, the only definition.

It is about angles, and we have two more versions. One is on the sides, and the other is on the combination of angles and sides. In the next lesson, we are going to cover the version on the sides eligible for a similar triangle.

Similar Triangles 4

Let's now move on to the next version, ***the group of sides eligible for a similar triangle***.

> First off, as a definition, we often say that ***similar triangles*** share the same shape but ***have*** different sizes, that is, ***different areas***.

Indeed, unless identical, ***similar triangles don't share the same area***, which is ***a part of the definition*** above.

Fig. 1

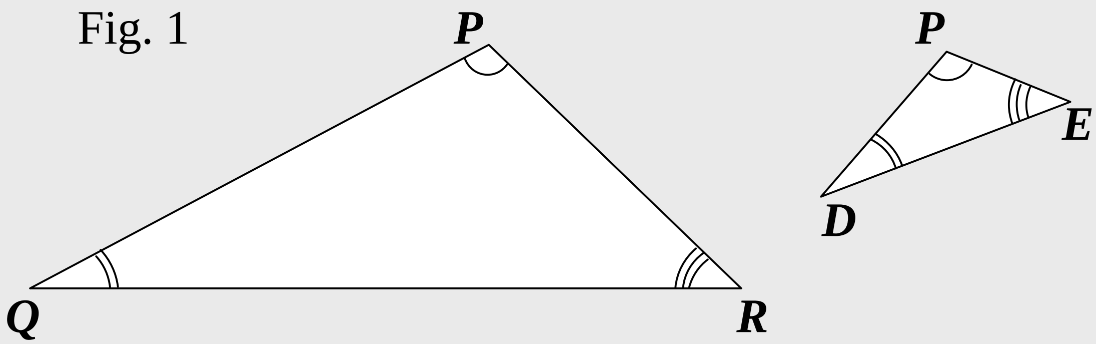
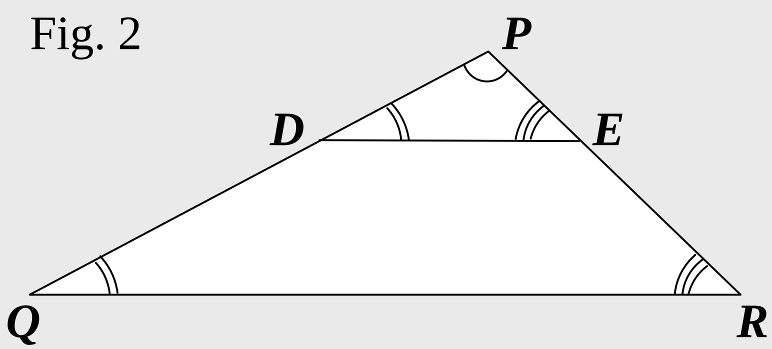


Fig. 2



All the triangles in each figure below are similar and have different areas.

Fig. 3

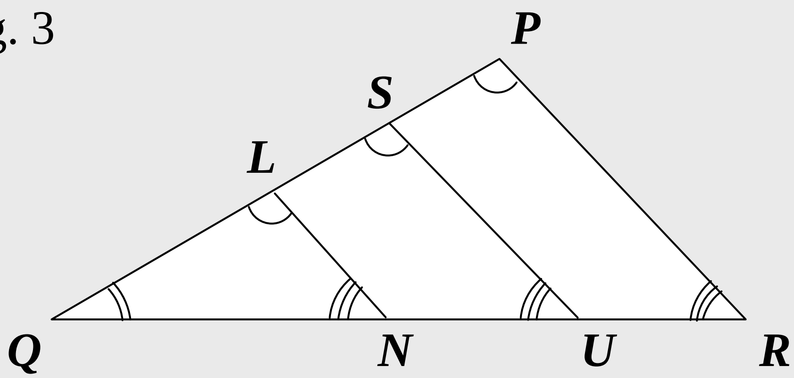


Fig. 4

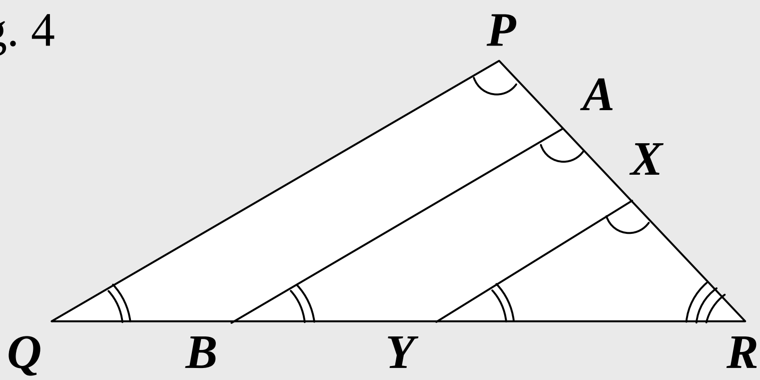
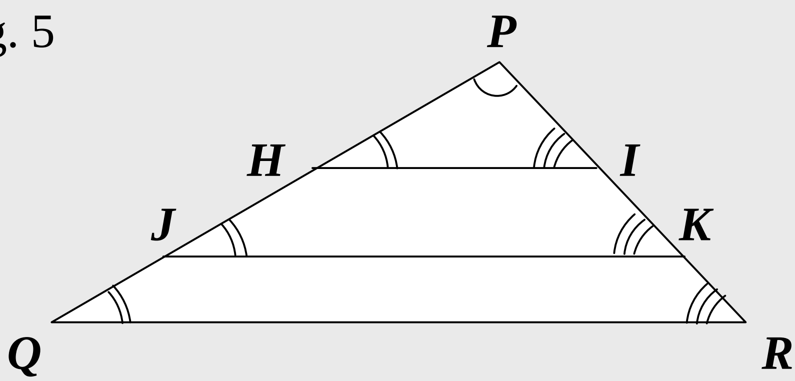


Fig. 5



So let’s now, start this version with getting answers to the questions as follows.

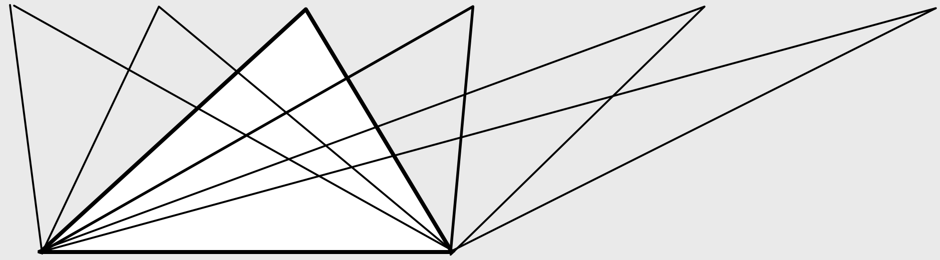
What makes different sizes, that is, different areas?

Calculating the area of a triangle, what do we use in a triangle?

What does the area of a triangle have to do with?

Getting answers to the questions above, we get to cover some important basics on triangles, along with the definitions and properties of similar triangles. So keeping that in mind, let’s now start getting the answers to the questions above.

Fig. 6

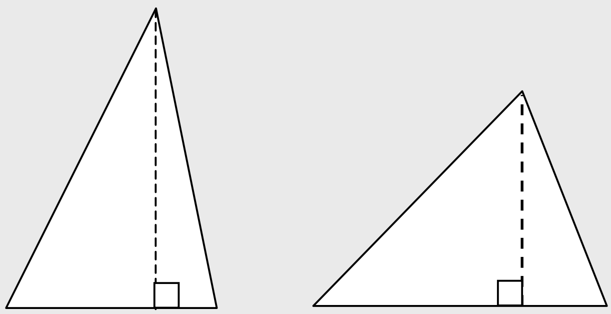


To begin with, as a formula, we have this:

> Taking ***the product of the base and the height divided by 2***, we get ***the area of a triangle***.

So the area of a triangle depends on the base and the height. What then, are the base and the height?

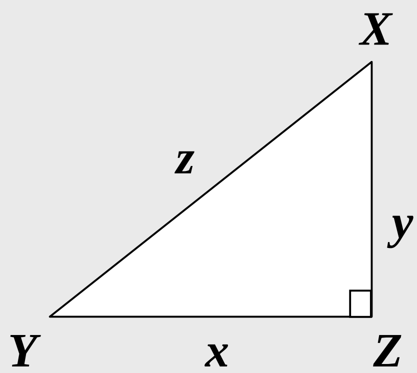
Fig. 7



In other words, in a triangle, what do we take as the base and the height?

The base is a side in a triangle, and if the triangle is a right triangle, the height is a side, too. So the area depends on the sides of a triangle.

Fig. 8



In the figure above, ΔXYZ is a right triangle, so not only the base but the height is a side, too. And the reason follows.

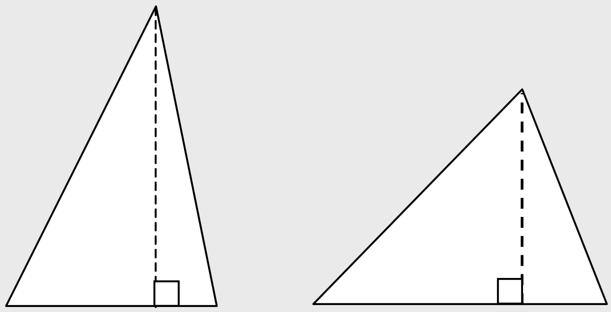
The height is perpendicular to the base, that is, the two make a right angle, 90°, so in the figure above, if taking the side y as the height, we take the side x as the base.

And, of course, instead of x, we can take y as the base; then, x is the height. So either way, in a right triangle, the base and the height both are sides.

What if, however, it's not a right triangle?

The height is the distance from the base to the vertex away from the base.

Fig. 7



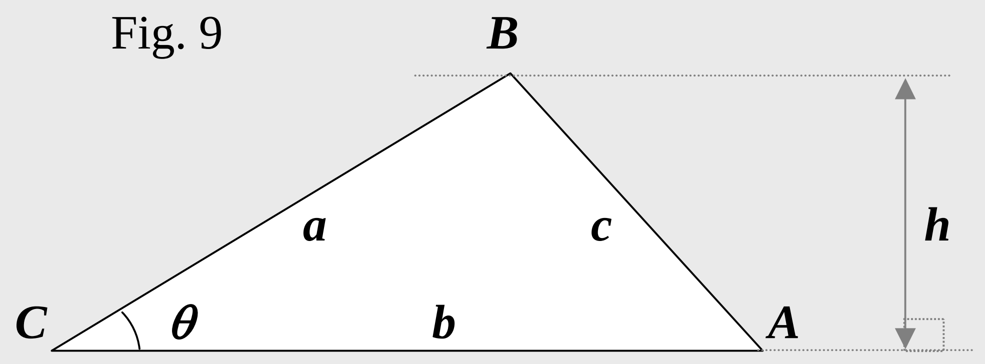
And if it's not a right triangle and the height is not given, we can use a side to find the height, which does, therefore, depend on a side. So we can still say that the area depends on the sides.

Why, though, does the height depend on a side?

How can we use a side to find the height?

If it's not a right triangle, but an angle is given, we can find the height using this: $\boldsymbol{h} = \boldsymbol{a} \sin\theta$, where $\boldsymbol{h}$ is the height, $\boldsymbol{a}$ is a side, and $\sin\theta$, called ***sine theta***, is a ratio, which is after all, a number as $\frac{1}{4}$ or $\frac{2}{3}$. And if $\theta = 90^{\circ}$, we get $\sin\theta = 1$.

Fig. 9



What in the world is $\sin\theta$?

Its practical meaning as well as mathematical meaning is explained in the lessons, ***The Sine***.

Don't worry about $\sin\theta$, too much, now. Just take it as a number, for now, indicating a ratio as $\frac{1}{2}$ or $\frac{1}{3}$.

And if $0^{\circ} < \theta < 180^{\circ}$, we get this: $0 < \sin\theta \leq 1$.

So the height depends on a side and an angle $\boldsymbol{\theta}$.

If given two sides and the angle between the two, we can find $\boldsymbol{R}$, the area of ΔABC using the formula as follows.

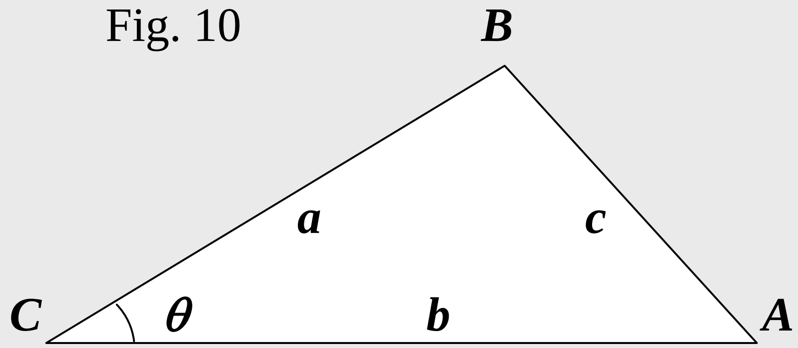


Fig. 10

$\boldsymbol{R} = \frac{1}{2}ab\sin\theta$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the two sides forming the angle $\boldsymbol{\theta}$, called ***theta***, and $0^\circ < \theta < 180^\circ$.

So finding, for instance, the area of a regular triangle where each side is of length 8, since every angle is 60°, we can set $\boldsymbol{a} = 8$, $\boldsymbol{b} = 8$, and $\boldsymbol{\theta} = 60^\circ$, and then, we get it this way:

$\boldsymbol{R} = \frac{1}{2}$ x 8 x 8 x **sin** 60°, where **sin** $60^\circ = \frac{\sqrt{3}}{2}$, so the area is $16\sqrt{3}$.

So the height depends on a side, along with an angle, and thus, we can say that the area depends on the sides.

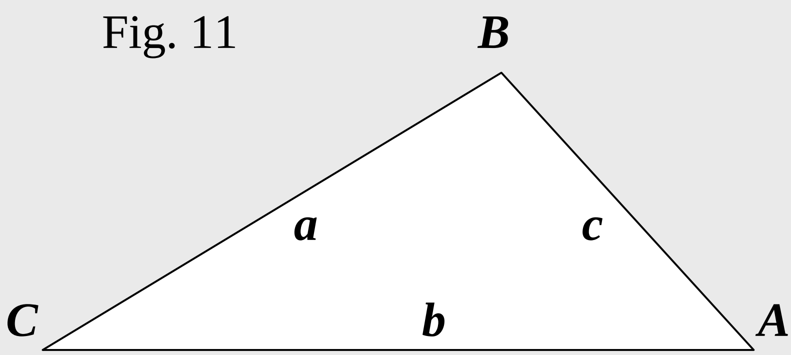
You might still wonder now, because angles seem to get involved, and might ask, "What if no angle is given?"

Well then, this time, we can use a tool, called Heron's formula, which is entirely made of sides, and is this:

$$\boldsymbol{R} = \sqrt{\boldsymbol{s}(\boldsymbol{s}-\boldsymbol{a})(\boldsymbol{s}-\boldsymbol{b})(\boldsymbol{s}-\boldsymbol{c})}, \text{ where } \boldsymbol{s} = \frac{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}}{2},$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are, of course, the three sides.

Fig. 11



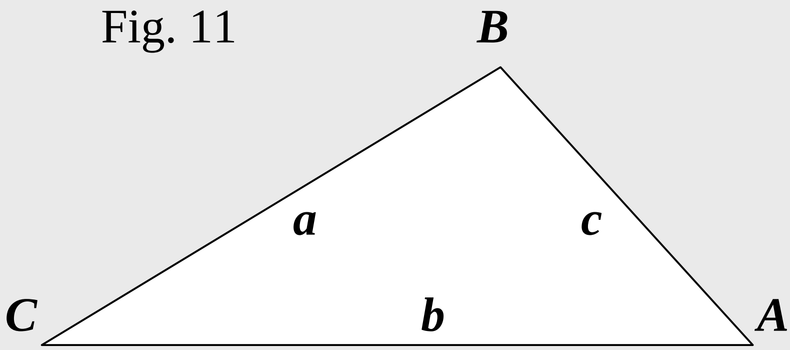
And let's now, find the area of the regular triangle, which we've found earlier using the formula with $\sin\theta$.

It's a regular triangle where each side is of length 8.

$$\boldsymbol{R} = \sqrt{\boldsymbol{s}(\boldsymbol{s}-\boldsymbol{a})(\boldsymbol{s}-\boldsymbol{b})(\boldsymbol{s}-\boldsymbol{c})}, \text{ where } \boldsymbol{s} = \frac{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}}{2},$$

where ***a***, ***b***, and ***c*** are, of course, the three sides.

Fig. 11



So for instance, if $\boldsymbol{a} = 8$, $\boldsymbol{b} = 8$, and $\boldsymbol{c} = 8$, we get this:

$$\boldsymbol{R} = \sqrt{\boldsymbol{s}(\boldsymbol{s}-8)(\boldsymbol{s}-8)(\boldsymbol{s}-8)}, \text{ where } \boldsymbol{s} = \frac{8+8+8}{2} = 12.$$

So we get this:

$$\boldsymbol{R} = \sqrt{12(12-8)(12-8)(12-8)} = \sqrt{12 \text{ x } 4 \text{ x } 4 \text{ x } 4} = 16\sqrt{3},$$

which is the area we found using this: $\boldsymbol{R} = \frac{1}{2}\boldsymbol{ab}\sin\boldsymbol{\theta}$.

So now, the answer to the questions below is the sides.

What makes different sizes, that is, different areas?

Calculating the area of a triangle, what do we use in a triangle?

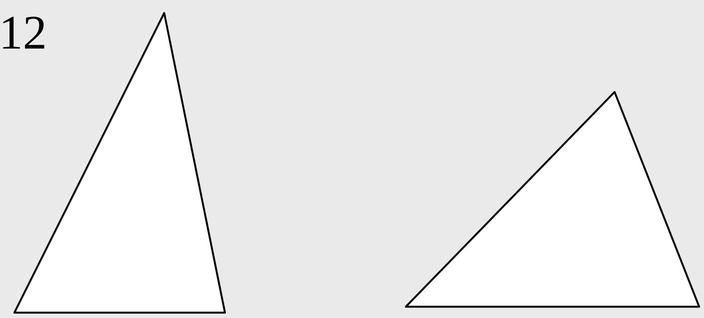
What does the area of a triangle have to do with?

And if we lengthen or shorten the sides, the area can increase or decrease.

In short, if we change the sides, the area ***can change***.

Why not just ‘changes’ instead of ‘***can change***’ in the sentence above?

Fig. 12

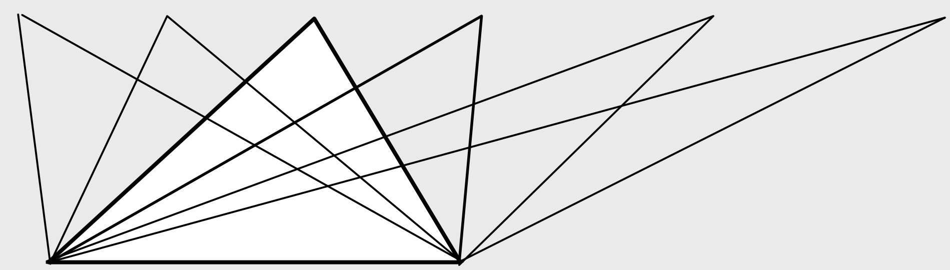


Changing the sides doesn't guarantee a different area.

In other words, it's ***not always*** the case that a triangle with a different side group has a different area.

Why not?

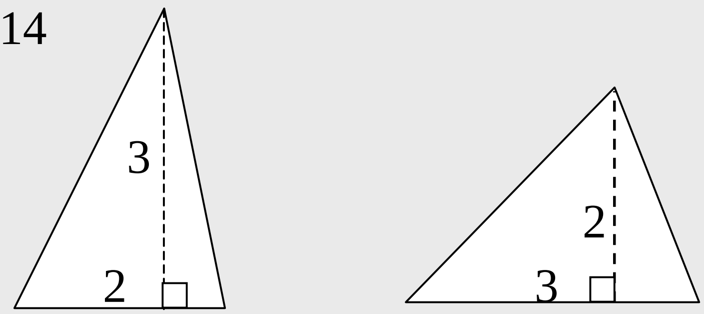
Fig. 13



Triangles with different side groups can share the same area.

As shown in Fig. 14 below, even if changing the sides, we can still get a new triangle with the same area.

Fig. 14



Taking ***the product of the base and the height divided by 2***, we get ***the area of the triangle***.

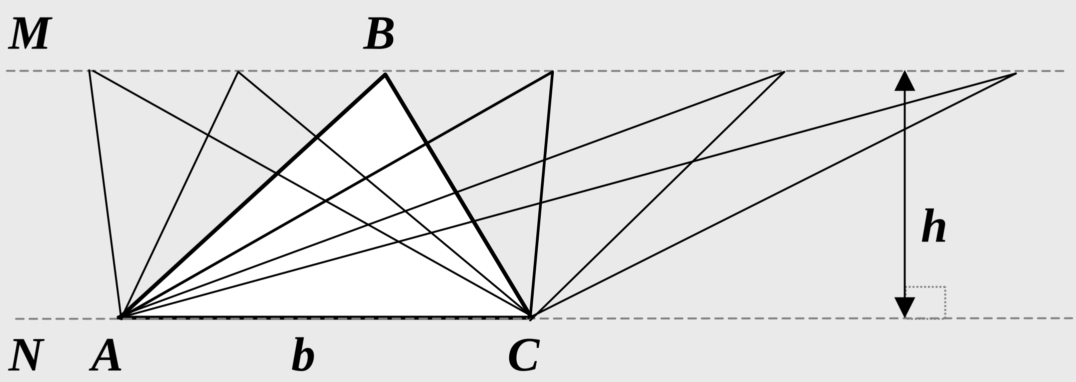
So the one on the left is $\frac{2 \text{ x } 3}{2}$, and the other is $\frac{3 \text{ x } 2}{2}$.

Both triangles, therefore, share the same area, which is 3.

And that’s not it.

In Fig. 15 below, if the two lines ***M*** and ***N*** are parallel, and we hold the base ***b*** and the height ***h*** constant, though we keep changing the two sides in ΔABC as shown in the figure, the area stays the same.

Fig. 15



It's because of this:

Taking the product of the base and the height divided by 2, we get this: $\frac{bh}{2}$, which is the area of every triangle above.

Thus, it's not the case different sides mean a different area. It is the case though, a different area means different sides.

Now, back to similar triangles.

Similar triangles have different areas, if not identical, of course. Though changing the sides of a triangle, that is, lengthening or shortening the sides, we can still get a triangle with the same area; it is still the case, however, we cannot get a triangle with a different area unless we change one or more sides of a triangle.

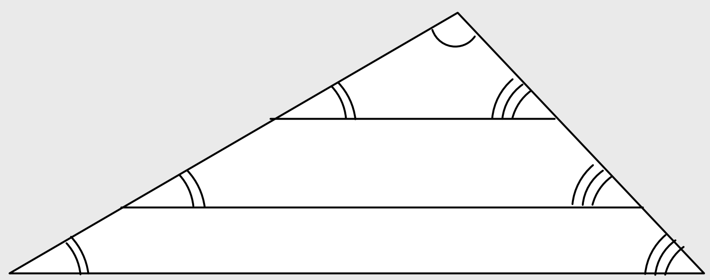
So we gotta use different sides if we need to make a triangle with a different area.

And if not identical, similar triangles have different areas. They are normally not identical. And, a different area means different sides.

If making thus, a similar triangle, use different sides.

What sides, though? ☺

Fig. 16



We don’t just make the sides change, of course. The sides should change to the sides eligible for a similar triangle.

So we should be able to find ***the way the sides change*** so that we get a similar triangle.

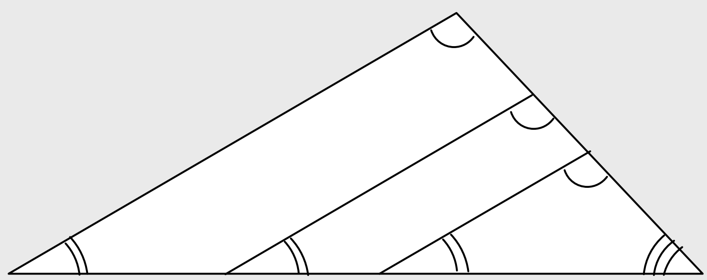
We can get a similar triangle ***changing the sides***.

Similar triangles share the same angle group.

So we should be able to get a triangle with the same angle group ***making the sides change***. In other words, we should be able to find ***the way the sides change*** so that we get a triangle with the same angle group.

What do we mean by ***the way the sides change***?

Fig. 17



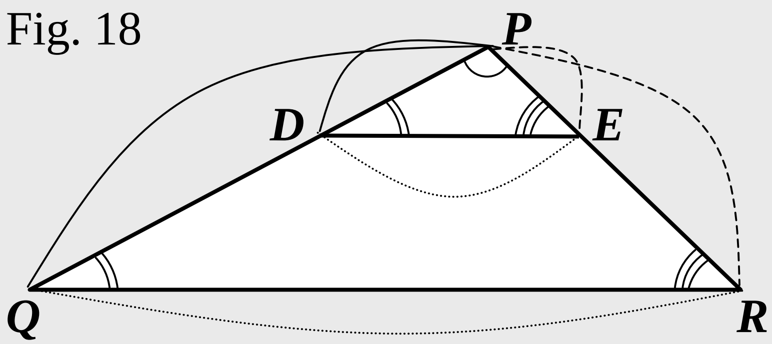
Saying sides, we mean their lengths, as well as line segments. Saying, therefore, that ***the sides change***, we mean that ***their lengths change***.

So ***the way the sides change*** means ***the way their lengths change***, the way they lengthen or shorten.

Let’s now, start finding the way.

You may want to first, take a close look at the two triangles below. And the two are ΔPQR and ΔPDE.

Fig. 18

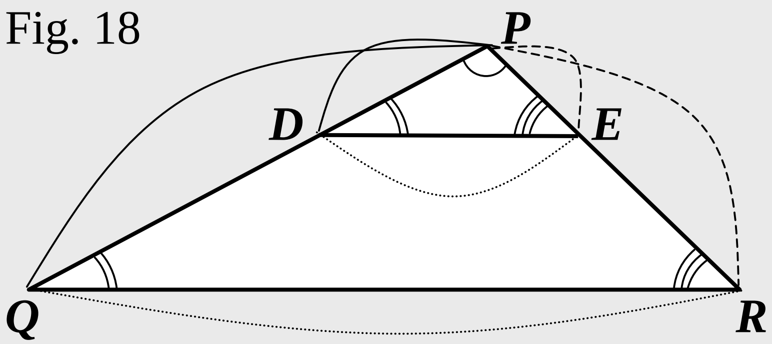


The two share the same angle group.

So?

Since sharing the same angle group, the two are similar.
And using the symbol, we can put them the way as follows.

Fig. 18



$$\Delta PQR \sim \Delta PDE.$$

Suppose now, we need to get ΔPQR from ΔPDE.

So we get ΔPQR making the sides in ΔPDE lengthen.
That is to say that in ΔPDE, the sides lengthen to the sides in ΔPQR. How do they lengthen, though?

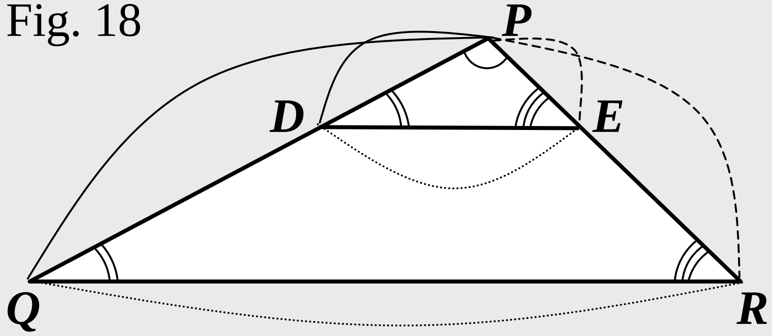
In other words, what is the way they change, and in this case, what is the way they get longer?

Let's now repeat the same discussion the way as follows.

Suppose this time, we want to get ΔPDE from ΔPQR.

That is, we need to get it shortening the sides in ΔPQR.

Fig. 18



That is to say that in ΔPQR, the sides shorten to the sides in ΔPDE. How do they shorten, though?

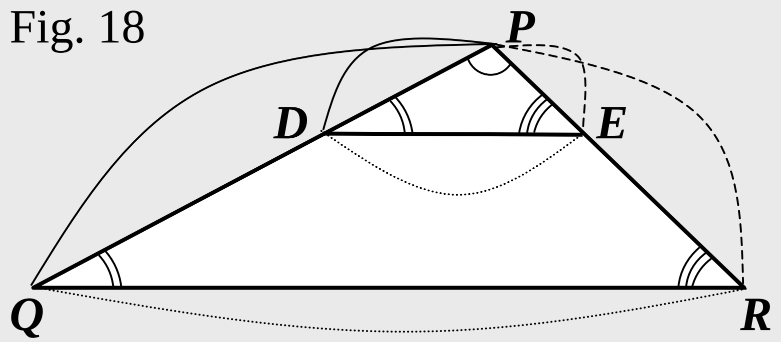
In other words, what is the way they change, and in this case, what is the way they get shorter?

So either way, we want to know the way the sides change. More specifically, we can put the ideas the way as follows.

We want to change the sides, yet need to keep the angles unchanged, that is, keep the same angle group. We need to see therefore, ***the way*** the sides change (get longer or shorter) keeping the angles unchanged.

Keeping the angles unchanged, that is, keeping the angle group the same, in what manner do you think the sides in a triangle have to change?

Fig. 18



In what manner does the change need to happen?

How about this: the ways the sides change are the same?

So what if the way each side changes is the same?

Pretty close, though not quite enough.

It's not quite wrong to say that the three sides change in the same manner, that is, lengthen or shorten the same way.

A bit too broad, and not specific enough.

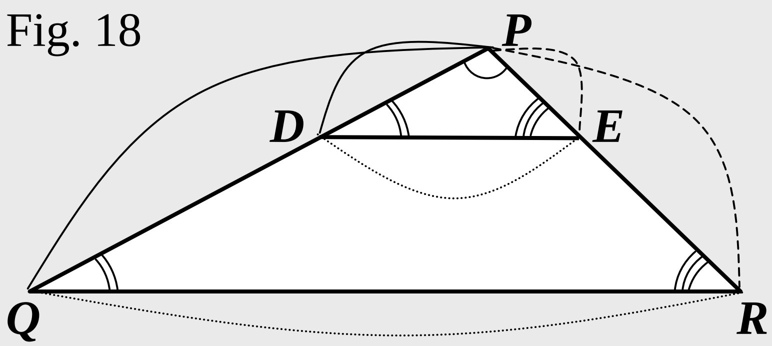
So we need some more elaboration.

How about then, the way as follows?

Each of the three sides increases in the same amount.

That is, we add the same length to each and every side.

Fig. 18



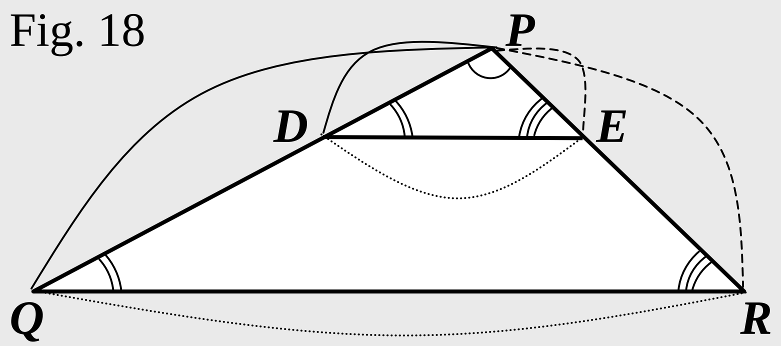
We cannot make a similar triangle that way.
With an exception, though.

So except for one case, adding the same length to, or subtracting it from each and every side, we don't make a similar triangle.

In other words, if every side lengthens or shortens by the same length, no similar triangle gets made, except one case.

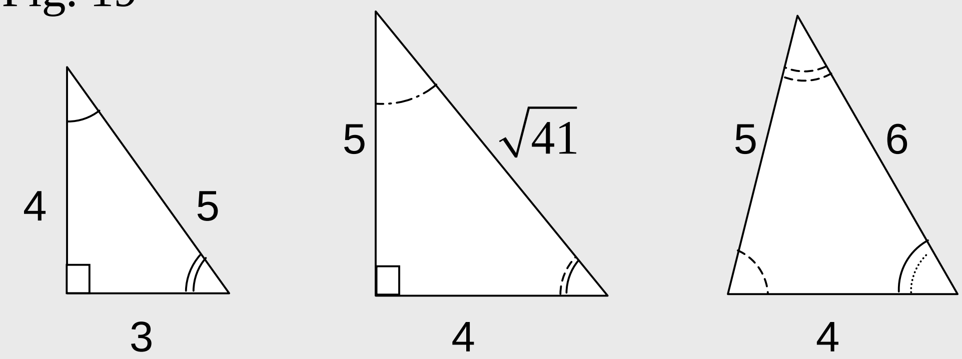
What's wrong with it?

Fig. 18



The angle group changes.

Fig. 19



$\sqrt{41} \approx 6.4031$, where ‘≈’ means approximately.

We’ll continue this in the next lesson.

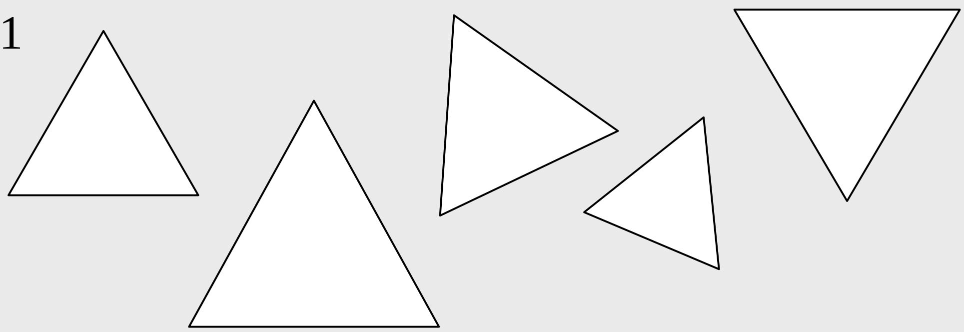
Similar Triangles 5

In any triangle except one kind, if every side changes in the same amount, the new triangle gets a different group of three angles, and thus, is not similar.

Adding the same length to each and every side in a triangle, we don't get a similar triangle, except one case.

What is the exception?

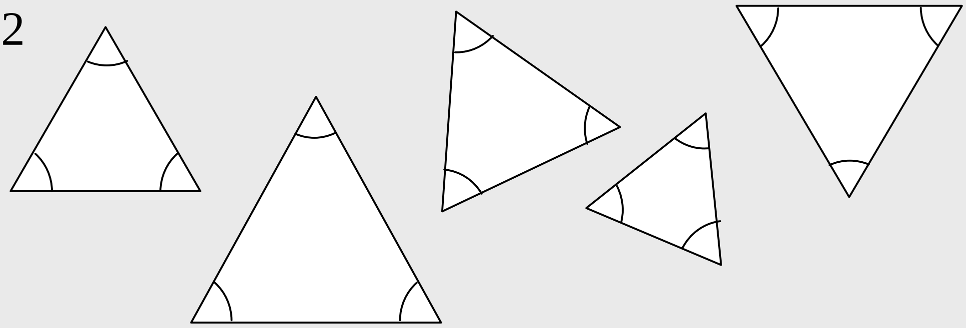
Fig. 1



If in a regular triangle, every side changes in the same amount, the new triangle is regular, too, since all the new sides are equal.

And all regular triangles are similar, so the new triangle is similar, also.

Fig. 2



Except for regular triangles, however, if every side changes in the same amount, the angle group changes, so the new triangle is not similar.

For instance, if the side group in a triangle is (3, 4, 5), and the group changes to this new group: (4, 5, 6), the new triangle is not similar.   Why not, though?   ☺

The triangle with (3, 4, 5) is a right triangle, but the triangle with (4, 5, 6) is not.

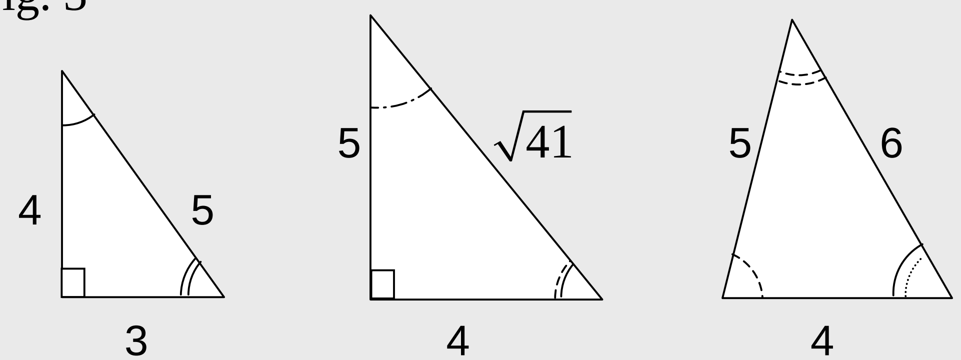
If you know the math tool called Pythagorean Theorem, a.k.a. The Distance Formula, you get the idea.

The tool says, "If it's a right triangle, the sum of the squares of the two sides making the right angle equals the square of the side called the hypotenuse facing the right angle."

For the triangle with (3, 4, 5), we get this: $3^2 + 4^2 = 5^2$.
And for the triangle with (4, 5, 6), we get this: $4^2 + 5^2 \neq 6^2$.

One is therefore, a right triangle, but the other isn't.

Fig. 3



$\sqrt{41} \approx 6.4031$, where '≈' means approximately.

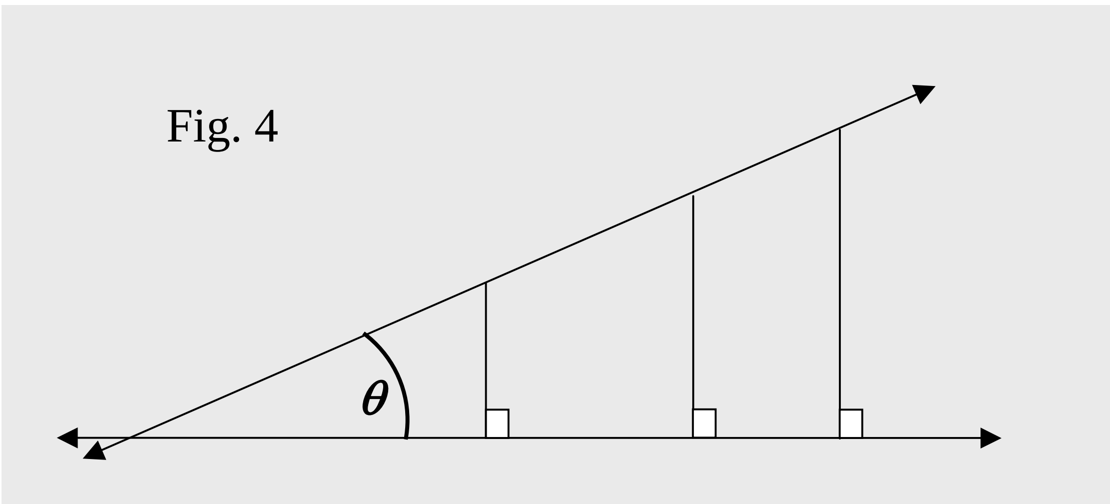
So different angle groups happen; therefore, it’s not the way the sides change to make a similar triangle.

What then, can be the way?

Getting back to the basics, we have this: Similar triangles share three angles, that is, the same angle group. So we need to see the way the sides change (get longer or shorter) keeping the angle group the same.

That is to say that we keep the angles unchanged when changing the sides.

What then, do we want to know about?



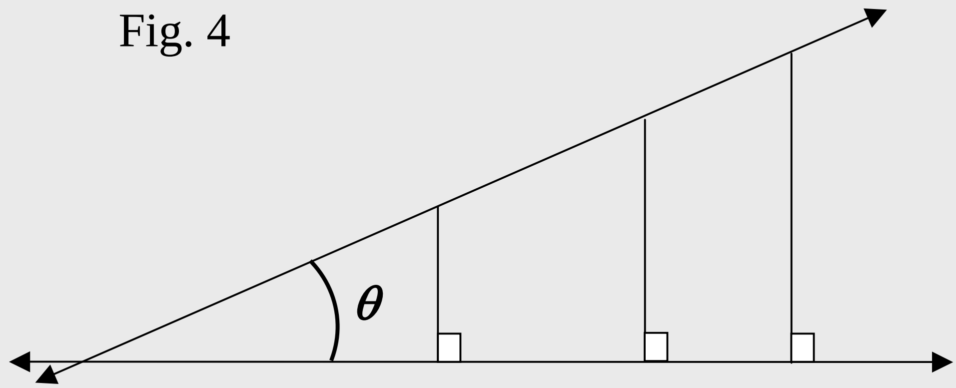
angles made by two lines

We want to know about angles made by two lines. And talking about angles, we often use right triangles. ∠ ☺

We use them as angle signs. And an angle has to do with a ratio between two sides called the two legs in a right triangle.

A line can make an angle against a horizontal line. And we can make many right triangles as shown in the figure below.

Fig. 4



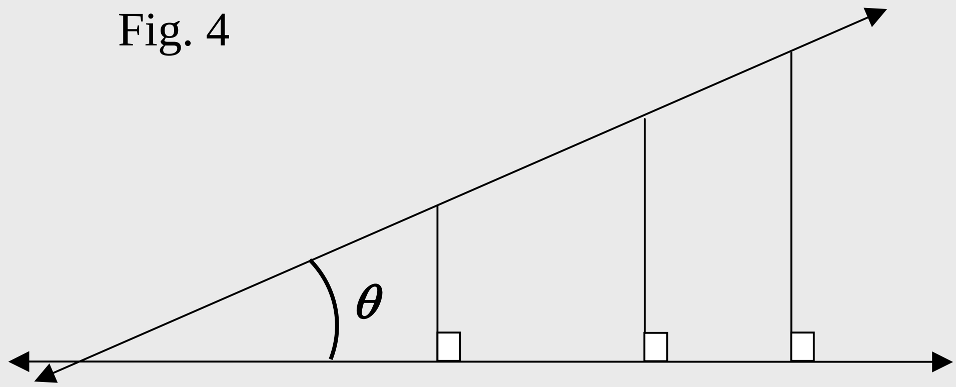
Then, the angle depends on the ratio between two sides, one is horizontal, and the other is vertical.

So it depends on the ratio of the vertical side to the horizontal side in every right triangle made the way above.

And in math, the ratio is often called the ***rise-over-run***.

Now, if the angle is fixed, the ratio doesn't change, and vice versa. So if the ratio is fixed, the angle doesn't change.

Fig. 4

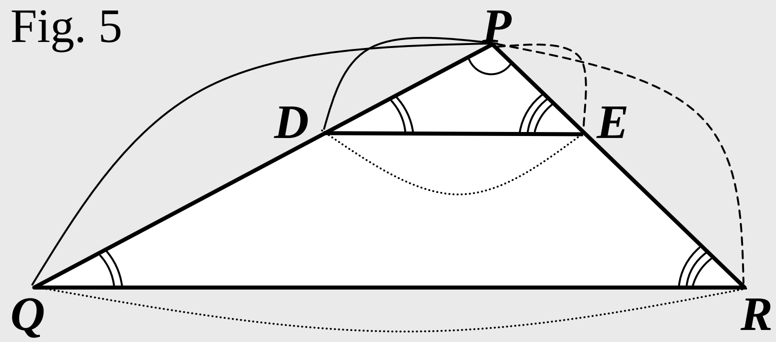


What then, can we do with the fact above?

We can use it making a similar triangle.
How then can we use it when making a similar triangle?

If keeping the ratio the same, we keep the angle the same, that is, unchanged.

Fig. 5



So keeping unchanged the ratio between two sides in a triangle when we change the sides, we keep the angles unchanged.   What then, do we get?

Similar triangles share the same group of angles.

Fig. 6

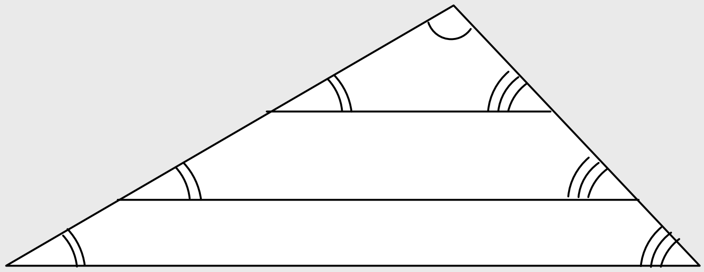
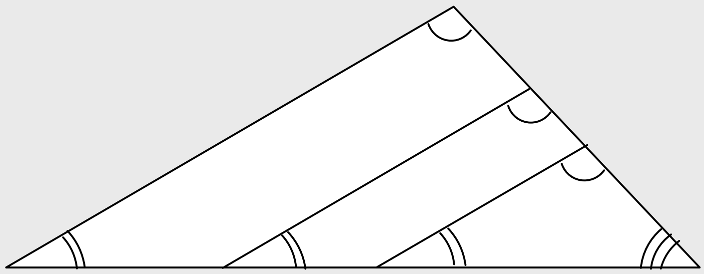


Fig. 7

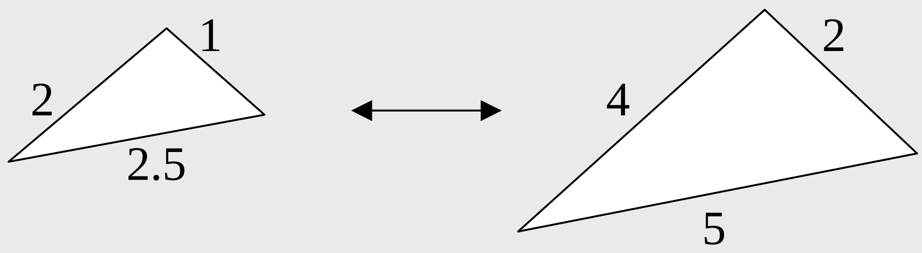


So we get a similar triangle.
If keeping unchanged the ratio between two sides in a triangle when we change the sides, we get a similar triangle.

Well then, how does each side in a triangle has to change to become a side in a similar triangle?

If all the sides get longer, or all the sides get shorter ***at the same ratio***, then a similar triangle gets made.

Fig. 8



And scaling up or down a value, we can do it using the math tool called a ratio.

So it can also, be called a ***scale*** or ***scale factor***.

For instance, scaling down 9 by a factor of a third, we get 3.

Scaling down 6 by a factor of a half, we get 3.

And scaling up 3 by a factor of 2, we get 6.

So if all the sides get longer, or all the sides get shorter ***on the same scale***, then a similar triangle gets made.

In other words, if the ***same scale factor*** is applied to all the three sides, then the new triangle is similar.

A ***scale factor*** is another name for a ***ratio*** in this case.

And applying it to a side, we multiply the side by it.

What then, about the reverse?

We know that if a similar triangle gets made, all the sides lengthen (or shorten) ***at the same ratio***.

In other words, if the new triangle is similar, the ***same scale factor*** is applied to all the three sides.

So, between two similar triangles, reversing the idea above, we do three times the process as follows.

> In one of the two triangles, divide a side by its corresponding side in the other triangle.

What then, do we get?

If in one of two similar triangles, dividing each side by its corresponding side in the other triangle, what do we get?

We get the same scale factor, that is, the same ratio.

The reverse is important, very important.

It’s because it is often used when you take an exam, test, and quiz, and of course, when doing homework.

Doing the problems, you often need to check to see if two triangles are similar. You can do it doing the reverse above.

And yet, better be careful, since you might get confused. It might not be clear to divide which side by which side.

How then, can you not get confused?

You’ve gotta do the exact opposite.
Opposite of what?

The process where you get a similar triangle.

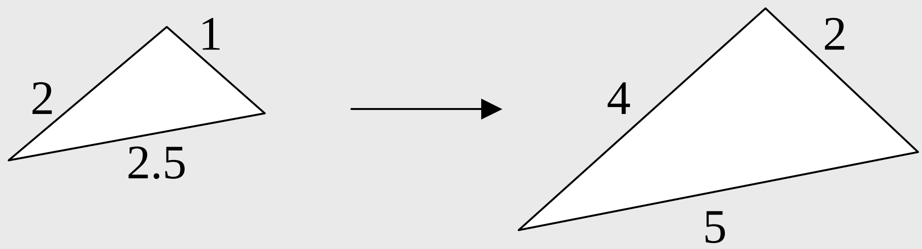
Not quit clear? Sill foggy?

Examples are the best teacher.

So let’s now, take some examples.

As shown in Fig. 9 below, each and every side increases twice as long, so each side in a similar triangle is twice its matching side in the triangle to be changed.

Fig. 9

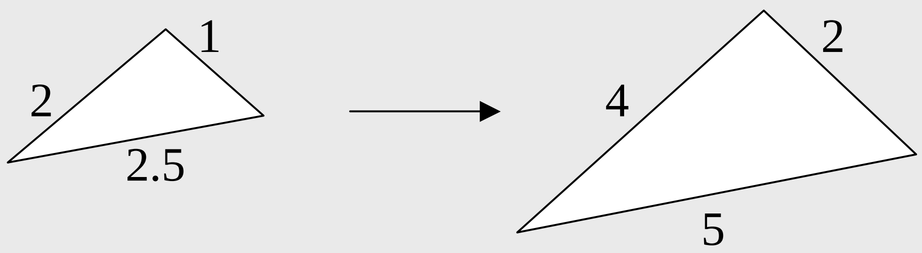


Each and every side in the triangle on the right is twice its matching side in the triangle on the left, and for instance, the side of length 2 matches the side of length 4.

What side then, does the side of length 2.5 match?

We call it the ***corresponding*** side.

Fig. 9



And it’s the side of length 5, which is therefore, the side ***corresponding*** to the side of length 2.5.

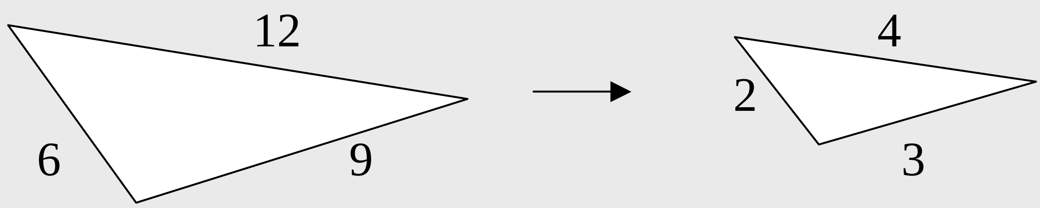
By the same token, the side of length 1 matches the side of length 2, which belongs to, of course, the triangle on the right hand side, and is the side ***corresponding*** to the side of length 1.

What then, is the corresponding side?

The matching side. It is the side that matches the side increased or decreased accordingly, according to the pattern the changes are made.

For another instance, every side shortens to its third, so each side in the similar triangle is a third of its matching side in the triangle to be changed as shown below.

Fig. 10



Each and every side in the triangle on the right is a third of its matching side in the triangle on the left. For instance, the side of length 9 matches the side of length 3, which is the side ***corresponding*** to the side of length 9.

And of course, in each of Fig. 9 and Fig. 10 above, the two triangles share the same group of angles, because the two are similar.

Well then, how does each side in a triangle has to change to become a side in a similar triangle?

Let’s now, reconsider the figures below, and compare the two sides in each pair of sides corresponding to each other.

Fig. 9

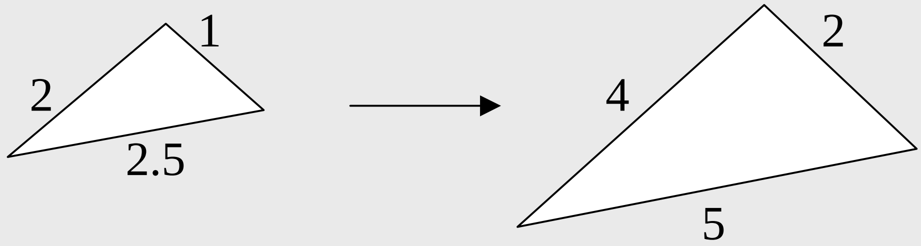
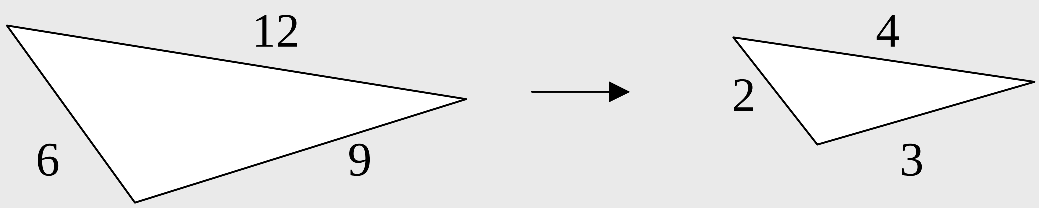


Fig. 10



In Fig. 9, multiplying by **2** every side in the triangle on the left, we get the triangle on the right.

In Fig. 10, multiplying by **a third** every side in the triangle on the left, we get the triangle on the right.

And we can call **2** and **a third** stated above ***scale factors***.

What then, do you think we need to do to each and every side in the triangle on the left to get the triangle on the right?

Fig. 9

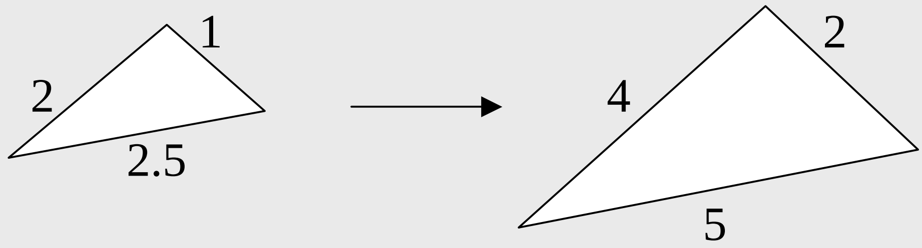
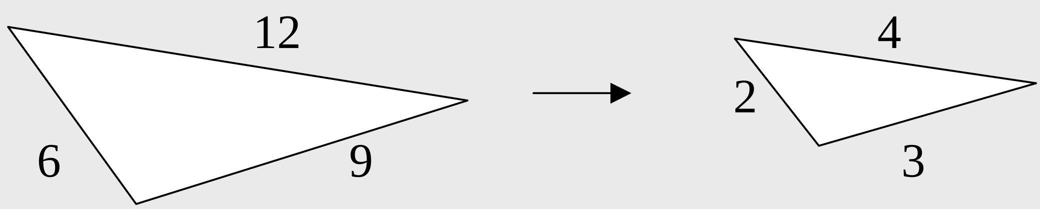


Fig. 10



We multiply each and every side by the same scale factor, that is, the same ratio.

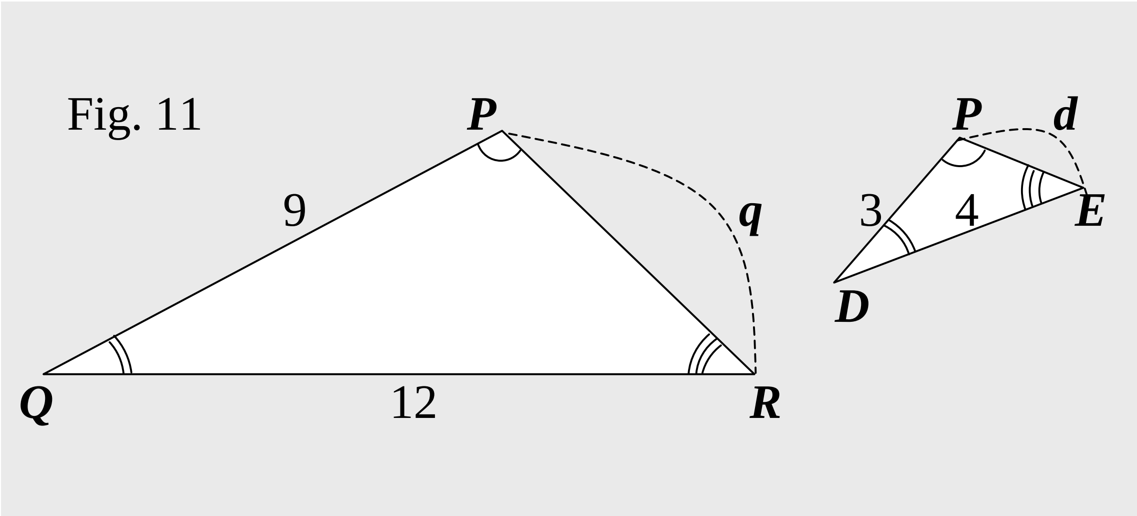
So multiplying each and every side by the same scale factor, that is, the same ratio, we get a side group that forms a similar triangle.

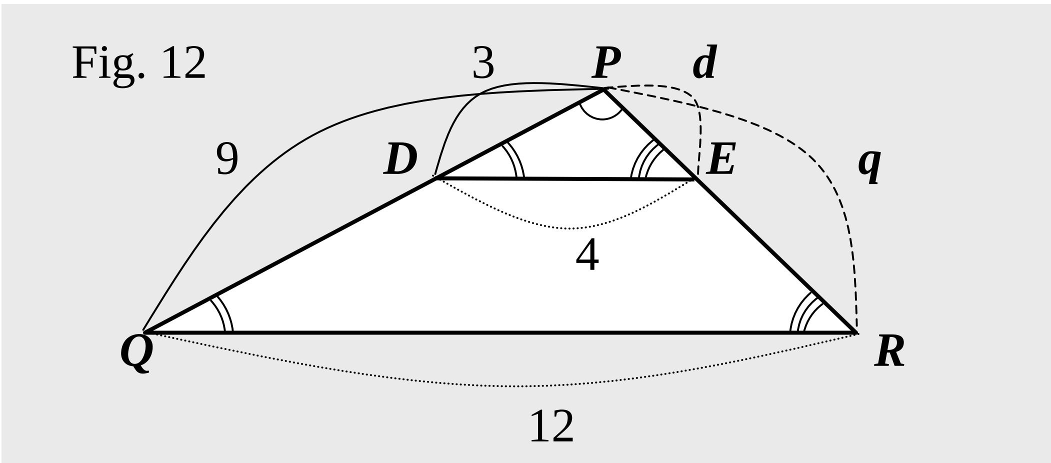
Applying the same scale factor to every side in a triangle, we multiply every side by the same scale factor, and then we get every side in a similar triangle, that is, we get the side group that makes a similar triangle.

In short, multiplying every side by the same scale factor, that is, the same ratio, we get a similar triangle.

Let’s now take an example.

Refer to the figures below. Both are the same.





The side of length 9 changes to the side of length 3, and the side of length 12 changes to the side of length 4.

What then, is $\dfrac{d}{q}$?

It is a third, so we get this: $\dfrac{\boldsymbol{d}}{\boldsymbol{q}} = \dfrac{1}{3}$. How?

Fig. 11

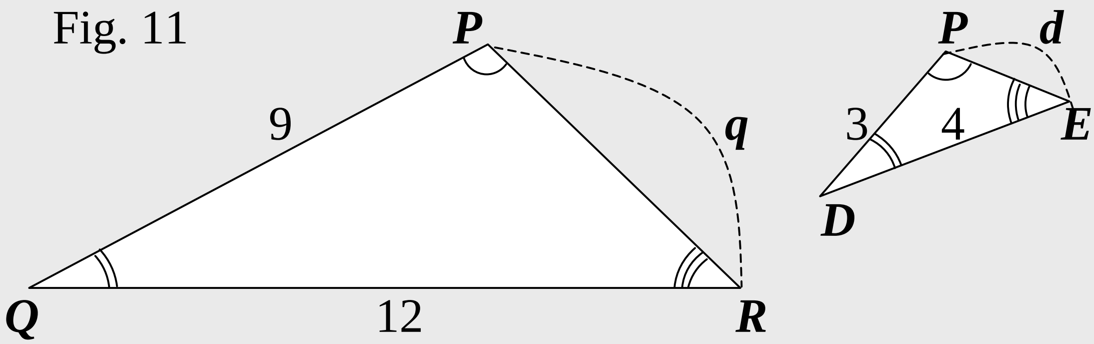
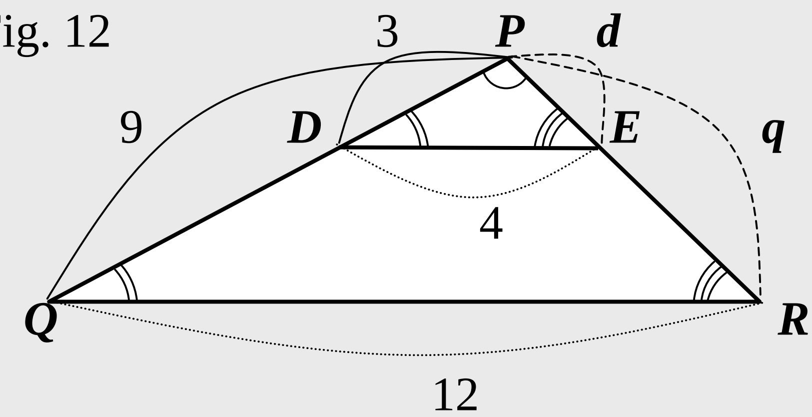


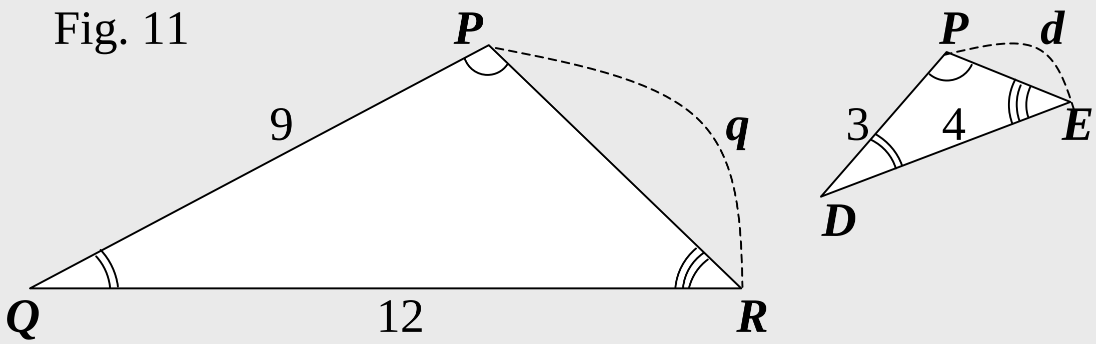
Fig. 12



We can see that a third of each side in $\Delta \boldsymbol{PQR}$ is a side in $\Delta \boldsymbol{PDE}$.

That is, we get $\frac{1}{3}$ x $9 = \frac{9}{3} = 3$ and $\frac{1}{3}$ x $12 = \frac{12}{3} = 4$.

Fig. 11



And we have $\frac{3}{9} = \frac{1}{3}$ and $\frac{4}{12} = \frac{1}{3}$.

Thus, we get this: $\frac{1}{3}$ x $\boldsymbol{q} = \frac{\boldsymbol{q}}{3} = \boldsymbol{d}$, and $\frac{\boldsymbol{d}}{\boldsymbol{q}} = \frac{1}{3}$.

We can put the operations above in plane language the way as follows.

A third of 9 is 3, and a third of 12 is 4.

And the two ratios are equal, and are $\frac{1}{3}$.

Thus, a third of $\boldsymbol{q}$ is $\boldsymbol{d}$, and the ratio $\frac{\boldsymbol{d}}{\boldsymbol{q}}$ is $\frac{1}{3}$. Now, what?

Now, the definition comes in.

> *By definition*, if each and every side changes with the same particular ratio, the new triangle is similar.

And we can put the same the way as follows, too.

> *By definition*, if each and every side changes on the same scale, the new triangle is similar.

And the reverse is true, too.

> So now again, *by definition*, if the new triangle is similar, all the three sides change on the same scale.

And we can put the same the way as follows, too.

> *By definition*, if each and every side changes with the same scale factor, the new triangle is similar.

We'll continue this in the next lesson.

Similar Triangles 6

Let's first, go over the last part in the previous lesson.

> *By definition*, if each and every side changes with the same particular ratio, the new triangle is similar.

And we can put the same the way as follows, too.

> *By definition*, if each and every side changes on the same scale, the new triangle is similar.

And the reverse is true, too.

> So now again, *by definition*, if the new triangle is similar, all the three sides change on the same scale.

And we can put the same the way as follows, too.

> *By definition*, if each and every side changes with the same scale factor, the new triangle is similar.

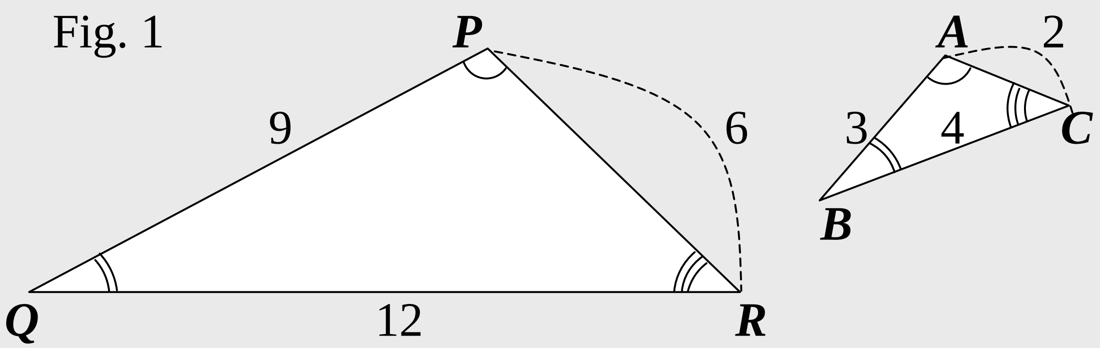
So what do we do to make a similar triangle?

We apply the same ratio to all the three sides.

A ratio is called a scale factor, too, so applying the same scale factor to all the three sides, we get a similar triangle.

So for instance, applying the same scale factor to each and every side of ΔPQR, we get a triangle similar to ΔPQR.

Fig. 1



If it becomes a third as long, the ratio is a third, and if it becomes twice as long, the ratio, i.e., the scale factor is two.

How then, do we apply the ratio to each side?

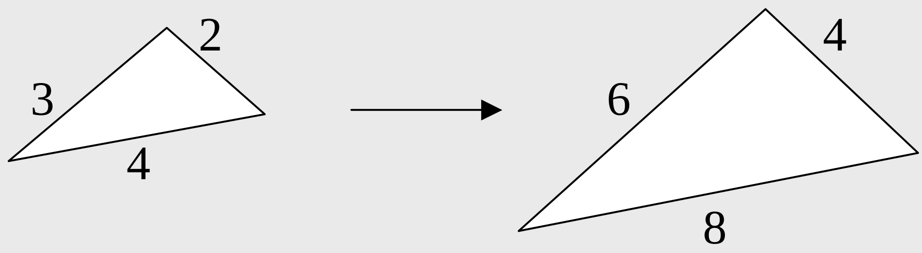
We multiply each side by the ratio.

What then is the product?

It is (the length of) a side in the similar triangle.

And we call it the *corresponding* side, which is the side that matches ***the side we apply the ratio to***.

Fig. 2



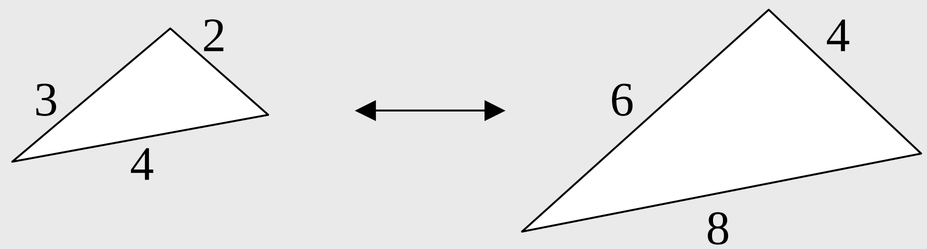
Each and every side in the triangle on the right is twice its matching side in the triangle on the left, and for instance, the side of length 3 matches the side of length 6.

The side we apply the ratio to is called ***the matching side*** in the statement regarding Fig. 2 above.

And of course, ***the side we apply the ratio to*** corresponds to its corresponding side, also.

It’s because both the two sides correspond to each other.

Fig. 3



The side of length 2 and the side of length 4 match and correspond to each other.

If, for another instance, this bolt corresponds to that nut, then that nut corresponds to this bolt, too.

So the bolt and the nut match, and correspond to each other.

Let’s now, discuss the idea above in the reverse manner.

What’s the idea, though?

The idea is this: Multiplying by the same ratio every side in a triangle, we get every side in a similar triangle.

So putting the idea in reverse, we can put it the way as follows.

First off, pair up the corresponding sides from two triangles. Then, we get three pairs.

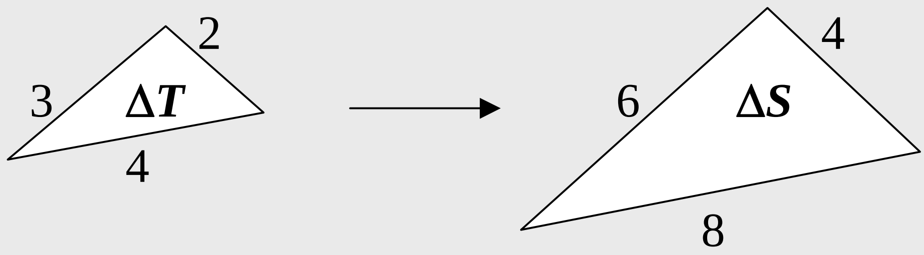
Next, take the ratio between the two sides in each pair. Then, we get three ratios. Do not just take those ratios, though. We take the three ratios reversing the way we make a similar triangle. The way we take the three ratios is crucial, and we'll cover it shortly.

And next, if the three ratios taken the way stated above are equal, the two triangles share the same shape, that is, the two are similar. Why, though?

It's because when taking the three ratios, we reverse the way we make a similar triangle. And the way is as follows.

When making **Δ*S*** that is similar to **Δ*T*,** for instance, we apply the same ratio to the three sides in **Δ*T***.

Fig. 4



Applying the same ratio to the three sides, we multiply by the same ratio each of the three sides.

The same ratio is 2 in the figure above.

And multiplying by the same ratio each of the three sides, we get the side corresponding to each of the three sides.

The side of length 4 corresponds to the side of length 2.

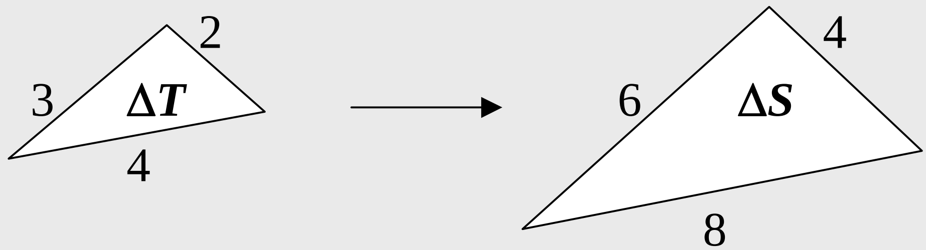
The side of length 6 corresponds to the side of length 3.

The side of length 8 corresponds to the side of length 4.

Then, the side we get belongs to $\Delta \boldsymbol{S}$, which is similar to $\Delta \boldsymbol{T}$.

So If dividing each side in $\Delta \boldsymbol{S}$ by its corresponding side in $\Delta \boldsymbol{T}$, we have to get the same ratio we applied to the three sides in $\Delta \boldsymbol{T}$ when making $\Delta \boldsymbol{S}$.

Fig. 4



Dividing the side of length 4 by the side of length 2, we get 2.

Dividing the side of length 6 by the side of length 3, we get 2.

Dividing the side of length 8 by the side of length 4, we get 2.

If, therefore, reversing the way we make a similar triangle, we do the division stated above three times, and get three equal ratios if, of course, the two triangles are similar.

And thus, we can put the definition regarding the group of sides this way: ***Similar triangles*** mean three ***equal ratios***, which mean also, similar triangles.

And using a math symbol, we can put the idea above the way as follows:

Similar Triangles ⇔ Equal Ratios

Or this way:

Similar Triangles ⇔ Ratios Unchanged

The ratio between two corresponding sides stays the same.

And equal ratios mean the same scale factor.

So we can put the definition the way as follows, too.

Similar Triangles ⇔ The Same Scale Factor

And thus, the three definitions below are all the same.

Similar Triangles ⇔ Equal Ratios

Similar Triangles ⇔ Ratios Unchanged

Similar Triangles ⇔ The Same Scale Factor

So we’ve got a definition now.
That’s what similar triangles mean in math, though it’s not the only one; we have some other ways to do it.

And it’s one thing. You want to know what you can do with it. That is to say that you want to know how to use it when solving problems, of course. Examples are the best teacher.

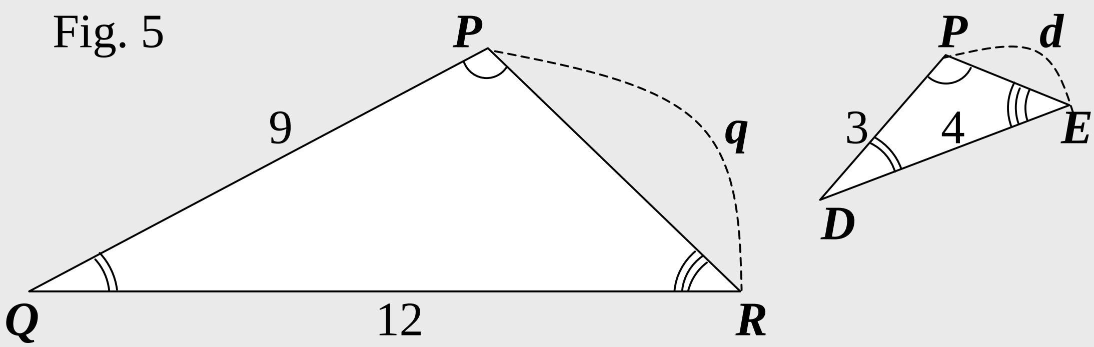
So let’s now take some examples, and see what we can do with the definition above.

We may want to start going over the example taken earlier.

Suppose first, that the two triangles below are similar.

So we have this: $\Delta PQR \sim \Delta PDE$.

Fig. 5



In Fig. 5 above, the side of length 9 in ΔPQR corresponds to the side of length 3 in ΔPDE, and the side of length 12 in ΔPQR corresponds to the side of length 4 in ΔPDE.

What then does the side q in ΔPQR correspond to?

It is the side $\boldsymbol{d}$ in $\Delta\boldsymbol{PDE}$.

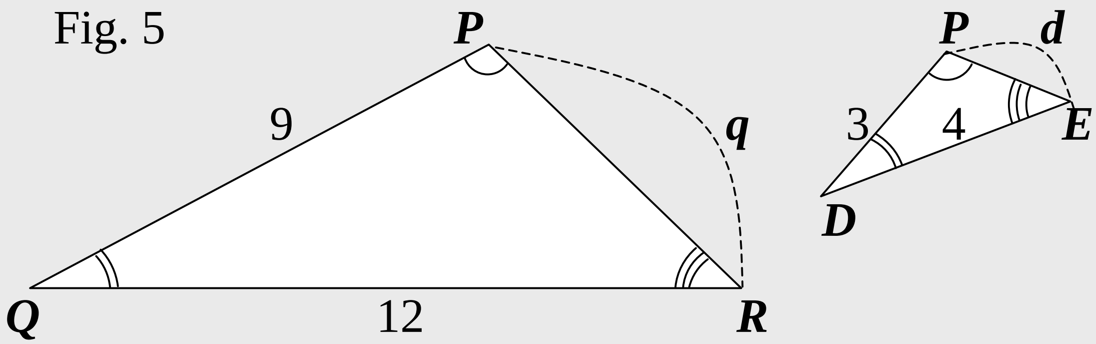


Fig. 5

And we can get this: $\frac{9}{3}=\frac{12}{4}=3$. Now, the definition says three equal ratios. So we can get this, too: $\frac{\boldsymbol{q}}{\boldsymbol{d}}=3$.

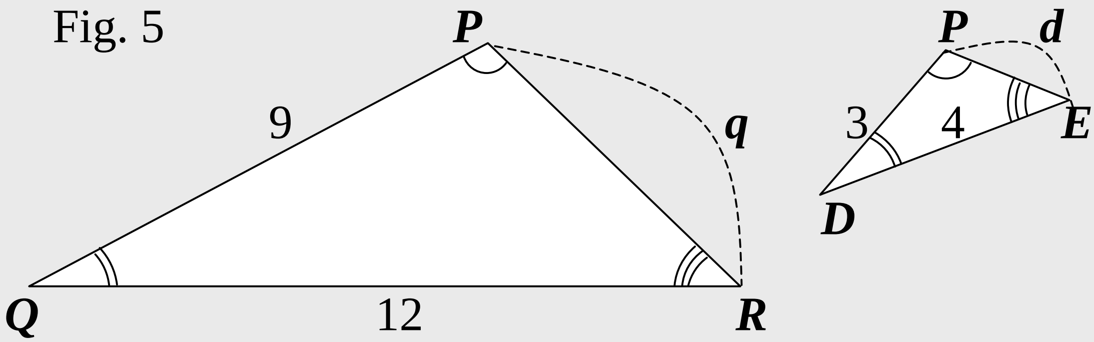
That is, we can get this: $\frac{9}{3}=\frac{12}{4}=\frac{\boldsymbol{q}}{\boldsymbol{d}}=3$.

Thus, we can get three equal ratios, each of which is a ratio of a side in $\Delta\boldsymbol{PQR}$ to its corresponding side in $\Delta\boldsymbol{PDE}$.

What then is the point?

We now, have this: $\frac{9}{3} = \frac{12}{4} = \frac{\boldsymbol{q}}{\boldsymbol{d}} = 3.$

Fig. 5



Therefore, the point is this:

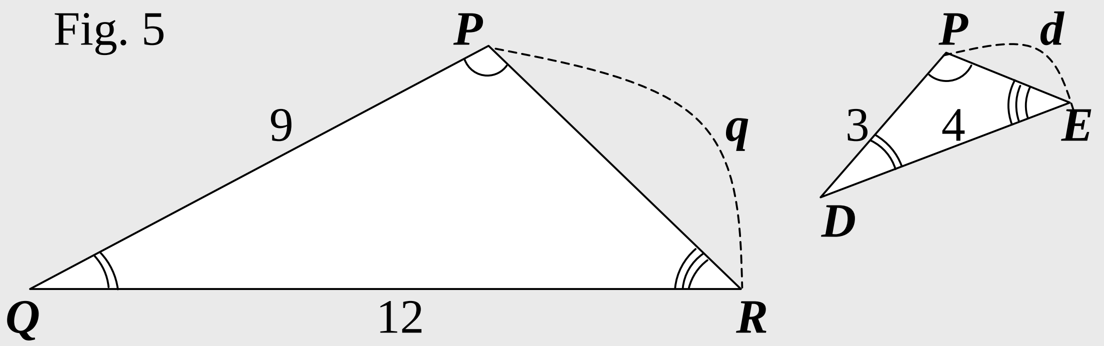
All the denominators are the sides from one triangle only, and the numerators are the sides from the other triangle only. And, of course, in each ratio, both the top and bottom corresponds to each other.

And you may want to keep in mind the point above

It's not the only way, though, where we can get three equal ratios. The other way is, however, basically the same way, or rather, the same idea, but is a bit different, because we just switch the point of reference. Switching the reference point, we just get their reciprocals. And they are as follows.

We can put the ratios a bit differently the way as follows.

Fig. 5



First, we can get this: $\frac{3}{9} = \frac{4}{12} = \frac{1}{3}$. So next, we can get this, too: $\frac{\boldsymbol{d}}{\boldsymbol{q}} = \frac{1}{3}$ by definition, which says three equal ratios.

Therefore, we can get three equal ratios, each of which is a ratio of a side in $\Delta \boldsymbol{PDE}$ to its corresponding side in $\Delta \boldsymbol{PQR}$.

And in $\Delta \boldsymbol{PQR}$, multiplying each side by the ratio, $\frac{1}{3}$, we get its corresponding side in $\Delta \boldsymbol{PDE}$.

And also, in $\Delta \boldsymbol{PDE}$, multiplying each side by the ratio 3, we get its corresponding side in $\Delta \boldsymbol{PQR}$.

So in a particular triangle, multiplying each side by the same ratio, we get its corresponding side in another triangle said to be similar to the particular triangle.

How then can we pair up the sides corresponding to each other?

We ***apply the same ratio to all the three sides*** when making a similar triangle.

So among the sides, the length order doesn’t change.

And thus, the correspondence goes by the length.

Of the two triangles similar to each other, the longest side in one corresponds to the longest in the other. And the same is true of the other two sides in each triangle.

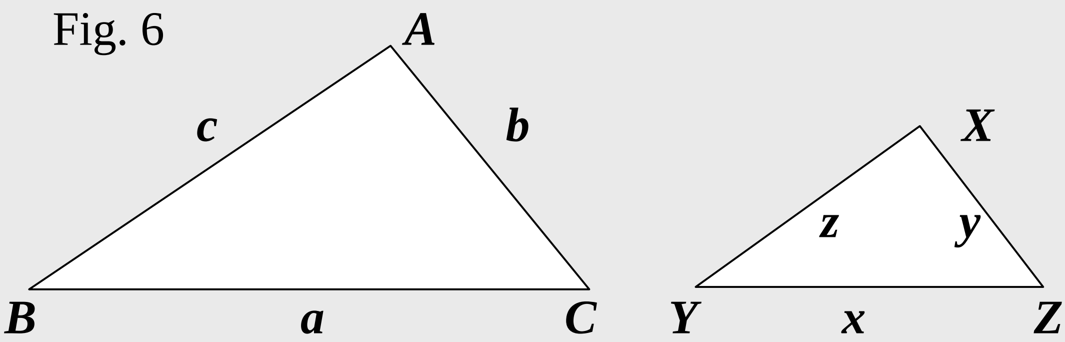
So the correspondence goes the way as follows.

> The longest side corresponds to the longest.
>
> The next longest corresponds to the next longest.
>
> And the shortest corresponds to the shortest.

And let’s see now again, how to check to see if two triangles are similar taking the three ratios.

First, we get three pairs of the sides that correspond to each other, and get three ratios the way as follows.

Fig. 6



The longest side corresponds to the longest.

The next longest corresponds to the next longest.

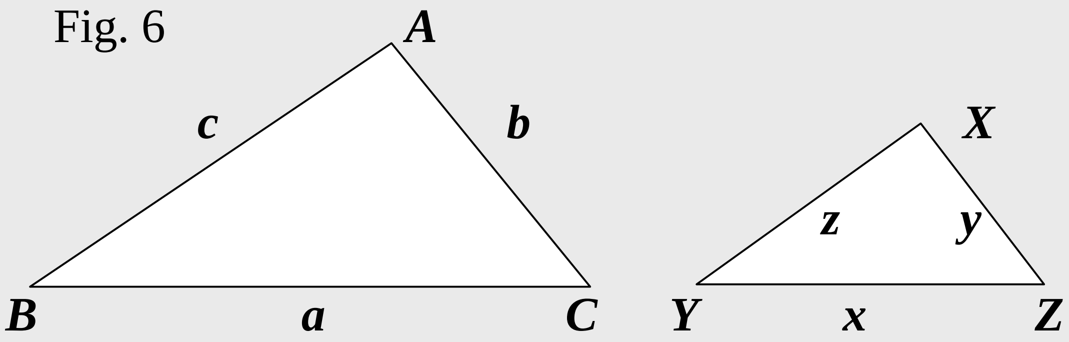
And the shortest corresponds to the shortest.

So the three pairs are (a, x), (b, y), and (c, z).

When we take the ratio between the two sides in each pair, we form a fraction where the numerator is a side in one triangle, and the denominator is a side in the other triangle.

So, the three ratios are these: $\frac{a}{x}$, $\frac{b}{y}$, and $\frac{c}{z}$.

Fig. 6



Of course, we don’t use the same side more than once when taking the three ratios.

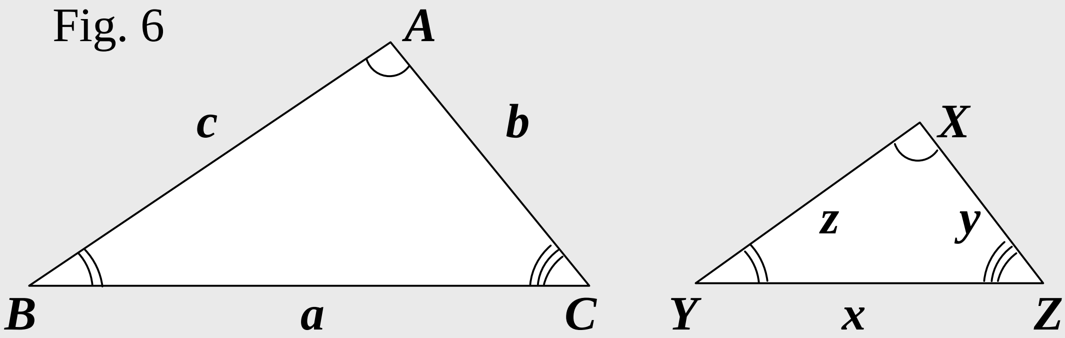
And make sure in every fraction, the numerator is a side in one triangle only, the denominator is a side in the other triangle only, and both sides correspond to each other.

So in the three ratios, ***all the numerators*** are the three sides in ***one triangle only***, all the denominators are the three sides in the other triangle only, and both sides in each ratio correspond to each other.

What ratios then do we get?

We get three equal ratios.

Fig. 6



In the three ratios, ***all the numerators*** are all the three sides in ***one triangle only***, all the denominators are all the three sides in the other triangle only, and also, both sides in each ratio correspond to each other.

And if we follow the rule stated in the box above, and the two triangles are similar, we get the ratios this way:

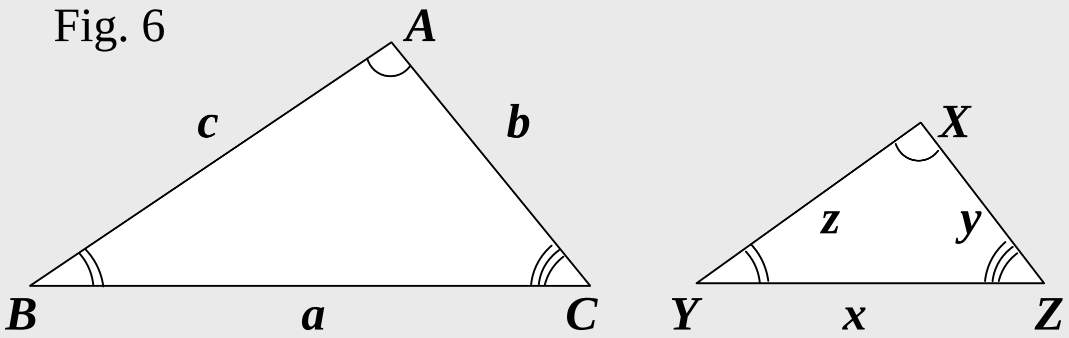
$$\frac{a}{x}=\frac{b}{y}=\frac{c}{z}.$$

So when we verify similar triangles without enough information on the angles, what matters first is the fact that the three ratios are equal.

It's because if multiplying each side by ***the same ratio***, that is, ***the same scale factor***, we get its corresponding side in a ***similar triangle***.

And we can get the three ratios the way as follows, too.

Fig. 6



$\frac{x}{a}$, $\frac{y}{b}$, and $\frac{z}{c}$, where consistency matters, that is, all the numerators have to be from one triangle only, all the denominators need to be from the other triangle only, and both sides in each ratio correspond to each other.

And of course, if we get the ratios this way: $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, the two triangles are similar. And also, if the two triangles are similar, we can get the ratios the way above.

Again, with not enough information on the angles, what matters first is the fact that the three ratios are equal.

And summing up, for now, we get the definitions as follows.

Similar Triangles ⇔ The Same Angle Groups

Similar Triangles ⇔ Equal Ratios

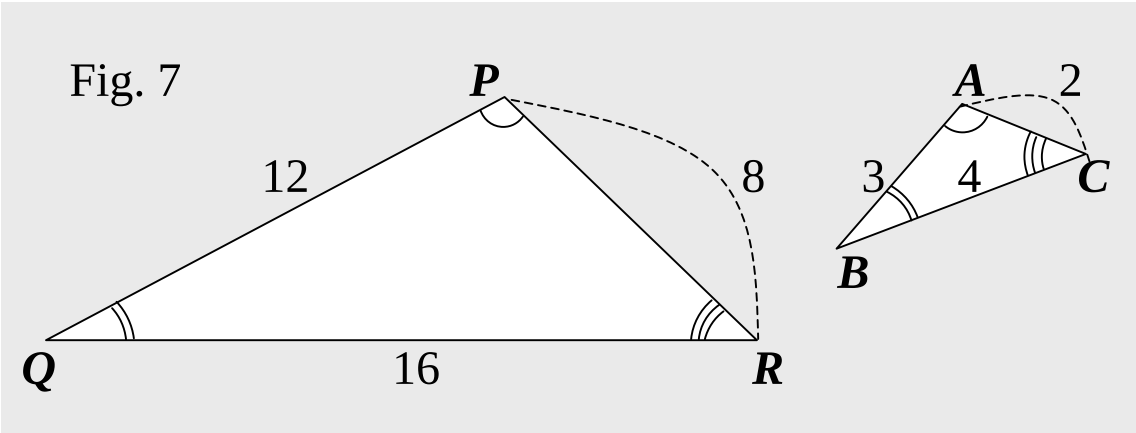
And we can put them the way as follows, too.

Similar Triangles ⇔ The Same Angle Groups

Similar Triangles ⇔ The Same Scale Factor

Let's now take the same example with different numbers.

Are the two triangles below similar?



Yes, they are, because we have $\angle \boldsymbol{P} = \angle \boldsymbol{A}$, $\angle \boldsymbol{Q} = \angle \boldsymbol{B}$, and $\angle \boldsymbol{R} = \angle \boldsymbol{C}$, that is, both triangles share the same angle group.

And we can put the idea this way: $\Delta \boldsymbol{PQR} \sim \Delta \boldsymbol{ABC}$.

Also, we get $\frac{12}{3} = \frac{16}{4} = \frac{8}{2} = 4$, or we get $\frac{3}{12} = \frac{4}{16} = \frac{2}{8} = \frac{1}{4}$.

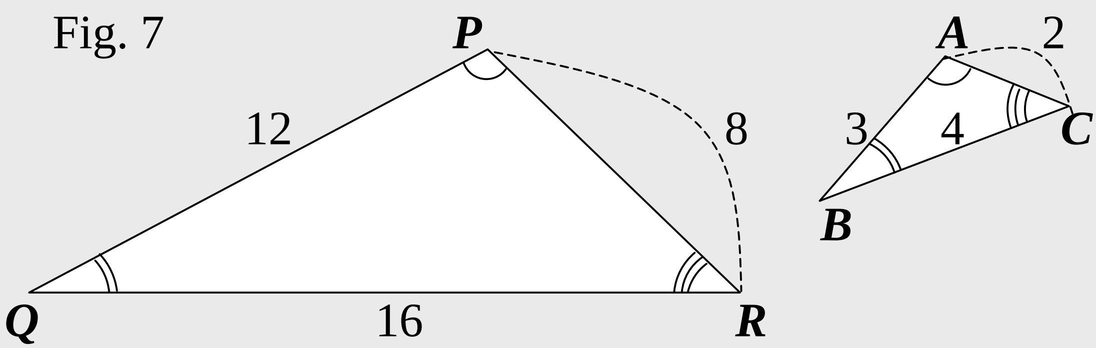
Either way, therefore, three equal ratios.

So now, what do we mean by the same shape with different size?

We mean similarity, which means, in case of triangles, the same group of three angles and the same ratio that applies to each and every side.

> So similar triangles share ***the same group of three angles***, and if multiplying by ***the same ratio*** each side in a particular triangle, we get to produce its ***corresponding*** side in a triangle ***similar*** to the particular triangle.

Fig. 7



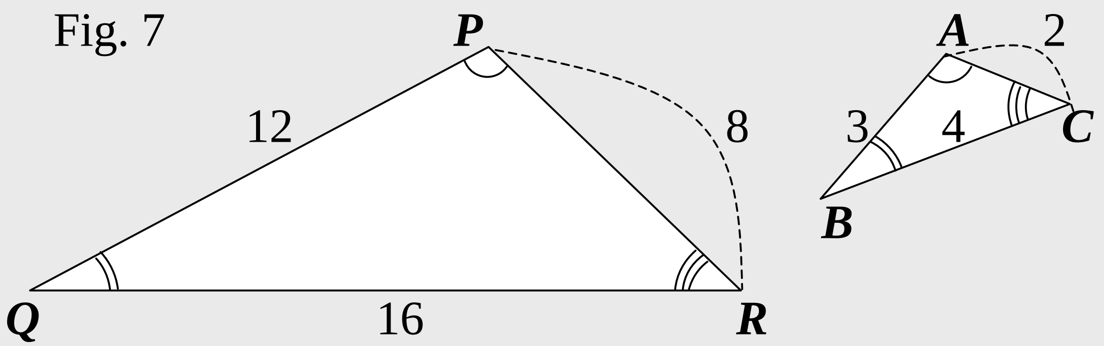
$\Delta \boldsymbol{PQR} \sim \Delta \boldsymbol{ABC}$, since $\angle \boldsymbol{P} = \angle \boldsymbol{A}$, $\angle \boldsymbol{Q} = \angle \boldsymbol{B}$, and $\angle \boldsymbol{R} = \angle \boldsymbol{C}$.

And equivalently, or rather, practically or efficiently, we can omit any one of the three equalities above.   Why?

It’s because if similar, the triangles share two angles, and vice versa, since the sum of the three angles in every triangle is the same, and is equal to 180°.

Now, if multiplying by *the same ratio* 4 (i.e., the same scale factor 4) each side in ΔABC, we get its *corresponding* side in ΔPQR, which is, of course, a triangle *similar* to ΔABC.

Fig. 7



So we get the side of length 8 corresponding to the side of length 2, get the side of length 12 corresponding to the side of length 3, and get the side of length 16 corresponding to the side of length 4.

Well then, how do we know if ΔPQR and ΔABC are similar? In other words, how do we check to see if the two are similar?

We can do so the way as follows.

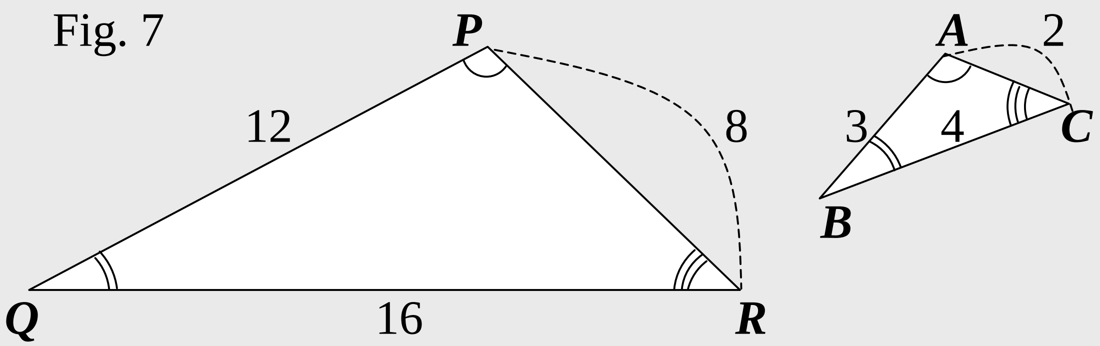


Fig. 7

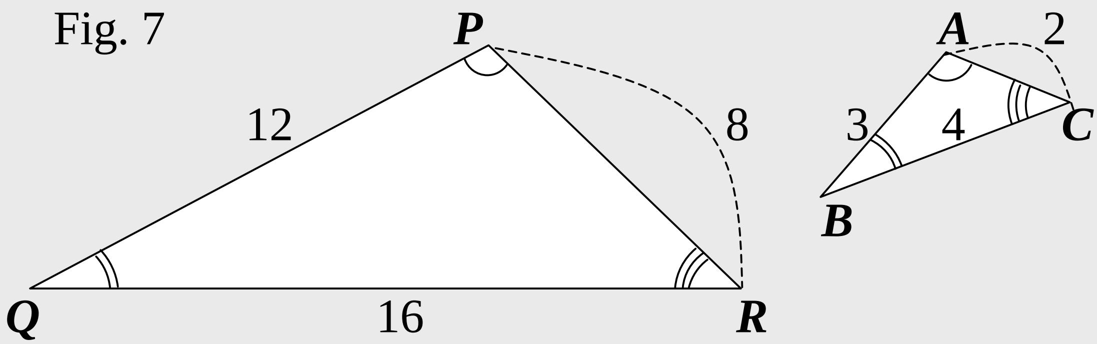
When we take the three ratios, the denominators are from one triangle only, the numerators are from the other triangle only, and the denominator and the numerator in each ratio are two sides corresponding to each other.

So taking the ratios, we get $\frac{12}{3} = \frac{16}{4} = \frac{8}{3} = 4$, and $\frac{3}{12} = \frac{4}{16} = \frac{2}{8} = \frac{1}{4}$.

So there is the same ratio, that is, the same scale factor, and it is 4 or $\frac{1}{4}$, and thus, we get this: $\Delta PQR \sim \Delta ABC$.

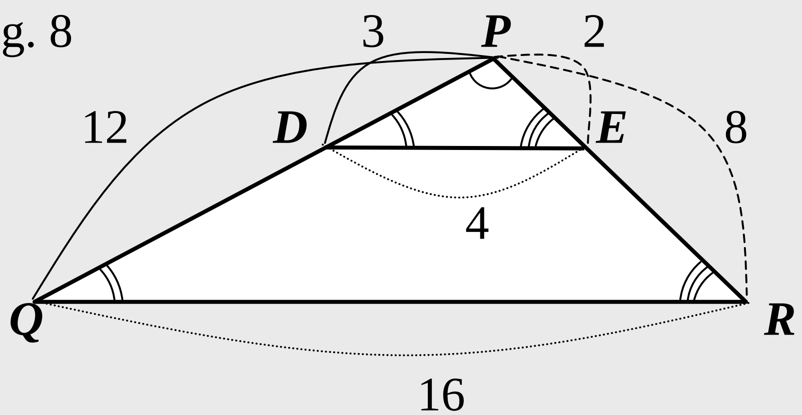
Now, multiplying, therefore, 4 by each side in ΔABC, we get its corresponding side in ΔPQR.

Fig. 7



And also, multiplying $\dfrac{1}{4}$ by each side in ΔPQR, we get its corresponding side in ΔABC. And renaming ΔABC to ΔPDE, we can put the figure above the way below, too.

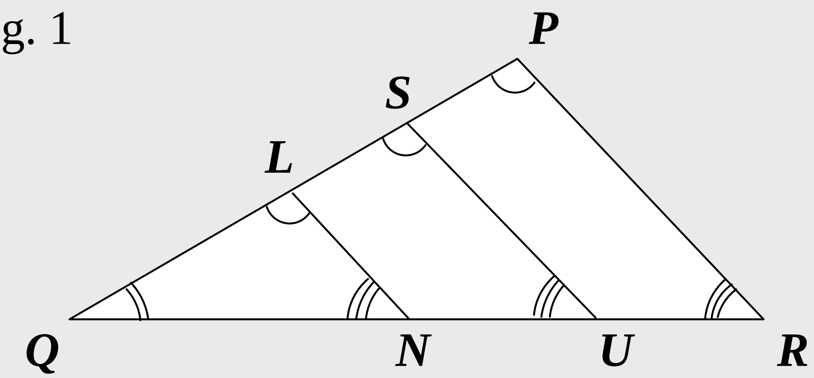
Fig. 8



Similar Triangles 7

Let's now move on to the next version, the eligibility regarding the combination of ***angles and sides***.

Fig. 1



This isn't just about the combination of angles and sides eligible for a similar triangle.

This is, actually, about those components that can determine a triangle, as well as the eligibility.

In short, it's about ***components determining a triangle***.

So a group of components can determine a triangle.

And we can call such a group a ***triangle determinant***.
What then, are those components?
What components can determine a triangle?

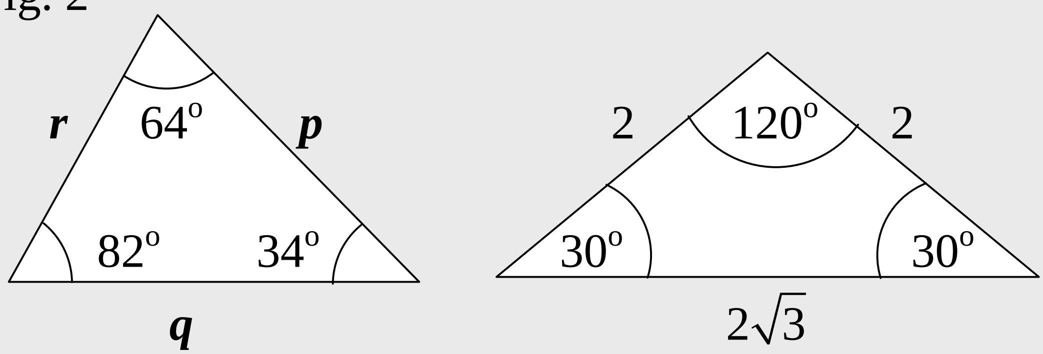
They are those components that we can specify by their values to define a particular triangle. So we can also say that if specified by their values, they are ***components defining a particular triangle***.

Note that we are ***not*** talking about a definition of a triangle, so not about a generic definition for triangles.

Such a definition says what a triangle is.

It can say, for example, that a triangle is made of three line segments closed end to end, and has three sides and three angles.

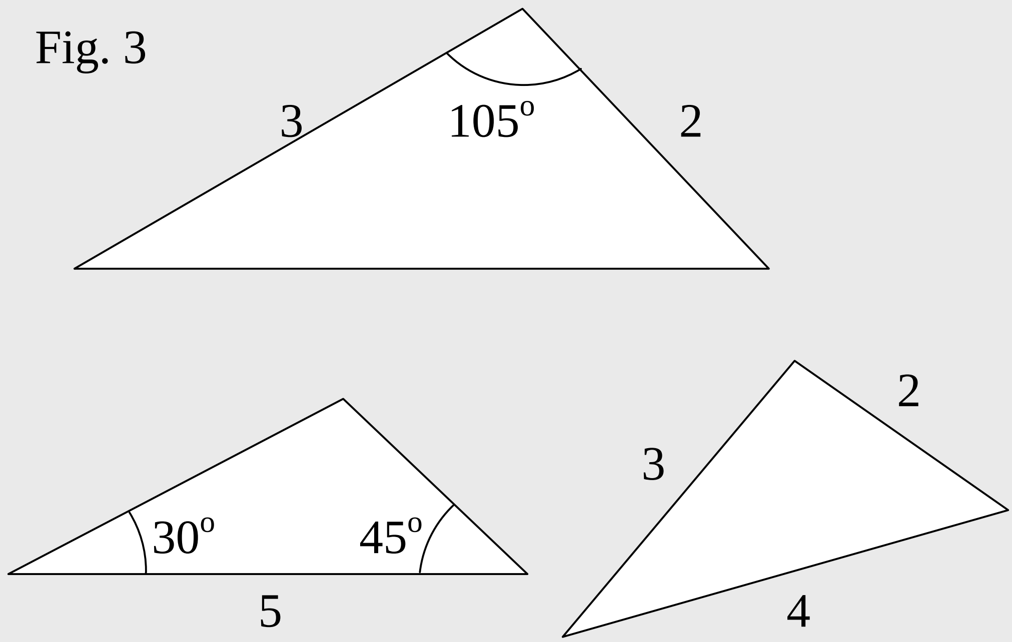
Fig. 2



What we are now talking about is ***triangle determinants*** and how to make ***a triangle definition***, that is, how to define a particular triangle.

A triangle definition provides specifics explaining a particular triangle, whereas a triangle determinant is a condition for a triangle to be determined.

Fig. 3



How then, do we define a particular triangle?

That is, how do we make a triangle definition?

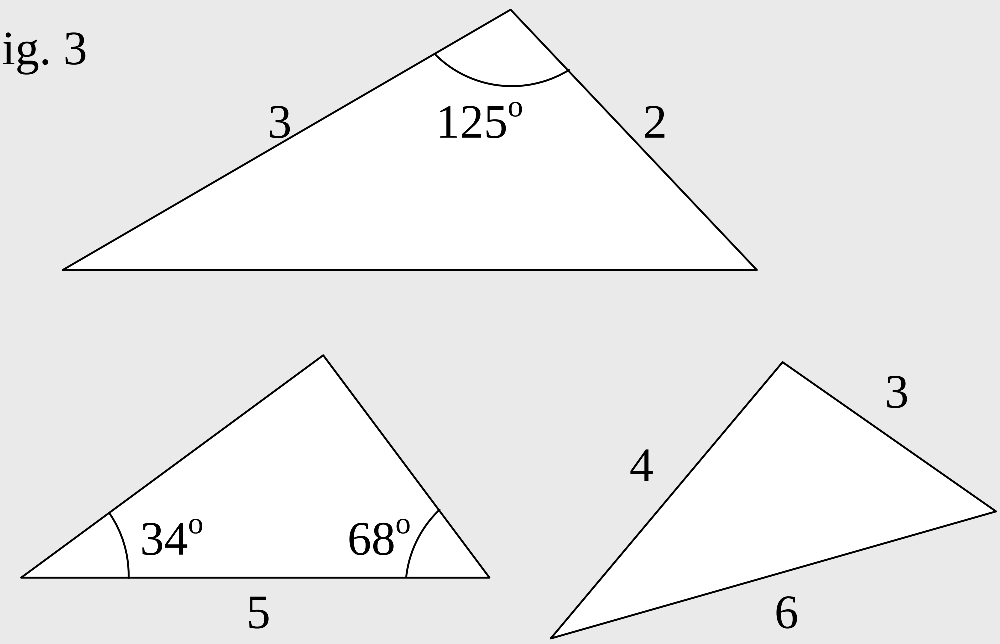
Do we just name it? So if defining it, for instance, can we just call it a triangle ***T*** or a triangle ***ABC***?

We may want to make sure, first, the difference between the two, a triangle definition and a triangle determinant.

First off, a triangle definition provides the values of those components that belong to a particular triangle so that people can identify the particular triangle.

And it specifies ***at least*** the values of those components that can determine the particular triangle.

Fig. 3



What then about a triangle determinant?

Literally, a triangle determinant just determines a triangle. It's a ***condition*** that determines a triangle. And such a condition is a group of components.

So a triangle determinant is a group of components that determines a triangle. And we only itemize the components. So we don't specify their values.

As you know, the components are in two kinds, one is in sides, and the other is in angles.

So when itemizing them, we just make a short list, which says the components in each kind if required to determine a triangle, but does not show their values as lengths of sides or magnitudes of angles.

Thus, if a group of components determines a triangle, we can call the group a triangle determinant. And its concept is quite close to, or rather, is the basis of the concept of a triangle definition.

So let's now get back to the question as follows.

How do we make a triangle definition?

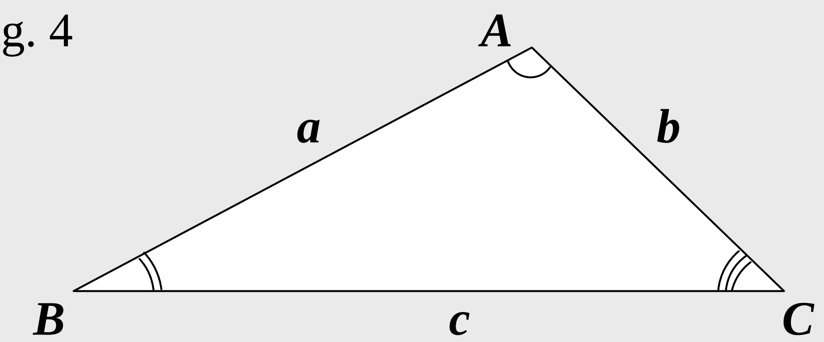
When we define a particular triangle, we don't just only name the triangle.

The name is only a part of the triangle definition.

Referring to a triangle, we can use its name.
So its name is convenient. It's, however, only for convenience, and it alone doesn't do the job, the definition.

Defining a particular triangle, we can specify by values only those components that determine the triangle, along with the name of the triangle. And we can call a group of those components a ***triangle determinant***, which is in three kinds.

Fig. 4



$\angle A = ?$ $\angle B = ?$ $\angle C = ?$ $a = ?$ $b = ?$ $c = ?$

How then can we define a particular triangle using the idea of such a determinant?

Suppose for instance, we want to define a triangle called T.

How then can we define the triangle T?

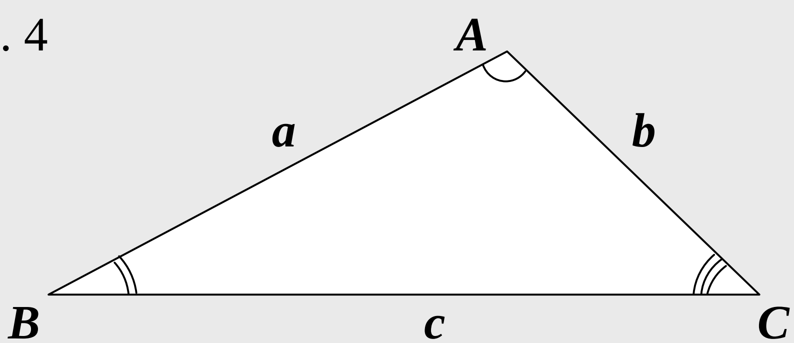
First off, its name is T. When naming a triangle, we often use a small triangle as a math symbol, so using the symbol, we can name the triangle this way: ΔT, read as a triangle T.

And of course, we can name a triangle this way, too: ΔABC assuming the three vertices of the triangle are A, B, and C.

That's an easy part; we can just name it using whatever letter we may want (as long as it's clear, of course).

What then about the other part we produce to define ΔT?

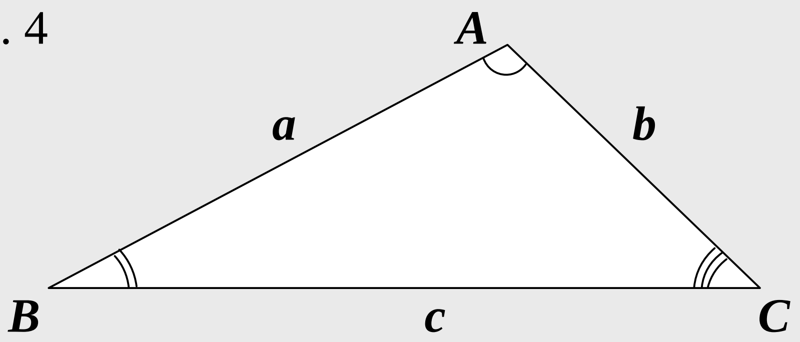
Fig. 4



$\angle A = ?$ $\angle B = ?$ $\angle C = ?$ $a = ?$ $b = ?$ $c = ?$

It can be the case that the other part is made of only those components that can determine ΔT, or of more than those.

Fig. 4



$\angle A = ?$ $\angle B = ?$ $\angle C = ?$ $a = ?$ $b = ?$ $c = ?$

So instead of producing all the six components, that is, three angles and three sides, we can just produce only those components that can determine ΔT.

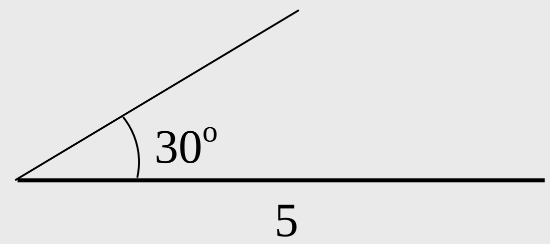
And producing those, we specify those by their values. Then, ΔT is determined specifically by the values of those components, that is, ΔT is defined.

Suppose for instance, one of the components is an angle, and another is a side. Then, producing them, we specify the magnitude of the angle and the length of the side.

So for instance, if producing an angle and a side, we can make a statement that in ΔT, an angle is 30^o, and a side is 5, or that ΔT has an angle of 30^o and a side of length 5.

And producing enough of those components, we define ΔT.

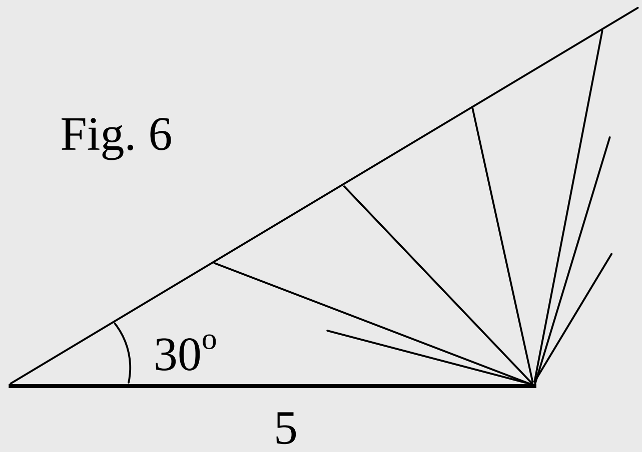
Fig. 5



Are the two enough, then? That is, are the two components produced in the example above enough to define ΔT?

Not enough yet. We need to produce more.
The two components produced in the example above are not sufficient to define a particular triangle.

Fig. 6



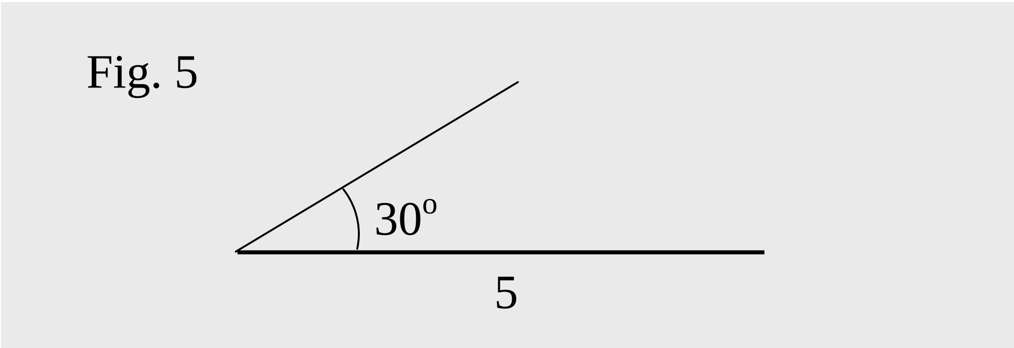
As shown above, there can be many triangles that can have an angle of 30^{o} and a side of length 5. Defining therefore, a particular triangle, we need to produce enough components that can determine the particular triangle ***only***.

Producing enough components, though, we don't need to produce all the components the triangle is made of.

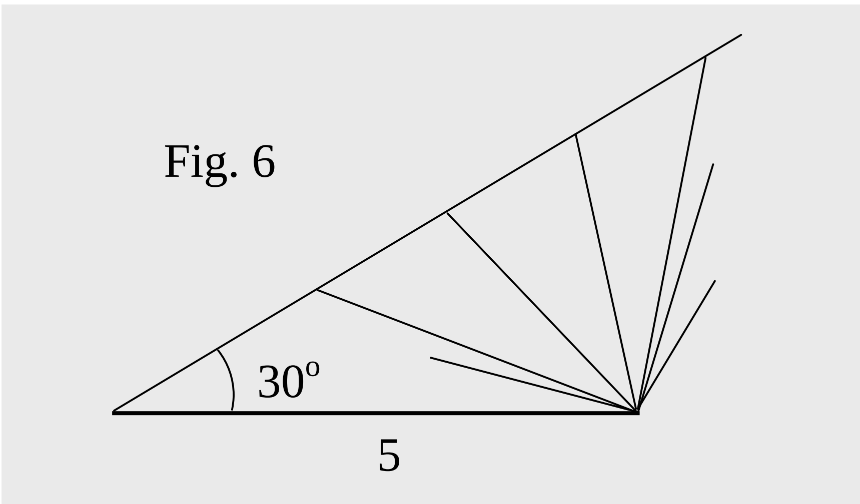
So we don't have to produce all the three angles and the three sides. How many components are enough, then?

And what components are they?

If producing enough components, we produce enough info that people can use to identify a particular triangle.



The info specified above is insufficient, that is, not enough to define a particular triangle, because more than one triangle have an angle of 30^{o} and a side of length 5 as shown below.



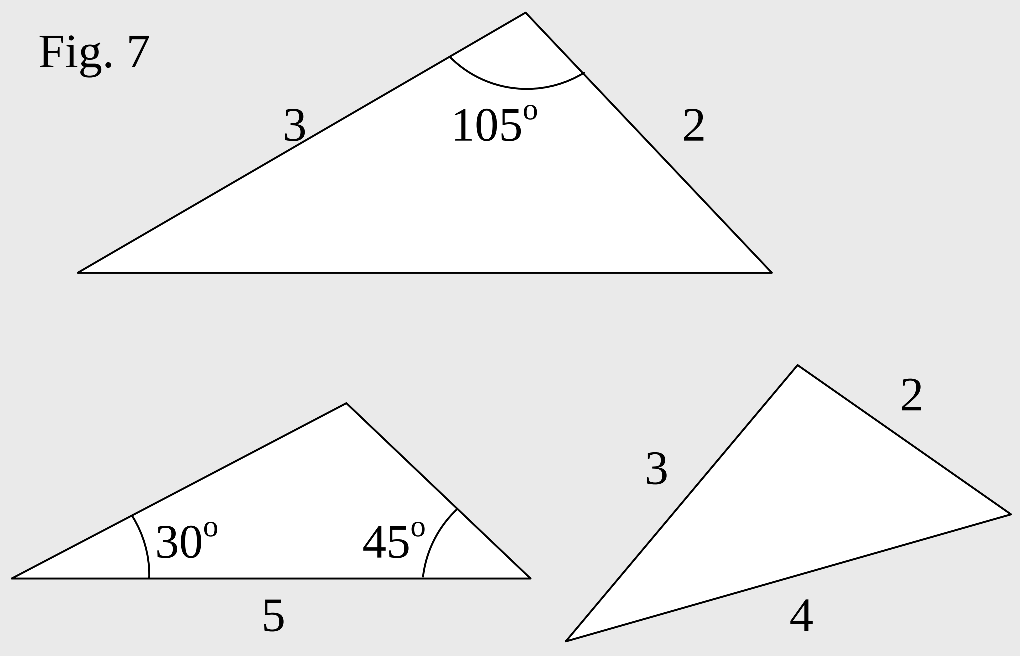
What then, do we mean by the enough components, that is, the enough info?

In mathematics, too, ***enough*** often means ***sufficient***. So in math, an enough amount often means the sufficient amount or the minimum required.

Producing, therefore, enough components, we produce the ***minimum*** components, that is, ***the least amount*** of components that can let people identify a particular triangle.

Thus, defining a particular triangle, we can produce the minimum components that can determine the triangle.

Fig. 7



Why not ‘produce’ but ‘can produce’ in the sentence above?

We can produce more than the minimum if we really want to, because the additional won't hurt the definition if belonging to the triangle. The minimum is enough, and is all required.

What then do we mean by determining a triangle?

That is, what do we mean by a triangle determinant?

Mentioned earlier that it's an itemized list of components that can determine a triangle. And the list indicates the number of components in each kind, sides or angles, if required to determine a triangle, but not their values.

And the number is the minimum. So the list shows the minimum components required to determine a triangle.

Thus, unlike a triangle definition, it is a condition that determines a triangle, and states the minimum number of components in each kind required to determine a triangle.

How many are those, then?   And what are they?

That is, what components does the list have?

Are they angles, sides, or a combination of both?

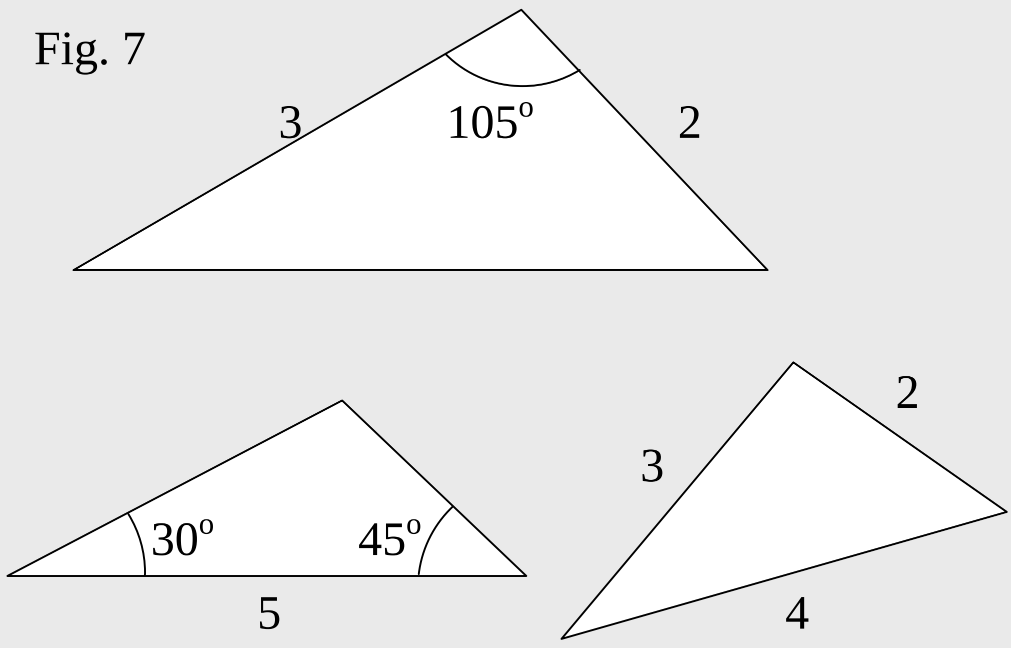
Looking at a bunch of triangles, we can see a pattern. And the pattern has something to do with three, 3.

In the word “triangle”, the prefix, “tri” means three.

And a triangle is made of three angles and three sides.

How many components then do you guess can determine a triangle?

Fig. 7



Three.

Thee components can ***determine a triangle***.

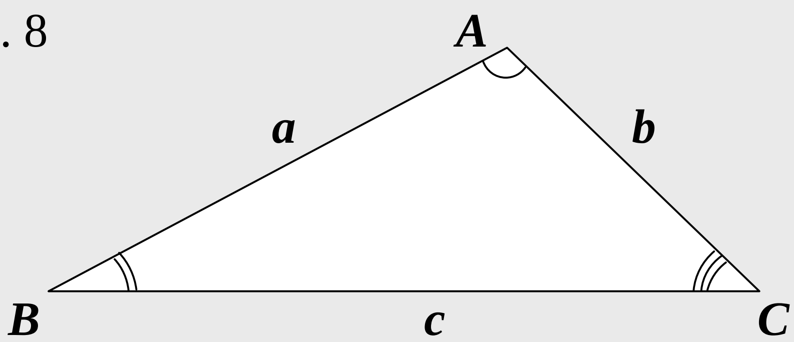
So a group of three components can be a triangle determinant.

Why ‘can be’ and why not just ‘is’ in the sentence above?

It’s because not all such groups are triangle determinants. That is, not any three components can determine a triangle.

What then, are the three?

Fig. 8



$\angle A$, $\angle B$, $\angle C$, a, b, and c

Of the six components above, which three can determine a triangle?

A triangle is made of three angles and three sides.

Fig. 8

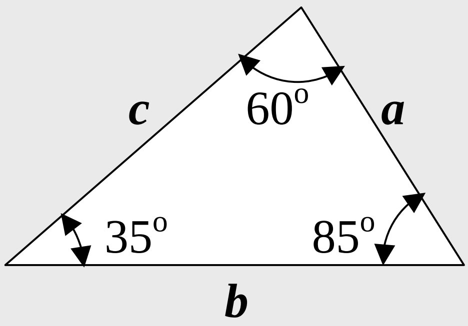


Fig. 9

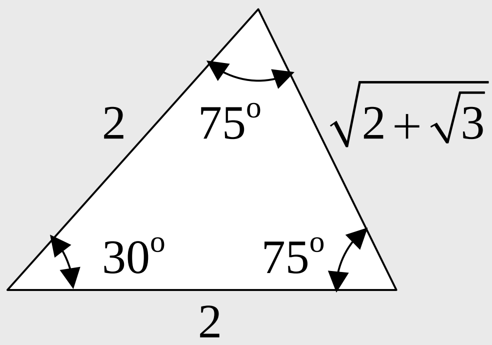
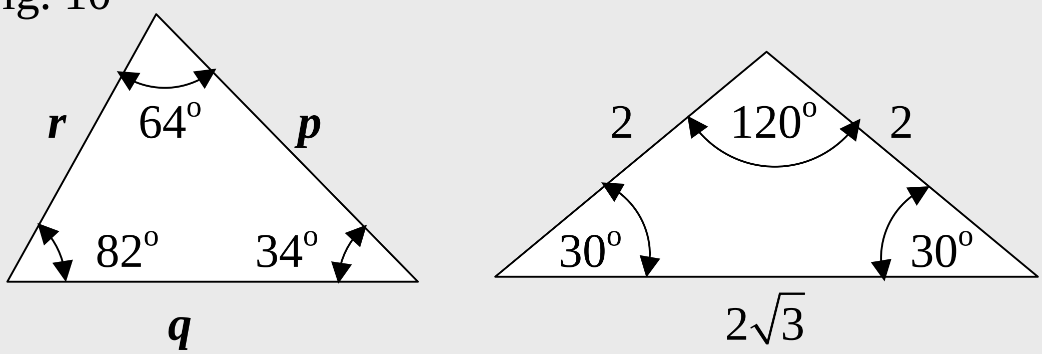


Fig. 10



So first off, can three angles determine a triangle?

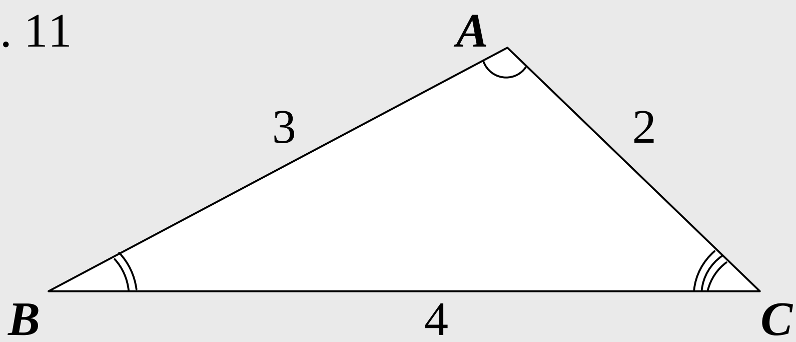
You know “They can’t.”, since many triangles share the same group of three angles. Infinitely many can share it.

And they are similar triangles.
Three angles, therefore, cannot determine a triangle.

So, let’s now, get back to the basics, where we can say again, “A triangle is made of three angles and three sides.”

So next, what about three sides?

Fig. 11

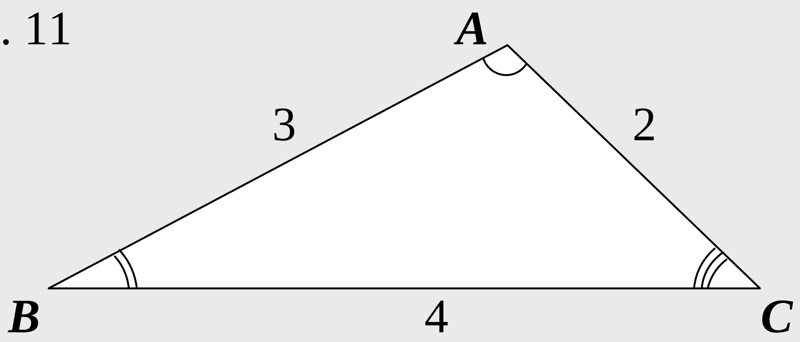


Yes, they do. Three sides do determine a triangle, so a group of three sides is a triangle determinant.

And if their lengths are given, that info is enough to define a particular triangle, because no other triangle has that info, and we don't need any other info to identify the triangle.

So for instance, if defining a triangle called ABC, shown below, we can put its definition the way as follows.

Fig. 11



A triangle ABC has a side of length 2, a side of length 3, and a side of length 4.

In short, ΔABC has three sides, which are 2, 3, and 4.

So the three sides are enough to define a particular triangle.

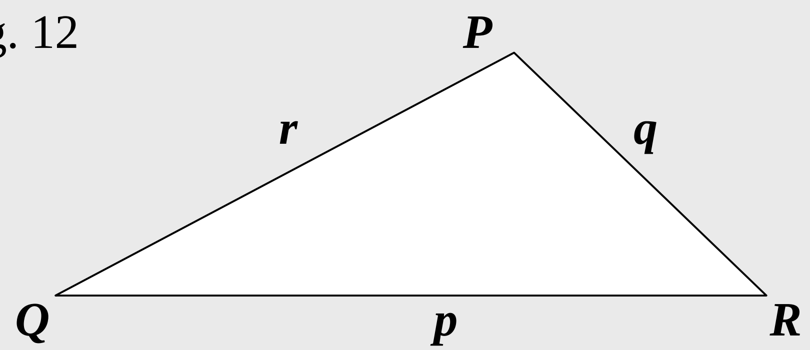
What about the three angles the triangle ABC has?

It is of course, the case that a triangle is made of not only three sides but three angles, too, which don't need, however, to be specified by their values, because the lengths of the three sides are enough for us to identify the triangle ABC.

Thus, three sides determine a triangle, and their lengths define the triangle.

What if the sides are in letters and not in numbers?

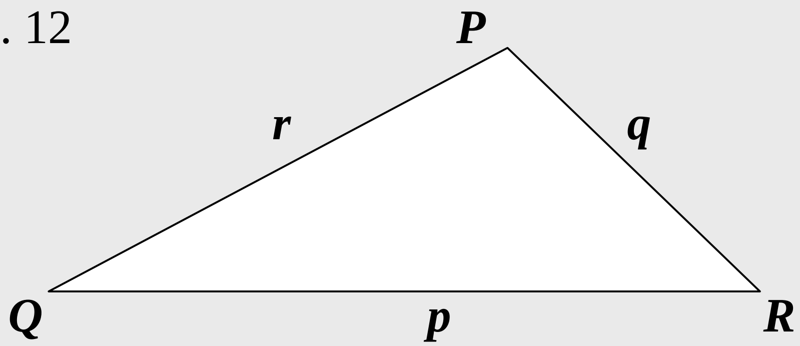
Fig. 12



Yes, they do, also. We can use letters for the lengths taking the letters as constants or variables. So we can specify the three sides by letters, too.

So for instance, if defining a triangle called ΔPQR using letters for the lengths, we can do it the way as follows.

Fig. 12



A triangle ***PQR*** has a side of length ***p***, a side of length ***q***, and a side of length ***r***.

In short, ΔPQR has three sides, which are ***p***, ***q***, and ***r***.

And of course, the sides are line segments.
So, how many different triangles can we make using three line segments?
And can we make a triangle using any group of three line segments?

Note that if making a triangle, we define a particular triangle, and vice versa, so if defining a particular triangle, we make a triangle.

Only one triangle can be made of three line segments ***if the three can make a triangle***.

So we can make ***only one particular triangle using three line segments if the three can make a triangle***.

We need to note, therefore, that ***not*** every group of three line segments can make a triangle.

And we'll talk about the reason why not in the next lesson.

Similar Triangles 8

Note that if making a triangle, we define a particular triangle, and vice versa, so if defining a particular triangle, we make a triangle.

Only one triangle can be made of three line segments ***if the three can make a triangle***.

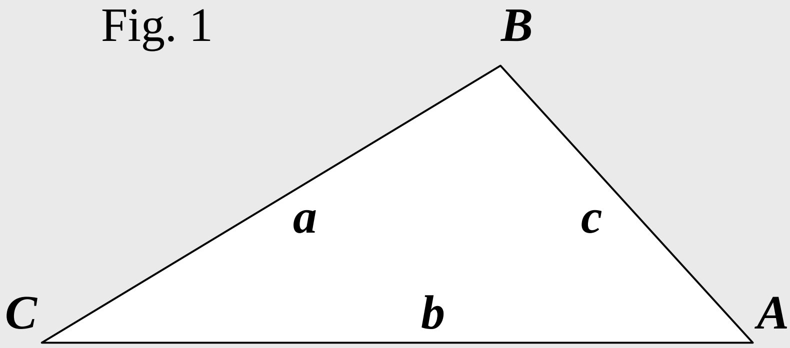
So we can make ***only one particular triangle using three line segments if the three can make a triangle***.

We need to note, therefore, that ***not*** every group of three line segments can make a triangle. Why not?

Let's now, draw a triangle, and put labels necessary as shown below. It's a good idea to make visible math objects or math ideas so that you can have visual aides. Then, you can learn them faster or think about solutions with less effort.

So don't just think; put it on paper.

Fig. 1



In the figure above, since the three sides ***a*, *b*,** and ***c*** make a triangle, ***any of the three sides is less than the sum of the other two***. It might sound quite obvious to some people, and yet, seems to be often neglected when they solve problems.

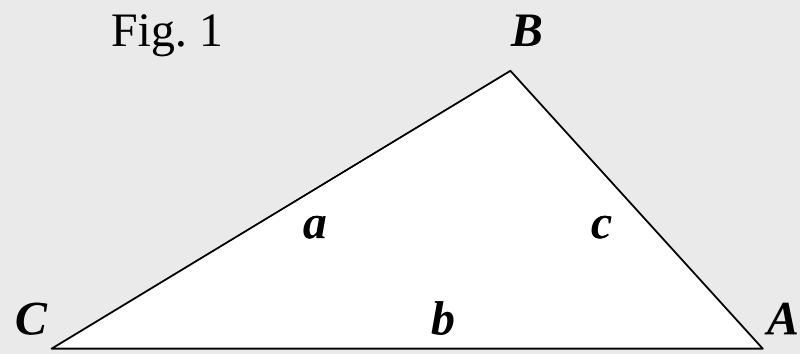
So, anyway, what do we mean by the statement below?

"Since the three sides ***a*, *b*,** and ***c*** make a triangle, ***any of the three sides is less than the sum of the other two***."

So in every triangle, the sum of any two sides is ***grater*** than the other, and we call the fact ***Triangle Inequality***.

In other words, the ***Triangle Inequality*** is saying that any one side has to be ***less*** than the sum of the other two.

Fig. 1



$$a < b + c, \quad b < c + a, \quad c < b + a$$

And the math property called ***Triangle Inequality*** is very important. It often helps get solutions when you solve problems related to triangles not only directly but indirectly, too. So you may want to be familiar with that.

If interested in more info on it, refer to ***Triangles*** in ***Basic Shapes***.

And now, getting back to triangle determinants, we can say that if producing three sides that can make a triangle, we define or determine a particular triangle.

In short, three sides ***determine*** a triangle.

So ***three sides*** are ***triangle determinant***.

A group of three sides is a triangle determinant.

And not all groups of three rods can make a triangle.

Isn’t there then, any other group of three components that can determine a triangle?

Yes, there is. ***An angle and two sides making the angle*** can determine a triangle. And in short:

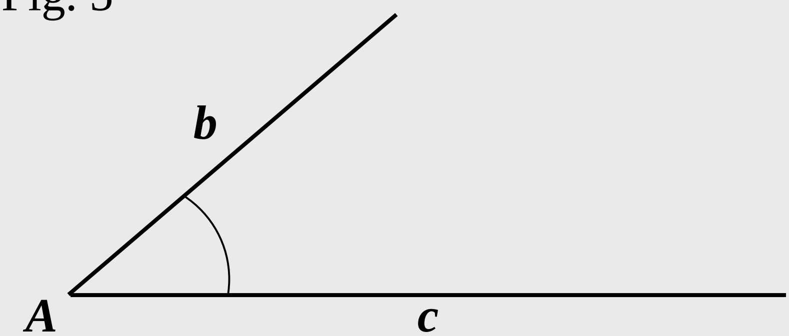
> Two sides and the angle in between determine a triangle.

So we now have ***two kinds*** in ***triangle determinants***.

One is a group of ***three sides***. And the other is a group of ***two sides and the angle between the two***.

Suppose for instance, we produce two sides $\boldsymbol{b}$ and $\boldsymbol{c}$, and put them together at an angle $\boldsymbol{A}$ as shown below.

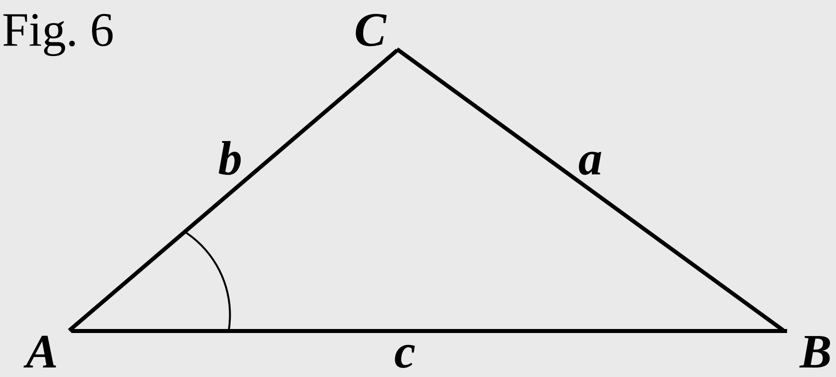
Fig. 5



As shown below, once an angle ***A*** and two sides ***b*** and ***c*** making the angle have been specified by values, the other side gets determined automatically, so a triangle is determined, and is defined, too.

The triangle determined is Δ***ABC***, and the other side is ***a***.

Fig. 6



Producing thus, at least two sides and the angle in between, we define a particular triangle.

And no matter what triangle it may be, it is determined by two sides and the angle between the two. In short, two sides and the angle in between determine a triangle.

Isn’t there then, any other group of three components that can determine a triangle?

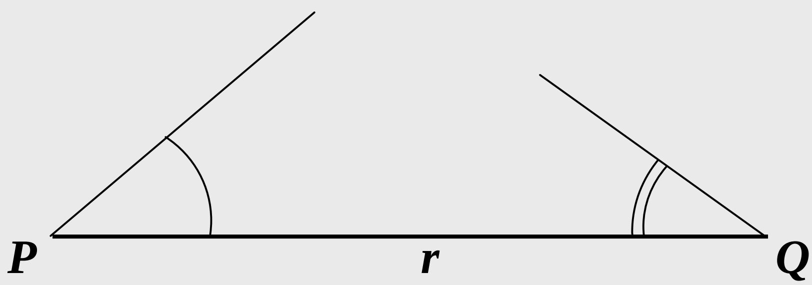
Yes, there is. ***Two angles and the side between the two*** can determine a triangle. And in short:

Two angles and the side in between determine a triangle.

How can they determine a triangle?

Suppose for instance, we produce two angles $\angle \boldsymbol{P}$ and $\angle \boldsymbol{Q}$, and put a side $\boldsymbol{r}$ in between as shown below.

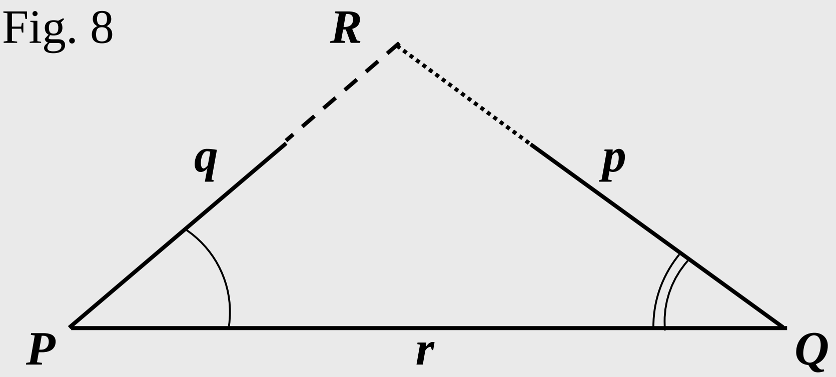
Fig. 7



As shown in Fig. 8 below, once one edge of the angle ***P*** and one edge of the angle ***Q*** have been extended, the other angle and the other two sides get determined automatically, so a triangle is determined.

The triangle determined is Δ***PQR***, the other angle is ∠***R***, and of course, the other two sides are ***p*** and ***q***.

Fig. 8



So we now have ***three kinds*** in ***triangle determinants***.

One is a group of ***three sides***. Another is a group of ***an angle and the two sides making the angle***. And the other is a group of ***two angles and the side in between***.

And that’s it.

There is one more thing, though. It is one of triangle basics, of course, and we'll talk about it a little later.

So now, back to similar triangles. We were talking about the combination of angles and sides eligible for a similar triangle.

And the combination has to do with a triangle determinant. It's because similar triangles are triangles, too. Actually, we can use one of the three kinds in triangle determinants to come up with a definition for similar triangles.

So, of the three kinds in triangle determinants, which one is the combination we can use to define similar triangles?

Well, since it's a combination of angles and sides, it's obviously not a group of three sides, which's been already covered. We have covered ***the group of sides eligible for a similar triangle***, and the definition is as follows:

Similar Triangles ⇔ Equal Ratios

And equivalently:

Similar Triangles ⇔ Equal Scale Factors

The definition is saying that in two similar triangles, the ratio between two corresponding sides stays the same. In other words, every ratio between two corresponding sides is equal. And in this case, we call the ratio the scale factor, too, which is therefore, another name for the ratio in this case.

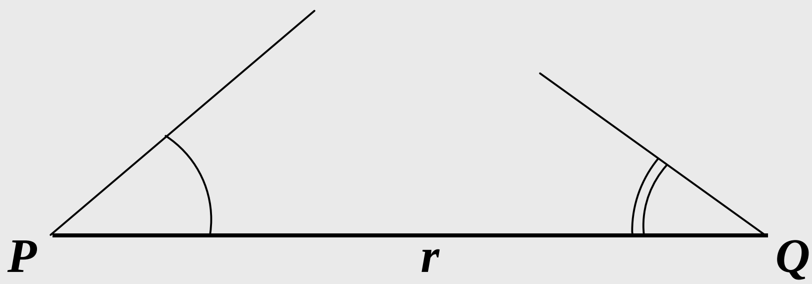
What then is the combination eligible for a similar triangle?

We have two determinants left, one is a group of ***two angles and the side between the two*** angles, and the other is a group of ***two sides and the angle between the two*** sides.

How about the first of the two determinants?

A group of two angles and the side in between?

Fig. 7



It’s not the combination, and in fact, we’ve already covered it.

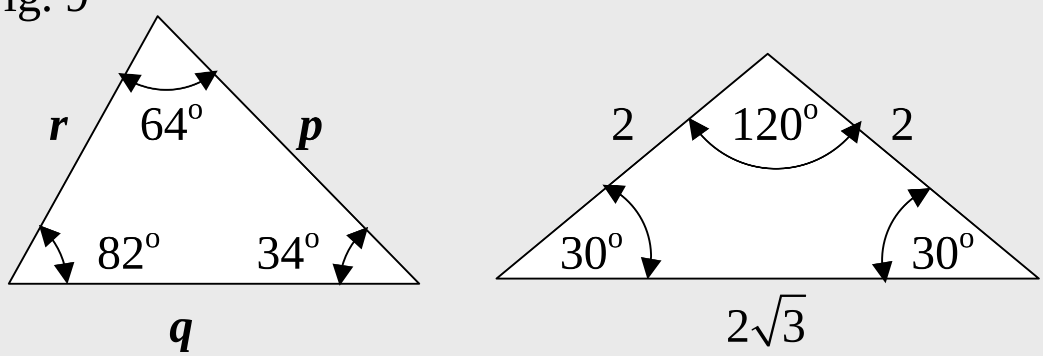
Why not, and when did we cover it?

It is no other than the case with the same group of three angles, and we covered the case when we covered the group of angles eligible for a similar triangle.

Why is it though, no other than the case?

For instance, we can have triangles as below.

Fig. 9



We can notice that the sum of the three angles in each is 180°.

So if knowing two angles in a triangle, we get to know all the three angles. In other words, two angles in a triangle can tell us the other angle, too. Why?

It's because the sum of the three angles in any triangle is the same, and is 180^o.

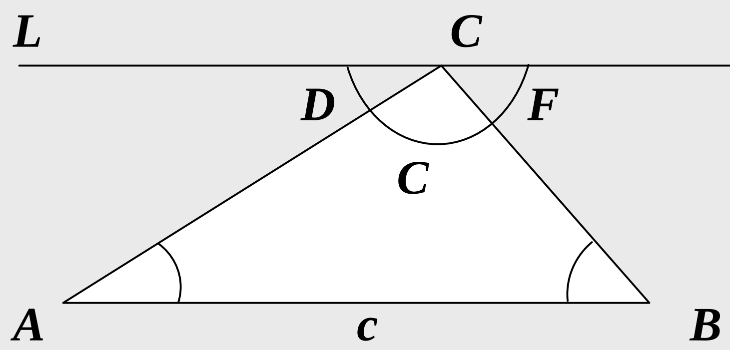
In every triangle, the sum of the three angles is 180^o.

How is that so? Why is it the case?

We've already covered why it is the case.
Let's now though, do it again.
It's not a bad idea to go over the case, and when going it over, we get to go over some other math basics, too.
This time though, we are going to do it a bit differently.

We can show that in $\Delta \boldsymbol{ABC}$, the sum of the three angles is 180°, that is, $\angle \boldsymbol{A} + \angle \boldsymbol{B} + \angle \boldsymbol{C} = 180^{\circ}$, the way as follows.

Fig. 10



Assuming the line $\boldsymbol{L}$ above is ***parallel*** to the side $\boldsymbol{c}$, we can have these: $\angle \boldsymbol{A} = \angle \boldsymbol{D}$, and $\angle \boldsymbol{B} = \angle \boldsymbol{F}$.

It's because both angles in each equality are alternate angles with parallel lines.

So we get this: $\angle \boldsymbol{A} + \angle \boldsymbol{B} + \angle \boldsymbol{C} = \angle \boldsymbol{D} + \angle \boldsymbol{F} + \angle \boldsymbol{C} = 180^{\circ}$, because the sum of the three angles make a straight angle, which is a half turn.

And $\Delta \boldsymbol{ABC}$ is arbitrary, so it represents all triangles. In every triangle therefore, the sum of the three angles is 180°.

Knowing thus, two angles in a triangle, we know all the three.

Taking the difference between 180^{o} and the sum of the two, we can get the third one, that is, the other angle.

And knowing that triangles share two angles, we can say that they are similar. It's because if sharing two angles, the triangles get to share three angles, that is, they have the same groups of three angles.

And again, the definition for similar triangles regarding angles is as follows.

Similar Triangles ⇔ The Same Angle Groups

And we have another definition regarding sides. So we now have two definitions for similar triangles, and we can put the two together the way as follows.

Similar Triangles ⇔ The Same Angle Groups

Similar Triangles ⇔ Equal Ratios

Or this way:

Similar Triangles ⇔ The Same Angle Groups

Similar Triangles ⇔ Equal Scale Factors

Now, we have checked two of the three kinds in triangle determinants. So we now have one left.

It’s a group of three components.

What then are the three?

The three are ***two sides and the angle between the two*** sides.

Two sides and the angle in between.

So the triangle determinant we are now going to consider is a group of two sides and the angle in between.

And it is the very combination we can use.

How then can we use the three components above to come up with a definition for similar triangles?

We may want to begin with going over how a group of the three components can determine a triangle, and if specified by their values, how the three components can define a particular triangle.

We’ll continue this in the next lesson.

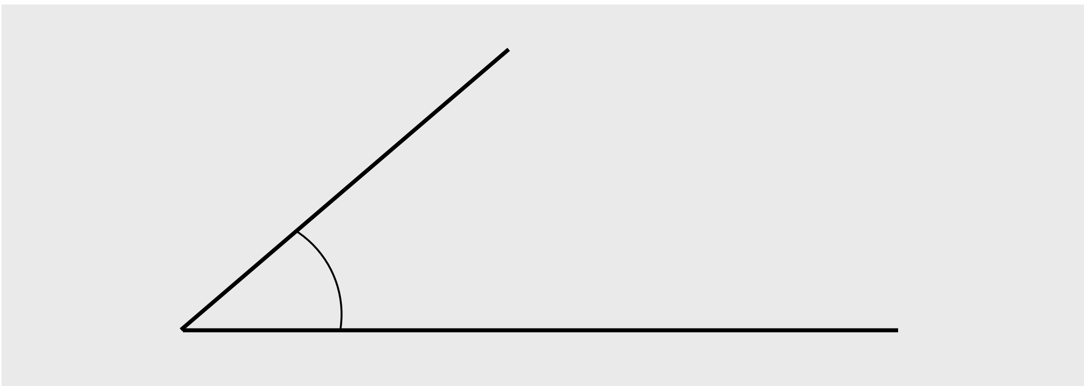
Similar Triangles 9

In the last lesson, we checked two of the three kinds in triangle determinants. So we now have one left.

It's a group of three components. And the three are ***two sides and the angle between the two*** sides.

Two sides and the angle in between.

And it is the very combination we can use.

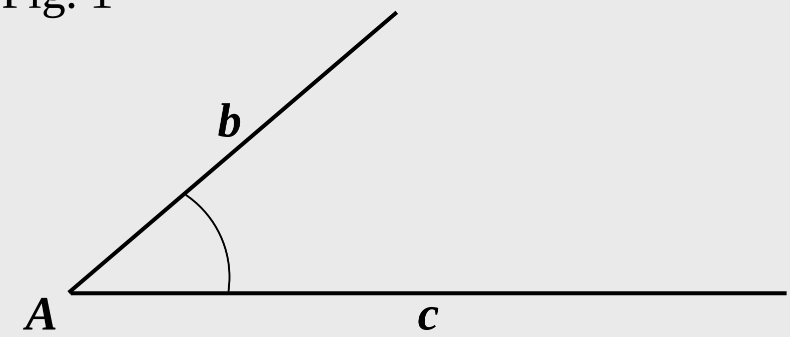


How then can we use the three components above to come up with a definition for similar triangles?

We may want to begin with going over how the three components can determine a triangle, and if specified by their values, how the three components can define a particular triangle.

So suppose again, for instance, we produce two sides ***b*** and ***c***, and put them together at an angle ***A*** as shown below.

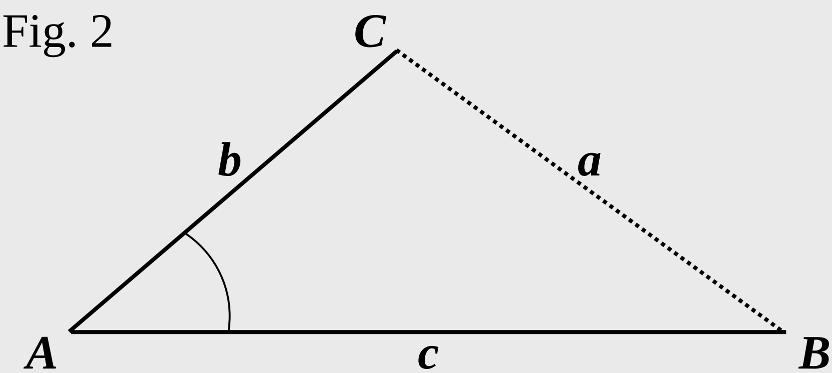
Fig. 1



Then, as shown in Fig. 2 below, since the angle ***A*** and the two sides ***b*** and ***c*** are determined, the other side gets determined automatically, so a triangle gets determined, and a particular triangle is defined.

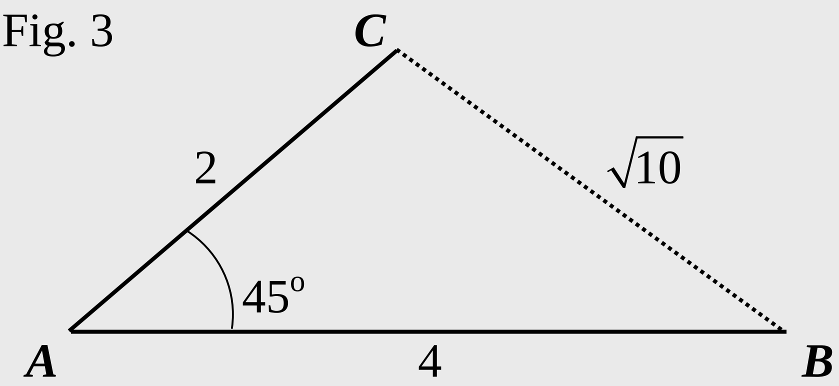
The particular triangle is Δ***ABC***, and the other side is ***a***.

Fig. 2



And for a particular example, as shown in Fig. 3 below, once an angle and the two sides making the angle have been specified by their values, the other side gets determined automatically, so a particular triangle is determined, and, it’s defined.

Fig. 3



So, if knowing two sides in a triangle and the angle between the two, we can practically see what the triangle is. That is, a group of the three components is a triangle determinant.

What then about similar triangles?

That is, how can we use the triangle determinant to come up with a definition for similar triangles?

The triangle determinant is a group of two sides and the angle in between. So let's now, first, consider the angle part.

What then can we say?

With regard to similar triangles, you can recall that if talking about angles, we consider the angles shared by triangles.

So when checking to see if two triangles are similar, we can try checking to see if the triangles share angles.
What then is the next?

The two sides that make the angle in each triangle.
So now, what do we consider if talking about sides regarding similar triangles?

With regard to similar triangles, you can recall that if talking about sides, we consider the same ratios, that is, the same scale factor.

How then can we see if two triangles are similar using, in each triangle, only two sides and the angle between the two?

> Two sides and the angle between the two

Mentioned that if talking about angles, we consider the angles shared by triangles. So we can try checking to see if the triangles share angles.

And mentioned also that if talking about sides, we consider the same ratios, that is, the same scale factor.

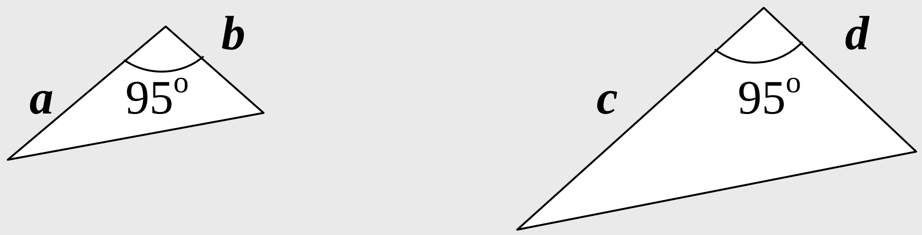
We're now talking about an angle and two sides in each of two triangles.

So we want to see if two triangles share an angle and see if two ratios can be equal, and of course, each ratio is the ratio between corresponding sides.

That is to say that we want to see if an angle is common and see if there is the same scale factor for two pairs of corresponding sides.

There are four ways to do the check the ratios.

Fig. 4



Doing the check on the triangles in the figure above, we can use any of the four as follows, because they are equivalent.

$$(1)\ \frac{a}{c}=\frac{b}{d} \qquad (2)\ \frac{a}{b}=\frac{c}{d} \qquad (3)\ \frac{c}{a}=\frac{d}{b} \qquad (4)\ \frac{b}{a}=\frac{d}{c}$$

And since equivalent, they can be put the way as follows.

$$\frac{a}{c}=\frac{b}{d} \Leftrightarrow \frac{a}{b}=\frac{c}{d} \Leftrightarrow \frac{c}{a}=\frac{d}{b} \Leftrightarrow \frac{b}{a}=\frac{d}{c}$$

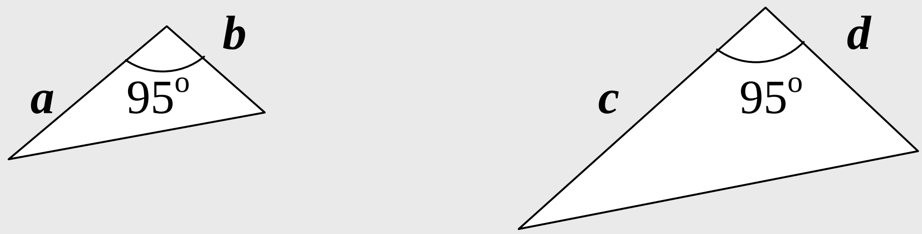
Doing some algebra, we can show that they're all the same.

For instance, dividing by b both sides of (1), and then multiplying by c both sides, we get (2). And taking the reciprocal of (2), we get (4), and vice versa.

Let's now see what each of the four is talking about.

(1) $\frac{a}{c} = \frac{b}{d}$ (2) $\frac{a}{b} = \frac{c}{d}$ (3) $\frac{c}{a} = \frac{d}{b}$ (4) $\frac{b}{a} = \frac{d}{c}$

Fig. 4



First off, (2) and (4) are saying that if in one triangle, the ratio between the sides making the angle ***can be*** equal to the ratio between the sides making the angle in the other triangle, the two triangles are similar.

Next, (1) and (3) are saying that if the two ratios between corresponding sides ***can be*** equal, the two triangles are similar.

Why 'can be', though, in the statements above?

We don’t just take those ratios.

We need to know exactly how to take the ratios.

It is ***critical*** to ***take the ratio the way as follows*****.**

If one ratio is ***the longer over the shorter***, the other ratio has to be the longer over the shorter, too.

And if one ratio is ***the shorter over the longer***, the other ratio has to be the shorter over the longer, also.

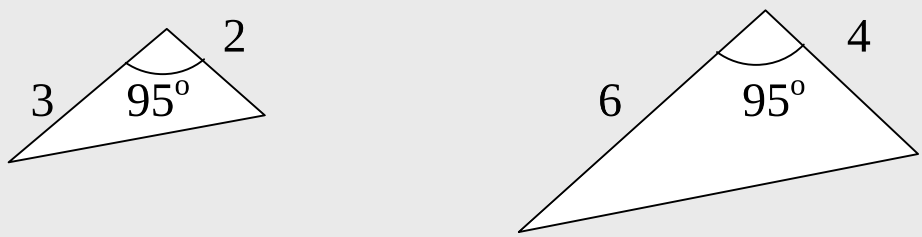
And of course, the same true of the scale factor, too, since it is another name for a ratio in this case.

So for instance, if the two sides given are 3 and 4 in one triangle, and in the other triangle, the two are 6 and 8, take the ratios this way: $\frac{4}{3}$ and $\frac{8}{6}$

or take the ratios this way: $\frac{3}{4}$ and $\frac{6}{8}$

For another instance, in two triangles, as shown below, if an angle is common, and the ratio between two sides making the angle can be the same, the two triangles are similar.

Fig. 5



95^{o} is common to both triangles, and we can have

$$\frac{3}{2}=\frac{6}{4} \quad \text{or} \quad \frac{2}{3}=\frac{4}{6}.$$

Note that if we take the ratio of the longer to the shorter in one triangle, make sure that we take the ratio of the longer to the shorter in the other triangle, too. And of course, if taking the ratio of the shorter to the longer in one, we take the ratio of the shorter to the longer in the other triangle, too.

And in fact, the correspondence between the sides works the same way as above, too. So it goes the way as follows.

In one triangle, the longest corresponds to the longest in the other triangle.

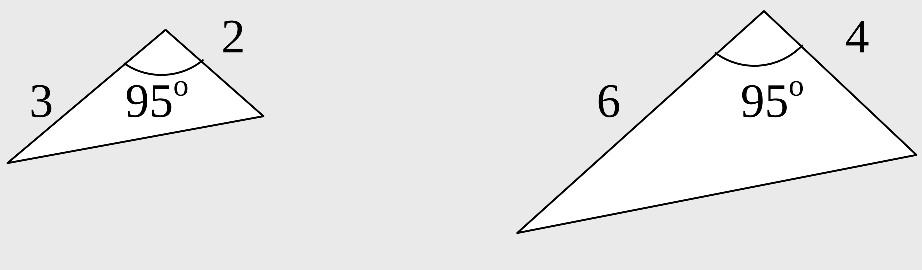
The next longest in one triangle corresponds to the next longest in the other triangle.

And the shortest in one triangle corresponds to the shortest in the other triangle.

In short, the shorter corresponds to the shorter, and the longer corresponds to the longer.

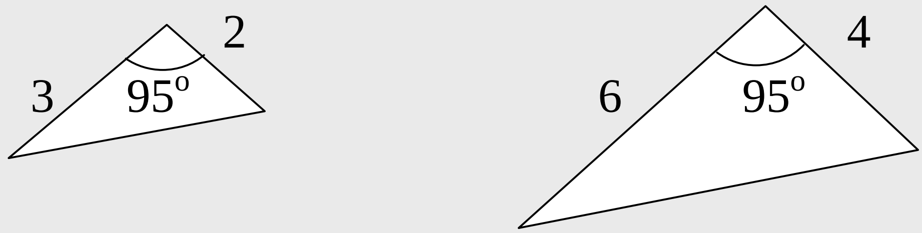
What then about using the corresponding sides in two triangles shown below?

Fig. 5



The shorter corresponds to the shorter, and the longer corresponds to the longer.

Fig. 5



Checking, therefore, to see if every ratio between two corresponding sides can be equal, we can do it the way as follows.

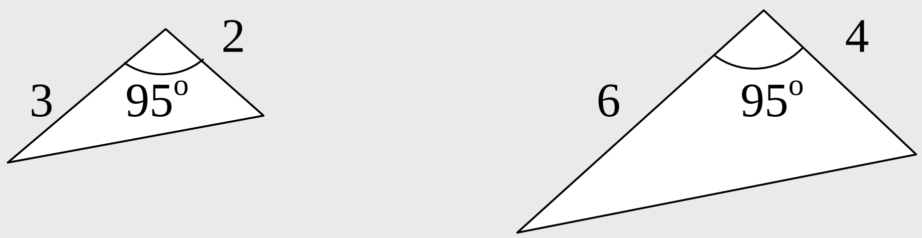
$\frac{3}{6}=\frac{2}{4}$, or this way: $\frac{6}{3}=\frac{4}{2}$.

And the ratios on both sides of the equal sign are called scale factors, too.

So the interpretation or the meaning of the example shown above can be taken the way as follows, too.

The two triangles share an angle 95^o, and applying the same scale factor 2 to the two sides 3 and 2 in the triangle on the left, we get 6 and 4, which are the two sides in the triangle on the right, and make 95^o, so we can say that the two triangles are similar.

Fig. 5



And we can put it the way as follows, too.

The two triangles share an angle 95^o, and applying the same scale factor 0.5 to the two sides 6 and 4 in the triangle on the right, we get 3 and 2, which are the two sides in the triangle on the left, and make 95^o, so we can say that the two triangles are similar.

And thus, either way, the bottom line is as follows.

If an angle is common, and the two ratios between the sides making the angle can be equal, the two triangles are similar.

In short,

Similar Triangles ⇔ Common angle, Two Equal Ratios

Thus, we have three kinds in definition for similar triangles, and we can put the three together the way as follows.

Similar Triangles ⇔ The Same Angle Groups

Similar Triangles ⇔ Equal Ratios (Scale Factors)

Similar Triangles ⇔ Common angle, Two Equal Ratios

What then, is the condition for similar triangles?

What is the condition that two triangles are similar?

It is any of the three below.

One is the same angle groups.

Another is equal ratios or scale factors.

And the other is a common angle with two equal ratios.

What then can we do with the three above?

We can use them when we need to check to see if two triangles are similar. So when doing the check, we need to see if between two triangles, any of the three below is true.

The Same Angle Groups

Equal Ratios (Scale Factors)

A Common Angle with Two Equal Ratios

And now, at the end of the discussions on triangle determinants, it was mentioned that there was one more thing, it was about triangle basics, and we would talk about it a little later. And it is time for us to talk about it.

First off, we have ***three kinds*** in ***triangle determinants***.

One is a group of ***three sides***. Another is a group of ***an angle and the two sides making the angle***. And the other is a group of ***two angles and the side in between***.

So what can we do with the three above?

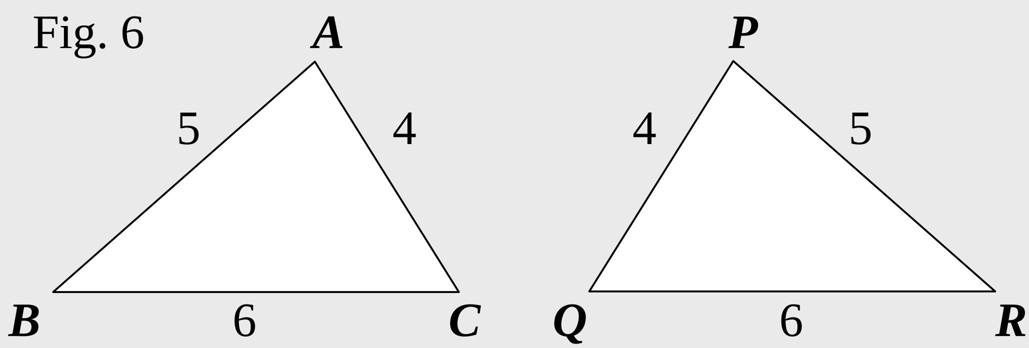
We can use them when we need to check to see if two triangles are identical.

To begin with, using the first of the three above, we can see if triangles are identical using the fact as follows.

> If triangles have the same groups of three sides, the triangles are identical.

It's because there exists only one triangle made of three particular sides if, of course, the three can make a triangle.

Fig. 6



The triangle ***ABC*** is identical to the triangle ***PQR***.
And using a math sign, we can put them the way as follows:

$\Delta ABC \equiv \Delta PQR$

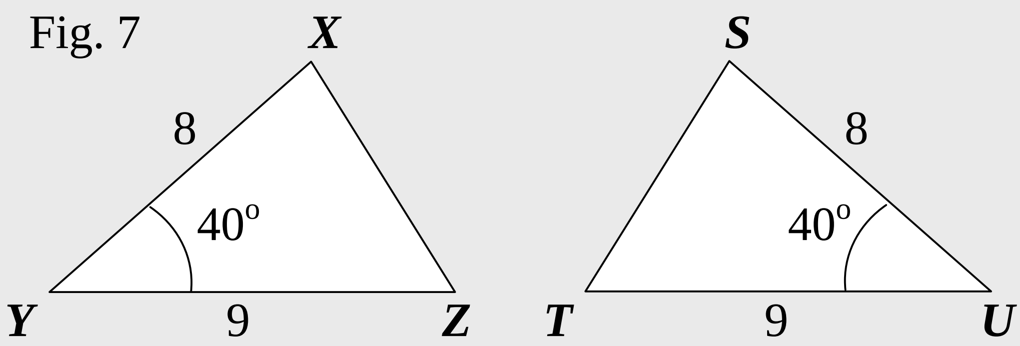
So '≡' means 'is identical to'.

Next, using the second of the three kinds, we can check to see if two triangles are identical the way as follows.

If sharing two sides and the angle between the two, the triangles are identical.

It's because an angle and two particular sides making the angle can make one triangle only.

Fig. 7



The triangle ***XYZ*** is identical to the triangle ***STU***.
And using a math sign, we can put them the way as follows:

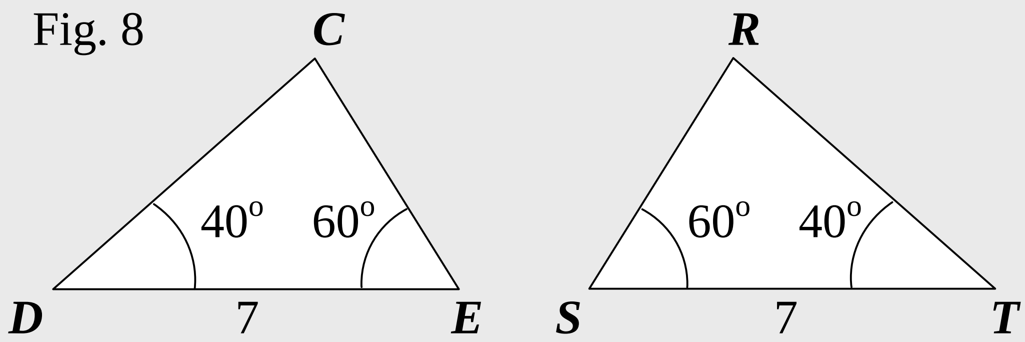
$\Delta STU \equiv \Delta XYZ$

And now, using the last one, that is, the third of the three kinds, we can check to see if two triangles are identical the way as follows.

If sharing two angles and the side between the two, the triangles are identical.

It’s because two particular angles and a particular side between the two can make one triangle only.

Fig. 8



The triangle ***CDE*** is identical to the triangle ***RST***.
And using a math sign, we can put them the way as follows:

$\Delta \boldsymbol{CDE} \equiv \Delta \boldsymbol{RST}$

Thus, using triangle determinants, we can see if triangles are identical.

Now, summing up, we can put the bottom line the way as follows.

First off, determining a triangle, we need three components, which are in three kinds.

So we have ***three kinds*** in ***triangle determinants***.

One is a group of ***three sides***. Another is a group of ***an angle and the two sides making the angle***. And the other is a group of ***two angles and the side in between***.

And we can use them when we need to check to see if two triangles are identical.

To begin with, using the first of the three above, we can see if triangles are identical using the fact as follows.

> If triangles have the same groups of three sides, the triangles are identical.

It’s because there exists only one triangle made of three particular sides if, of course, the three can make a triangle.

Next, using the second of the three kinds, we can check to see if two triangles are identical the way as follows.

> If sharing two sides and the angle between the two, the triangles are identical.

It’s because an angle and two particular sides making the angle can make one triangle only.

And now, using the last one, that is, the third of the three kinds, we can check to see if two triangles are identical the way as follows.

> If sharing two angles and the side between the two, the triangles are identical.

It’s because two particular angles and a particular side between the two can make one triangle only.

We have three kinds in definition for similar triangles, and we can put the three together the way as follows.

Similar Triangles ⇔ The Same Angle Groups

Similar Triangles ⇔ Equal Ratios (Scale Factors)

Similar Triangles ⇔ Common angle, Two Equal Ratios

What then, is the condition for similar triangles?

What is the condition that two triangles are similar?

It is any of the three below.

One is the same angle group.

Another is three equal ratios or the same scale factor.

And the other is a common angle with two equal ratios.

What then can we do with the three above?

We can use them when we need to check to see if two triangles are similar. So when doing the check, we need to see if between two triangles, any of the three below is true.

The Same Angle Groups

Equal Ratios (Scale Factors)

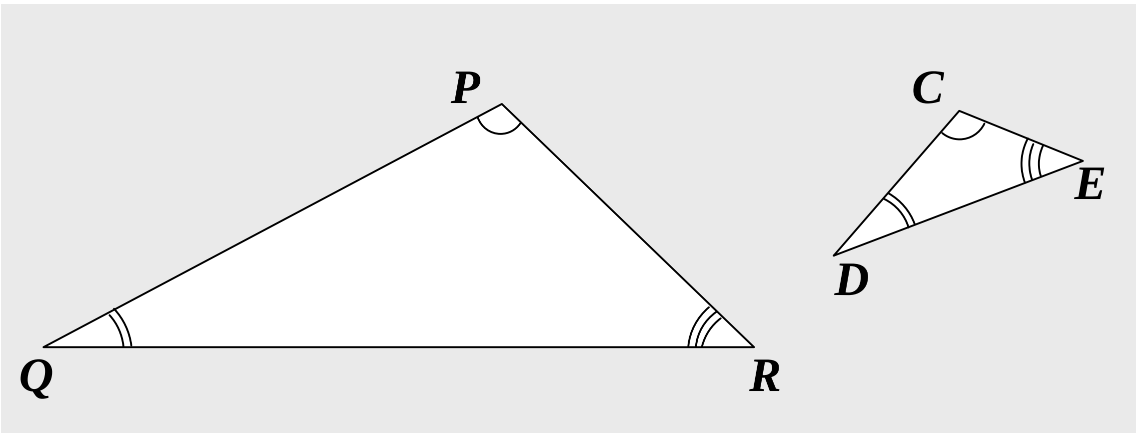
A Common Angle with Two Equal Ratios

Is that all about the definitions for similar triangles?

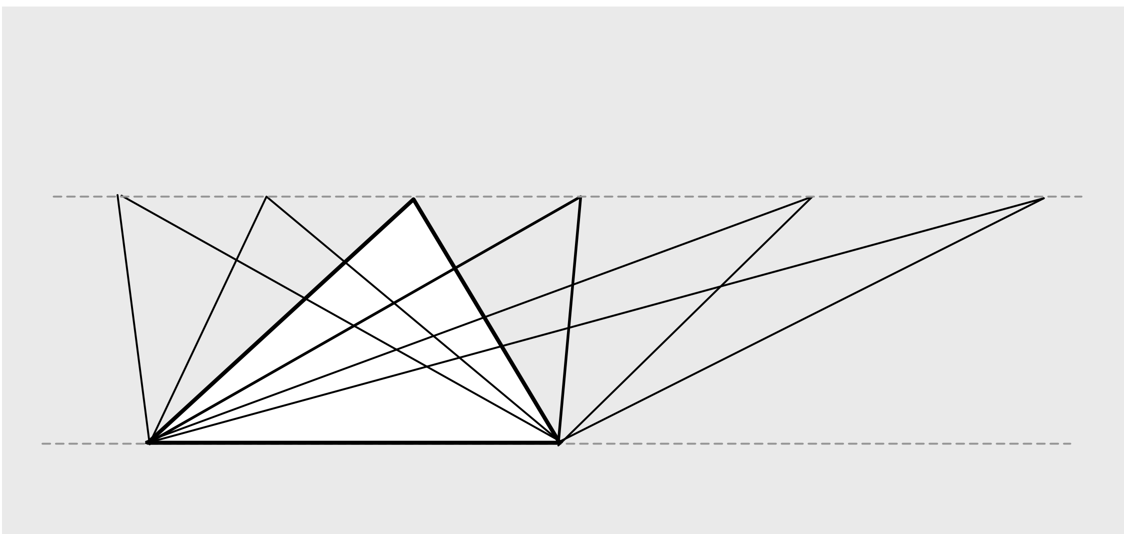
We need to cover one more important part, which is an algebra part. And it will be covered in the next lesson.

Similar Triangles 10

As stated earlier, if we define similar triangles in plane language, we may be able to say that ***similar triangles share the same shape with different sizes***, which might sound however, a bit too broad or somewhat loose.

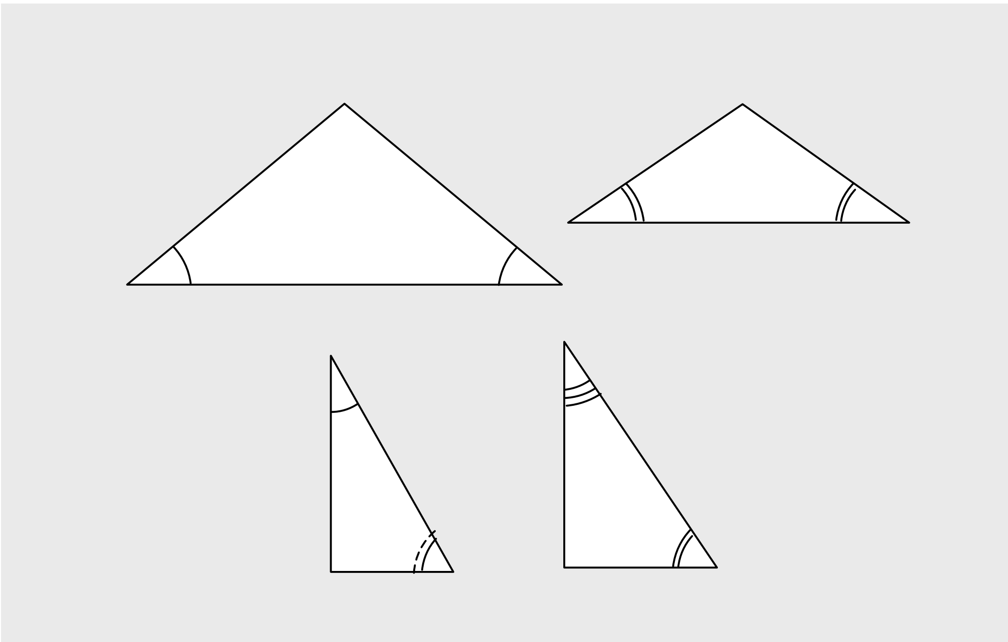


Different sizes? Many triangles can have the same areas but can have different sizes.



The same shape? A triangle is a shape, and so is a rectangle. So do all triangles share the same shape, and is the same true of rectangles, too?

Besides, many triangles not similar can look quite alike.



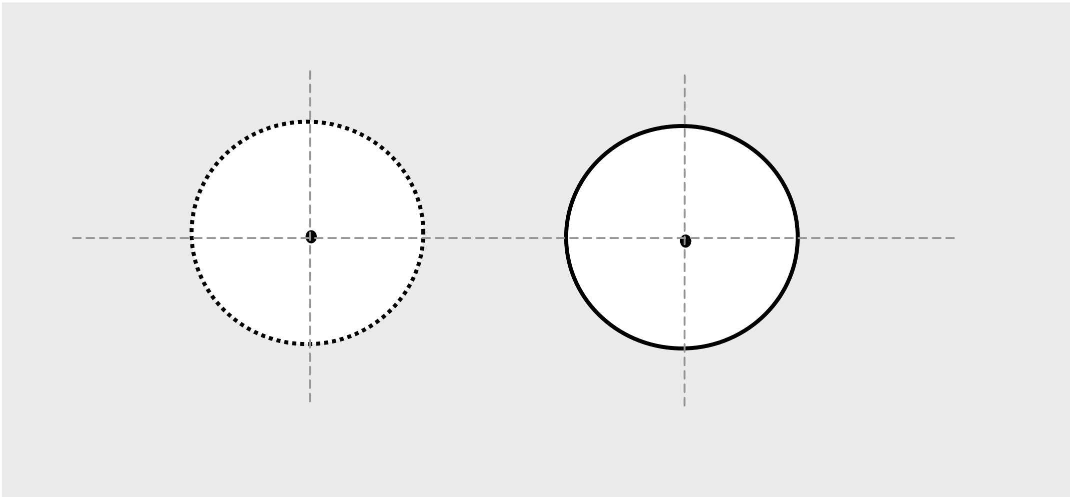
It’s somewhat like when defining a circle in math, we just say that ***a circle is round***, which is not wrong, and yet, doesn’t sound quite right, either.

It doesn’t in math, of course.

What seems to be the problem?

A bit too broad, and not specific enough.

We don’t’ make a definition for circles that way in math.



As you might have noticed already, if we define a math object in plane language as English or Chinese, we could experience some ambiguity, iffiness, deficiency, or excess in the description of the definition.

For instance, things round can be spheres like balls, some dishes, rings, etc. How then, can we avoid such defects?

We can use another language called mathematics, often called the language of science, to avoid or reduce such ambiguity. So we may want to use the language called math to define the ***math objects*** called ***similar triangles***.

We call such a definition a symbolic or formal definition.

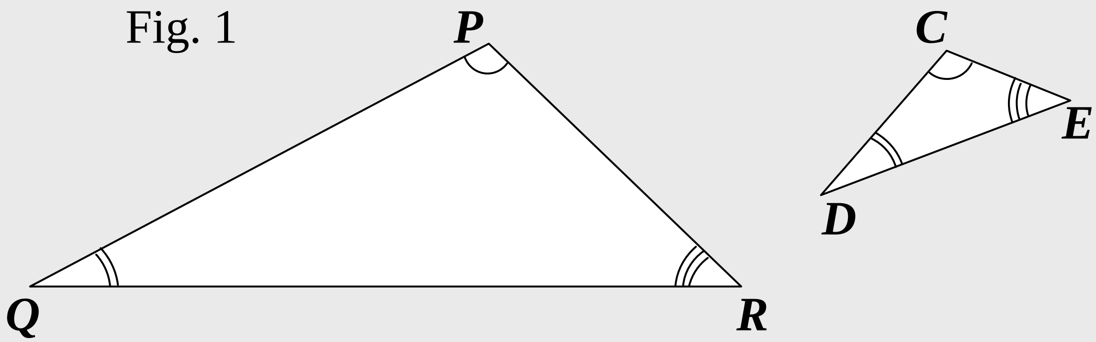
Such a definition is made of math expressions and symbols or signs, along with few words in plane language.

We are now going to use some specific examples, along with math expressions and symbols or signs to define similar triangles and continue approaching the concept.

So symbolically or formally, we can define similar triangles the way as follows. Note however, that it’s not the final version, yet.

In Fig. 1 below, by definition, ΔPQR and ΔCDE are similar if and only if we get ***any*** of (1), (2), (3-1), (3-2), and (3-3).

Fig. 1



(1) $$\frac{|DE|}{|QR|} = \frac{|CE|}{|PR|} = \frac{|CD|}{|PQ|}$$

(2) $\angle P = \angle C$, $\angle Q = \angle D$, and $\angle R = \angle E$

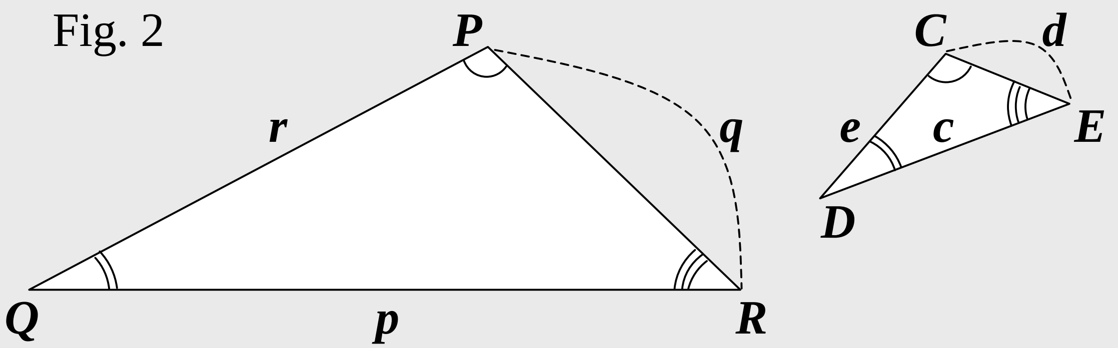
(3-1) $\angle P = \angle C$, and $\dfrac{|CD|}{|PQ|} = \dfrac{|CE|}{|PR|}$ i.e., $\dfrac{|PR|}{|PQ|} = \dfrac{|CE|}{|CD|}$

(3-2) $\angle Q = \angle D$, and $\dfrac{|DE|}{|QR|} = \dfrac{|CD|}{|PQ|}$ i.e., $\dfrac{|PQ|}{|QR|} = \dfrac{|CD|}{|DE|}$

(3-3) $\angle R = \angle E$, and $\dfrac{|DE|}{|QR|} = \dfrac{|CE|}{|PR|}$ i.e., $\dfrac{|PR|}{|QR|} = \dfrac{|CE|}{|DE|}$

The definition seems complicated because of many letters. No need to get annoyed, though. We can simplify it by setting up a labeling convention as shown in the figure below so that we can use fewer letters in the definition.

Fig. 2



In Fig. 2 above, ***p*** indicates the side ***QR***, ***q*** indicates the side ***PR***, and as is expected, ***r*** indicates the side ***PQ***.

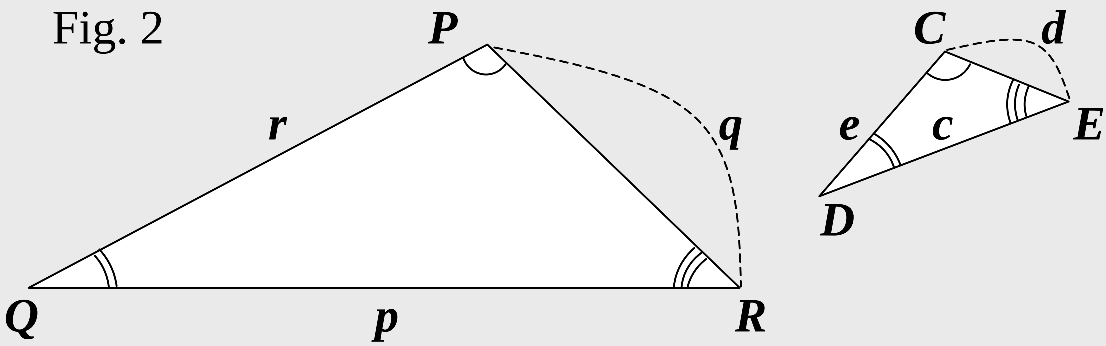
Also, ***p***, ***q***, and ***r*** indicate the lengths of their own, too.
So the convention is if indicating the side facing an angle, we use the lower case letter of the letter used for the angle.

Thus, by the convention above, ***c*** indicates the side ***DE***, ***d*** indicates the side ***CE***, and ***e*** indicates the side ***CD***.

So we can now simplify a bit more the definition using the convention above, and put the definition the way as follows.

Now again, by definition, ΔPQR and ΔCDE are similar if and only if we get ***any*** of (1), (2), (3-1), (3-2), and (3-3).

Fig. 2



(1) $\dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

(2) $\angle P = \angle C$, $\angle Q = \angle D$, and $\angle R = \angle E$

(3-1) $\angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$ i.e., $\dfrac{q}{r} = \dfrac{d}{e}$

(3-2) $\angle Q = \angle D$, and $\dfrac{c}{p} = \dfrac{e}{r}$ i.e., $\dfrac{r}{p} = \dfrac{e}{c}$

(3-3) $\angle R = \angle E$, and $\dfrac{c}{p} = \dfrac{d}{q}$ i.e., $\dfrac{q}{p} = \dfrac{d}{c}$

This is one of the basics in math, is extremely important, and is a very convenient tool when we solve problems. So you may want to grab the concept and the use of it.

First off, what do we mean by ‘if and only if’?

It’s often put this way: ‘iff’, and is other than ‘if’. And the difference between ‘if’ and ‘if and only if’ is as follows.

Suppose it is true that ***you win if and only if I win***.

Then, it is true that ***you win if I win***, and it is also true that ***I win if you win***.

So ***my winning means your winning***, and also, ***your winning means my winning***.

And also, the same is true of the statement as follows:
“If and only if you win, I win.” It means that your winning means my winning, and vice versa.

Suppose now, it is just true that you win if I win.

Then, it ***may not*** be true that I win if you win.

That is to say that my winning means your winning, but your winning may not mean my winning.

Next, this definition is said to be one of basics, which means simple or easy. It seems, though, cluttered with many expressions. Why is it then, made of so many expressions?

Granted the definition looks quite involved, and seems cluttered, too, so you may feel burdened or even annoyed, relax though. The expressions are rather simple and quite straightforward. And in fact, for educational purposes, it contains some redundant, that is, unnecessary expressions.

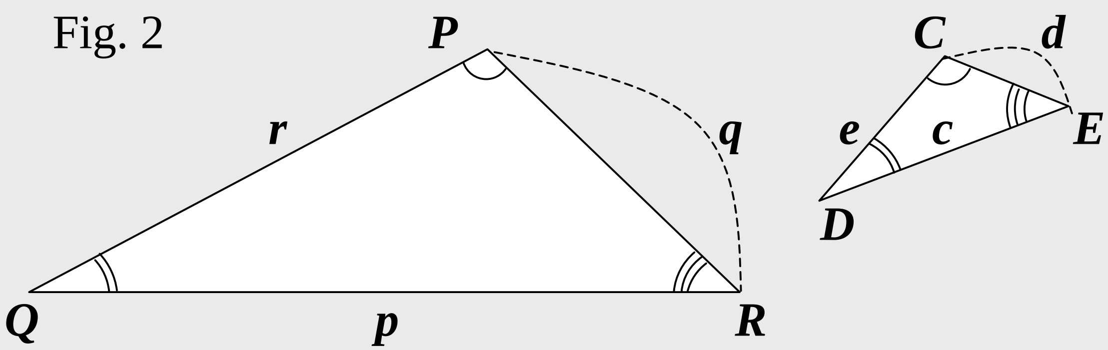
So, it will be much simplified so that you will get to see ***just three simple*** expression sets after all. We’ll narrow down to the three as strengthening the basics continuing this discussion, and shortly, will get them all the three clear.

And as stated in the definition, we do not have to satisfy all the expression sets at once, and not even two sets at the same time, to verify that two triangles are similar. Only one set is enough for the verification. So if either of the sets is true, the triangles are similar.

And if you want to see now the spoiler, the final three sets, they are as follows. If however, not want to see the spoiler now, just skip the next three pages.

Now, by definition, $\Delta PQR \sim \Delta CDE$ iff we get ***any*** of (1), (2), and (3). Note that '~' is a math sign meaning 'is similar to'.

Fig. 2



(1) $\dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$ Three equal ratios

(2) $\angle P = \angle C$, $\angle Q = \angle D$, and $\angle R = \angle E$

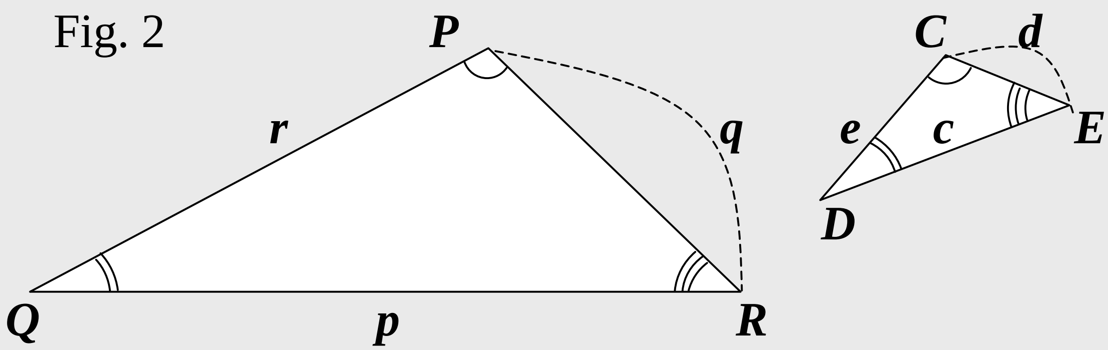
Same angle groups (Three common angles)

(3) $\angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$

A common angle and two equal ratios

It can even be simplified further the way as follows.

Fig. 2



$\Delta PQR \sim \Delta CDE \Leftrightarrow$ ***any*** of (1), (2), and (3) is true.

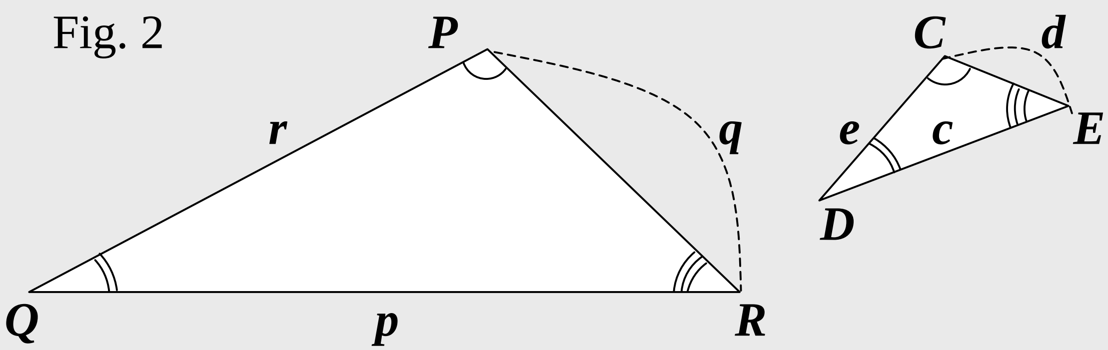
(1) $\dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

(2) $\angle P = \angle C$, and $\angle Q = \angle D$

(3) $\angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$

And we can divide the definition into three versions the way as follows, too.

Fig. 2



(1) $\Delta PQR \sim \Delta CDE \Leftrightarrow \dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

(2) $\Delta PQR \sim \Delta CDE \Leftrightarrow \angle P = \angle C$, and $\angle Q = \angle D$

(3) $\Delta PQR \sim \Delta CDE \Leftrightarrow \angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$

Now, continuing this discussion, you will get to see ***how the final sets get made*** and get to see better how the sets can be used in problem solving processes so that you can practically use them solving problems in various situations.

The definition that seems cluttered is made of many expressions. Some of them are redundant, though.

So the definition will be simplified much more.

You will get to see how those redundancies get removed.

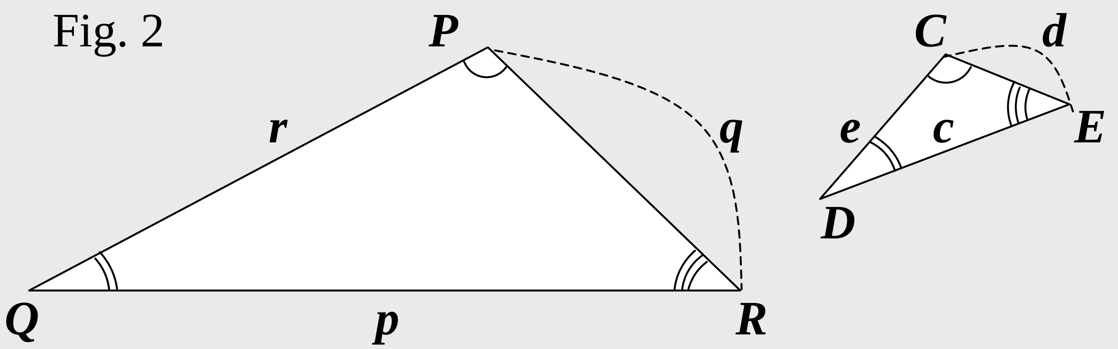
And in math, we often use symbols or signs to replace wordings in plane language as English or Chinese.

So, if $\boldsymbol{A}$ is similar to $\boldsymbol{B}$ in math, using a math symbol, we can put it this way: $\boldsymbol{A \sim B}$, which is way simpler. So math makes your life simpler, and thus, easier. Simpler the better.

Let’s now spread out each of the five expression sets in the definition cluttered with expressions so that we can look at each of those five sets closely, and remove some of those.

Now, we are going to begin with the first three sets, (1), (2), and (3-1), and can put the sets the way as follows.

Fig. 2



(1) $\Delta PQR \sim \Delta CDE$ if and only if we get $\dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

(2) $\Delta PQR \sim \Delta CDE$ if and only if we get

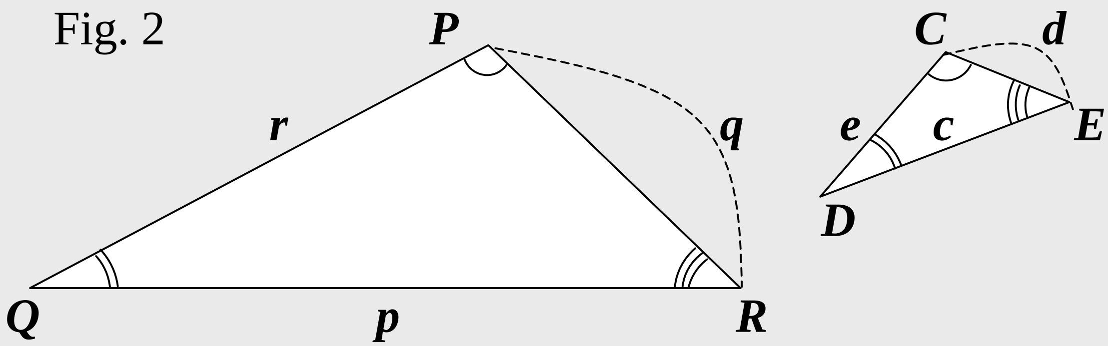
$\angle P = \angle C$, $\angle Q = \angle D$, and $\angle R = \angle E$

(3-1) $\Delta PQR \sim \Delta CDE$ if and only if we get

$\angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$ i.e., $\dfrac{q}{r} = \dfrac{d}{e}$

And we have two more, which are (3-2) and (3-3), but these two are redundant, and are as follows.

Fig. 2



(3-2) $\Delta PQR \sim \Delta CDE$ if and only if we get $\angle Q = \angle D$, and

$$\frac{c}{p} = \frac{e}{r} \quad \text{i.e.,} \quad \frac{r}{p} = \frac{e}{c}$$

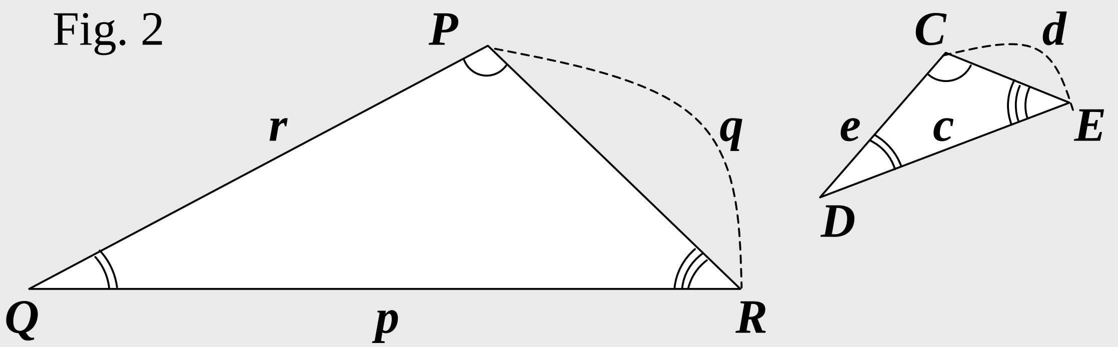
(3-3) $\Delta PQR \sim \Delta CDE$ if and only if we get $\angle R = \angle E$, and

$$\frac{c}{p} = \frac{d}{q} \quad \text{i.e.,} \quad \frac{q}{p} = \frac{d}{c}$$

The two (3-2) and (3-3) above are no other than (3-1) shown earlier. So we don't need those two, and shortly, will see why not.

Now, we are going to expand each of the three sets (1), (2), and (3-1), and see what we can do about each of those.

Fig. 2



(1) The two triangles ΔPQR and ΔCDE are similar if and only if we get $\dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

Now, expanding (1), we can say that

$\Delta PQR \sim \Delta CDE$ if we get $\dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$, and also, reversely

if $\Delta PQR \sim \Delta CDE$, we get $\dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

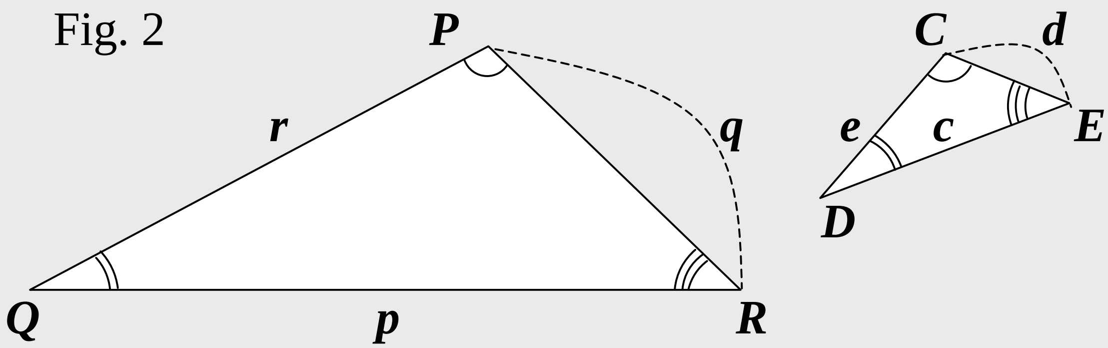
So ‘if and only if’ indicates “Two Way Traffic” and not “One Way”. Thus, it works either way. Very nice.

It is often shortened to ‘iff’, read as ‘if and only if’. And we call it ***the biconditional***, another name for ***equivalence***.

And using a math sign or symbol for it, we use this: ⇔, which is thus, read as ‘if and only if’.

So both sides of ⇔ are equivalent.

Fig. 2



And we can simply put the definition the way below.

$$(1) \quad \Delta PQR \sim \Delta CDE \Leftrightarrow \frac{c}{p} = \frac{d}{q} = \frac{e}{r}$$

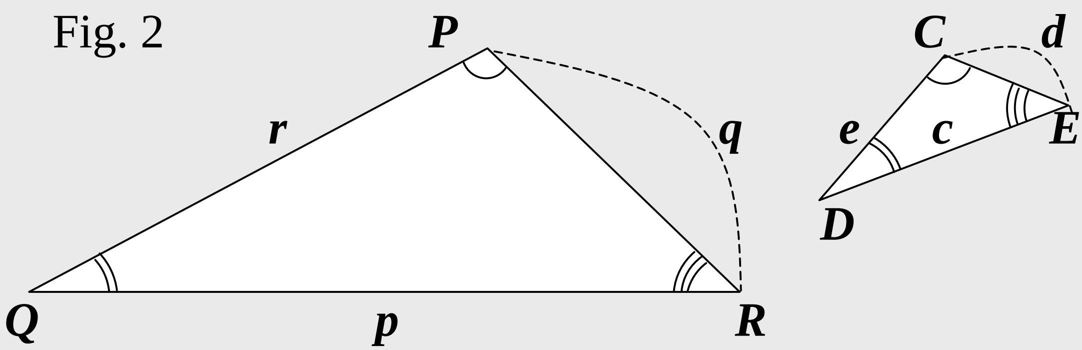
It is about the case of ***three equal ratios***, that is, ***the same scale factor***, and is one of the final three.

And the next is about the case of ***the same angle groups***, and is the expression set (2), which is as follows.

(2) $\Delta PQR \sim \Delta CDE$ if and only if we get

$\angle P = \angle C$, $\angle Q = \angle D$, and $\angle R = \angle E$.

Fig. 2



Now, expanding (2), we can say that $\Delta PQR \sim \Delta CDE$ if we get $\angle P = \angle C$, $\angle Q = \angle D$, and $\angle R = \angle E$,

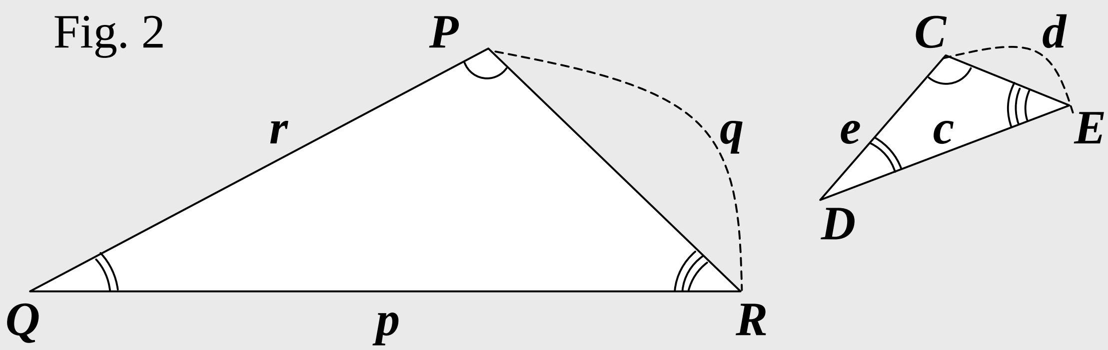
and also, if $\Delta PQR \sim \Delta CDE$, we get $\angle P = \angle C$, $\angle Q = \angle D$, and $\angle R = \angle E$.

So both ways, it works. It's not done yet, though.

We know the fact that in every triangle, the sum of all the three angles is equal to 180^{o}, which means the sum is the same for every triangle. So?

So if we get these two: $\angle \boldsymbol{P} = \angle \boldsymbol{C}$, and $\angle \boldsymbol{Q} = \angle \boldsymbol{D}$, we get this, too: $\angle \boldsymbol{R} = \angle \boldsymbol{E}$, automatically. Thus, two common angles mean three common angles, the same angle groups.

Fig. 2



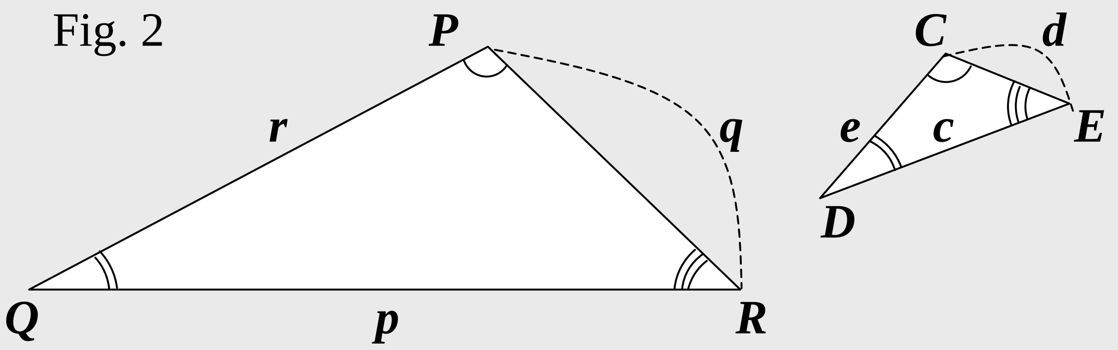
Thus, we can now say that we get this: $\boldsymbol{\Delta PQR \sim \Delta CDE}$ if we have these: $\angle \boldsymbol{P} = \angle \boldsymbol{C}$, and $\angle \boldsymbol{Q} = \angle \boldsymbol{D}$, and also, if we have this: $\boldsymbol{\Delta PQR \sim \Delta CDE}$, we get these: $\angle \boldsymbol{P} = \angle \boldsymbol{C}$, and $\angle \boldsymbol{Q} = \angle \boldsymbol{D}$, which means that we get all these: $\angle \boldsymbol{P} = \angle \boldsymbol{C}$, $\angle \boldsymbol{Q} = \angle \boldsymbol{D}$, and $\angle \boldsymbol{R} = \angle \boldsymbol{E}$.

And simply, we can put the definition the way below.

(2) $\boldsymbol{\Delta PQR \sim \Delta CDE} \Leftrightarrow \angle \boldsymbol{P} = \angle \boldsymbol{C}$, and $\angle \boldsymbol{Q} = \angle \boldsymbol{D}$

Thus, we now have the two of the final three as follows.

Fig. 2



(1) $\Delta PQR \sim \Delta CDE \Leftrightarrow \dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

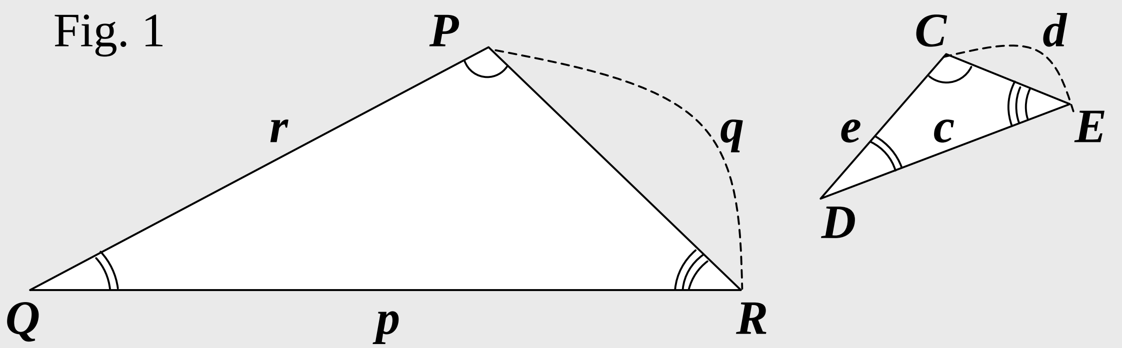
(2) $\Delta PQR \sim \Delta CDE \Leftrightarrow \angle P = \angle C$, and $\angle Q = \angle D$

And the next is about the case of ***the combination***, and is the expression set (3-1), which will be covered in the next lesson.

Similar Triangles 11

We now have the two of the final three as follows.

Fig. 1



(1) $\Delta PQR \sim \Delta CDE \Leftrightarrow \dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

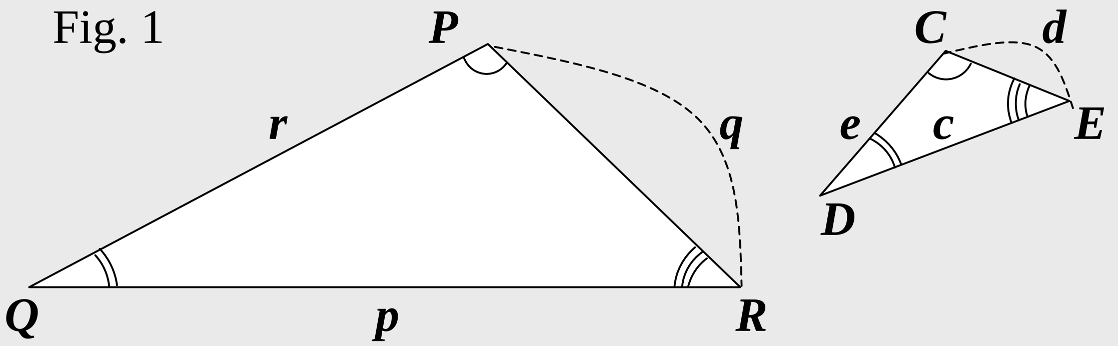
(2) $\Delta PQR \sim \Delta CDE \Leftrightarrow \angle P = \angle C$, and $\angle Q = \angle D$

And the next is about the case of ***the combination***, and is the expression set (3-1), which is as follows.

(3-1) $\Delta PQR \sim \Delta CDE$ if and only if we get $\angle P = \angle C$, and

$$\frac{e}{r} = \frac{d}{q} \text{ i.e., } \frac{q}{r} = \frac{d}{e}.$$

Fig. 1



Now, expanding (3-1), we can say that $\Delta PQR \sim \Delta CDE$ if we get $\angle P = \angle C$, and $\frac{e}{r} = \frac{d}{q}$ i.e., $\frac{q}{r} = \frac{d}{e}$, and also, if $\Delta PQR \sim \Delta CDE$, we get $\angle P = \angle C$, and $\frac{e}{r} = \frac{d}{q}$ i.e., $\frac{q}{r} = \frac{d}{e}$.

And now, why do we have the case as follows?

$$\frac{e}{r} = \frac{d}{q} \text{ i.e., } \frac{q}{r} = \frac{d}{e}$$

Both are equivalent.

That is to say that $\frac{e}{r} = \frac{d}{q}$ is no other than $\frac{q}{r} = \frac{d}{e}$. Why?

Dividing by e both sides of $\frac{e}{r} = \frac{d}{q}$, and then, multiplying both sides by q, we get this: $\frac{q}{r} = \frac{d}{e}$. More specifically,

dividing by e both sides of $\frac{e}{r} = \frac{d}{q}$, we get this:

$$\frac{e}{re} = \frac{d}{qe} \Rightarrow \frac{1}{r} = \frac{d}{qe}$$, and next,

multiplying both sides by q, we get this:

$$\frac{1}{r} \times q = \frac{d}{qe} \times q \Rightarrow \frac{q}{r} = \frac{dq}{qe} = \frac{d}{e} \Rightarrow \frac{q}{r} = \frac{d}{e}.$$

In short:

$$\frac{e}{r} = \frac{d}{q} \Rightarrow \frac{e}{re} = \frac{d}{qe} \Rightarrow \frac{eq}{re} = \frac{dq}{qe} \Rightarrow \frac{q}{r} = \frac{d}{e}.$$

And we can reverse the process above.

Dividing by $\boldsymbol{q}$ both sides of $\frac{q}{r} = \frac{d}{e}$, and then, multiplying both sides by $\boldsymbol{e}$, we get this: $\frac{e}{r} = \frac{d}{q}$. More specifically,

dividing by $\boldsymbol{q}$ both sides of $\frac{q}{r} = \frac{d}{e}$, we get this:

$$\frac{q}{rq} = \frac{d}{eq} \Rightarrow \frac{1}{r} = \frac{d}{eq}$$, and next,

multiplying both sides by $\boldsymbol{e}$, we get this:

$$\frac{1}{r} \text{ x } e = \frac{d}{eq} \text{ x } e \Rightarrow \frac{e}{r} = \frac{de}{eq} = \frac{d}{q} \Rightarrow \frac{e}{r} = \frac{d}{q}.$$

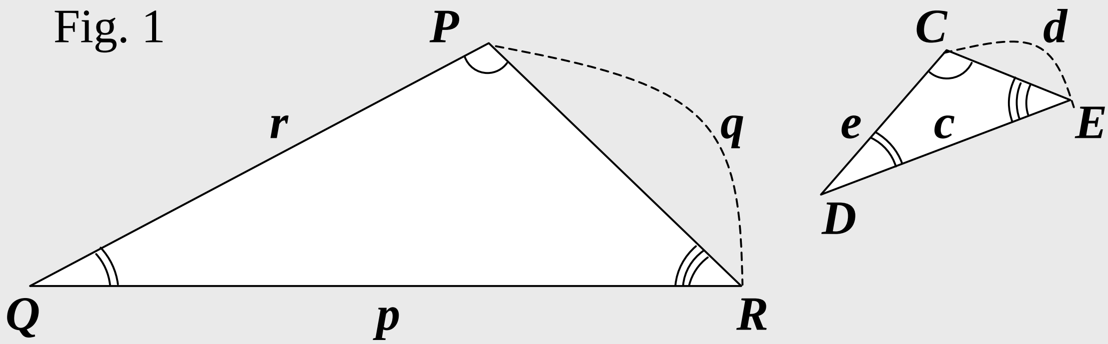
In short:

$$\frac{q}{r} = \frac{d}{e} \Rightarrow \frac{q}{rq} = \frac{d}{eq} \Rightarrow \frac{qe}{rq} = \frac{de}{eq} \Rightarrow \frac{e}{r} = \frac{d}{q}.$$

Thus, $\frac{e}{r} = \frac{d}{q}$ is no other than $\frac{q}{r} = \frac{d}{e}$; both are equivalent, so we can put them this way, too: $\frac{e}{r} = \frac{d}{q} \Leftrightarrow \frac{q}{r} = \frac{d}{e}$.

We now have all the final three. And the three are as follows.

Fig. 1



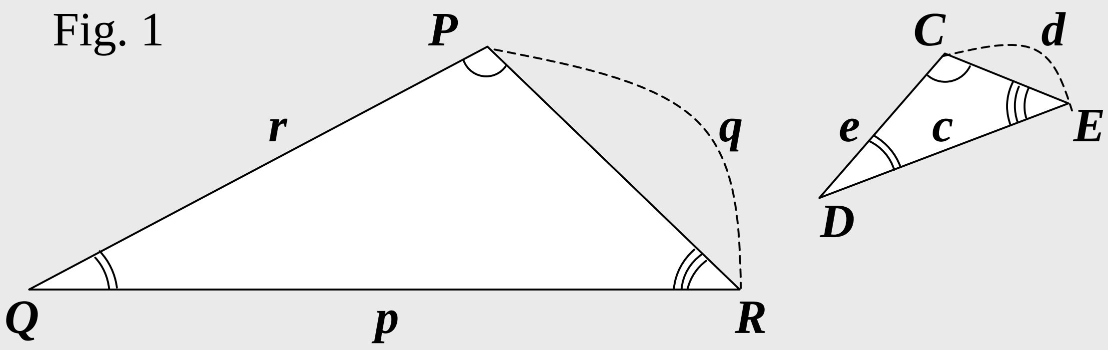
(1) $\Delta PQR \sim \Delta CDE \iff \dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

(2) $\Delta PQR \sim \Delta CDE \iff \angle P = \angle C$, and $\angle Q = \angle D$

(3) $\Delta PQR \sim \Delta CDE \iff \angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$

And as mentioned earlier, the definition cluttered is now simplified the way as follows.

Fig. 1



$\Delta \boldsymbol{PQR} \sim \Delta \boldsymbol{CDE}$ iff we get any of (1), (2), and (3).

(1) $\dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

(2) $\angle P = \angle C$, and $\angle Q = \angle D$

(3) $\angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$

And we can say that each of the three above is a condition for similar triangles. Very nice and convenient tool.

What about, though, these two sets: (3-2) and (3-3)?

(3-1) $\angle P = \angle C$, and $\frac{e}{r} = \frac{d}{q}$ i.e., $\frac{q}{r} = \frac{d}{e}$

(3-2) $\angle Q = \angle D$, and $\frac{c}{p} = \frac{e}{r}$ i.e., $\frac{r}{p} = \frac{e}{c}$

(3-3) $\angle R = \angle E$, and $\frac{c}{p} = \frac{d}{q}$ i.e., $\frac{q}{p} = \frac{d}{c}$

We've already covered how we get (3) from (3-1).

We did some algebra to each of $\frac{e}{r} = \frac{d}{q}$ and $\frac{q}{r} = \frac{d}{e}$ to get one from the other.

And we can do the same to each of (3-2) and (3-3), too. That is, we can put each of (3-2) and (3-3) exactly the same way as we put (3) the way below.

(3) $\angle P = \angle C$, and $\frac{e}{r} = \frac{d}{q}$

That's because all the three expression sets (3-1), (3-2), and (3-3) are saying the same story. In each and every set, the two sides make the common angle.

Just different letters are used for angles and sides.

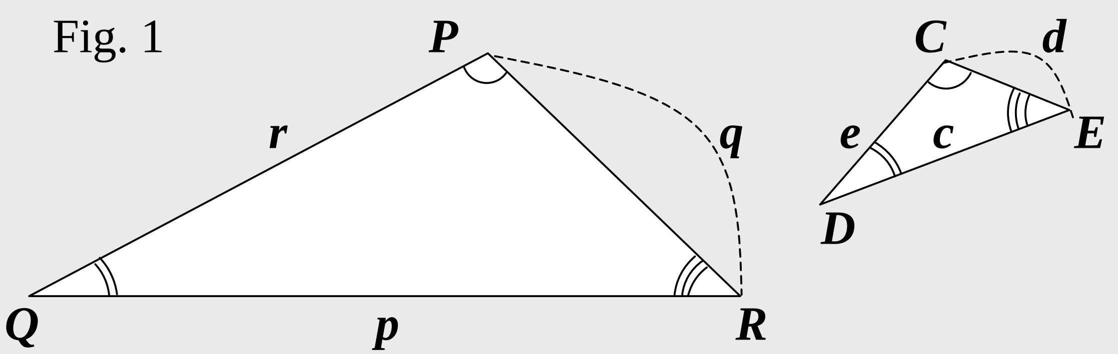
Practically, therefore, they are the same expressions.

(3-1) $\angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$ i.e., $\dfrac{q}{r} = \dfrac{d}{e}$

(3-2) $\angle Q = \angle D$, and $\dfrac{c}{p} = \dfrac{e}{r}$ i.e., $\dfrac{r}{p} = \dfrac{e}{c}$

(3-3) $\angle R = \angle E$, and $\dfrac{c}{p} = \dfrac{d}{q}$ i.e., $\dfrac{q}{p} = \dfrac{d}{c}$

Fig. 1



So (3-1), (3-2), and (3-3) are redundant.

Let's actually see now, in each of (3-2) and (3-3), how we get one equality from the other doing simple algebra.

First off, in $\frac{c}{p} = \frac{e}{r}$, dividing both sides by c and multiplying by r, we get $\frac{r}{p} = \frac{e}{c}$.

And reversely now, in $\frac{r}{p} = \frac{e}{c}$, dividing both sides by r and multiplying by c, we get $\frac{c}{p} = \frac{e}{r}$. So, $\frac{c}{p} = \frac{e}{r} \Leftrightarrow \frac{r}{p} = \frac{e}{c}$

Next, in $\frac{c}{p} = \frac{d}{q}$, dividing both sides by c and multiplying by q, we get $\frac{q}{p} = \frac{d}{c}$.

And reversely now, in $\frac{q}{p} = \frac{d}{c}$, dividing both sides by q and multiplying by c, we get $\frac{c}{p} = \frac{d}{q}$. So, $\frac{c}{p} = \frac{d}{q} \Leftrightarrow \frac{q}{p} = \frac{d}{c}$

And (3-1) is now reduced to (3). Thus, (3) is enough.

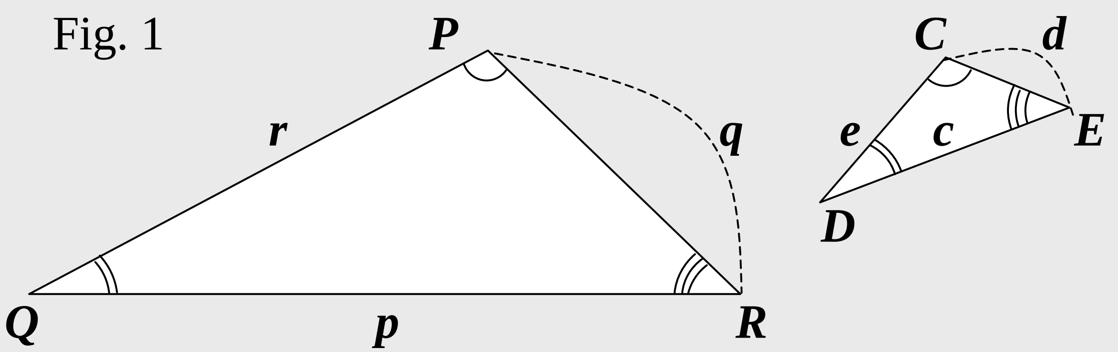
And we can use as (3) any of the three as follows.

(3) $\angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$

(3) $\angle Q = \angle D$, and $\dfrac{c}{p} = \dfrac{e}{r}$

(3) $\angle R = \angle E$, and $\dfrac{c}{p} = \dfrac{d}{q}$

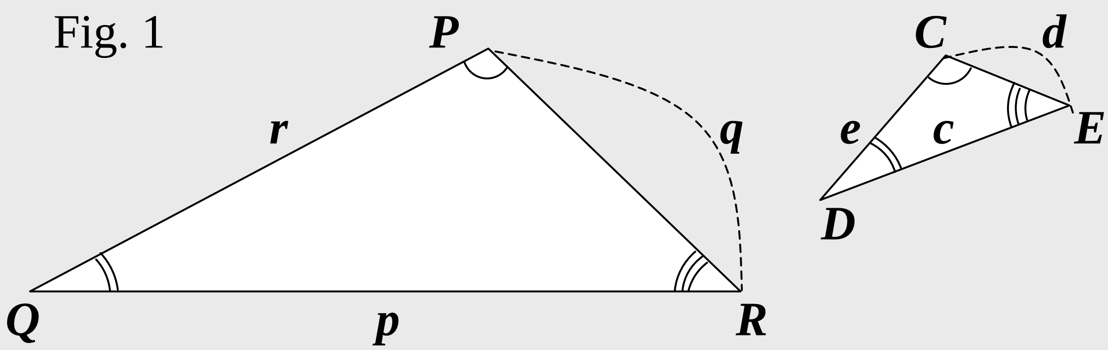
Fig. 1



Thus, as mentioned earlier, the definition cluttered is now simplified the way as follows.

By definition, $\Delta PQR \sim \Delta CDE$ iff we get any of (1), (2), and (3).

Fig. 1



(1) $\dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

(2) $\angle P = \angle C$, and $\angle Q = \angle D$

(3) $\angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$

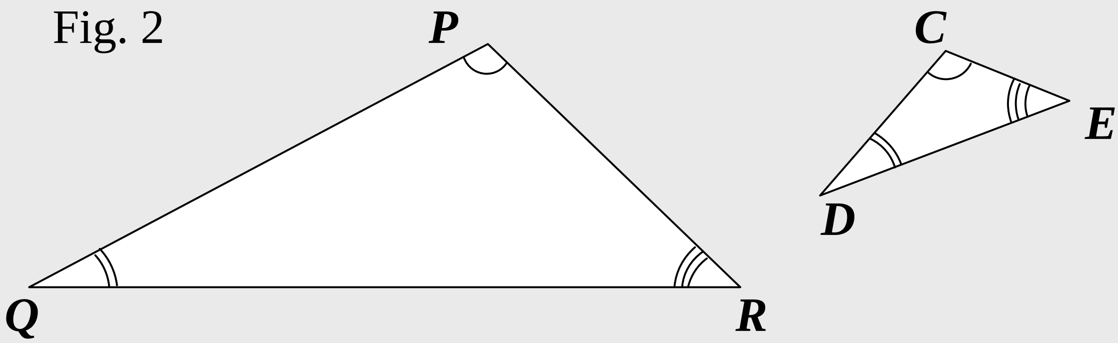
And again, each of the three above is a condition for similar triangles. Very nice and convenient tool.

Well, what if we get this: $\dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r} = 1$?

Then, we get these: $c = p$, $d = q$, and $e = r$, so we can say that one is identical to the other.

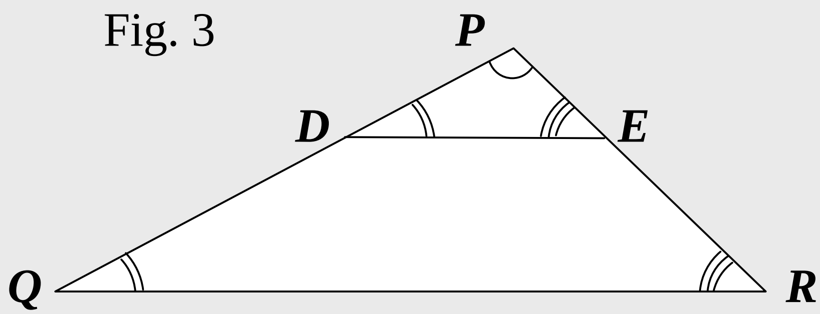
We have another convenience and will cover it now. We've been talking about triangles placed away from each other as shown in the figure as follows.

Fig. 2



We are now going to cover the case where ΔCDE is nested, that is, placed inside ΔPQR as shown in Fig. 3 below, so that it's now called ΔPDE.

Fig. 3



Thus, in this case, ΔPDE in Fig. 3 is the same as ΔCDE in Fig. 1.

Fig. 3

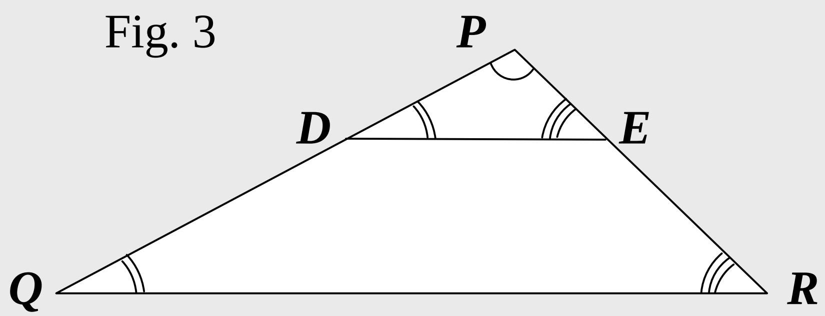
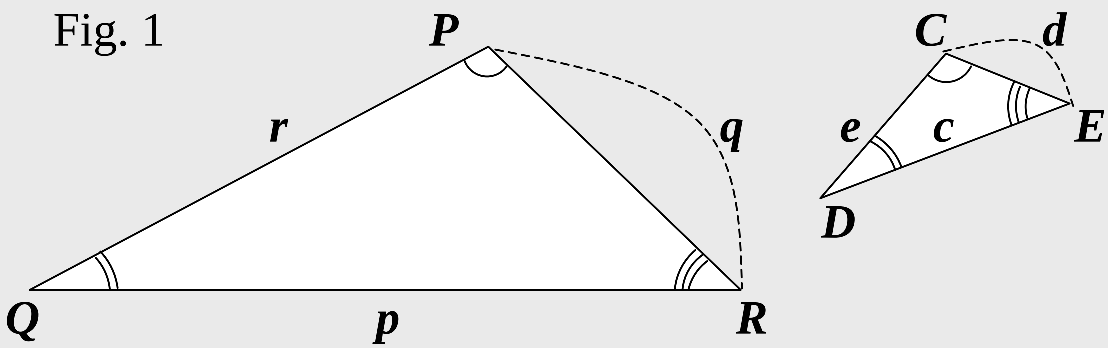


Fig. 1

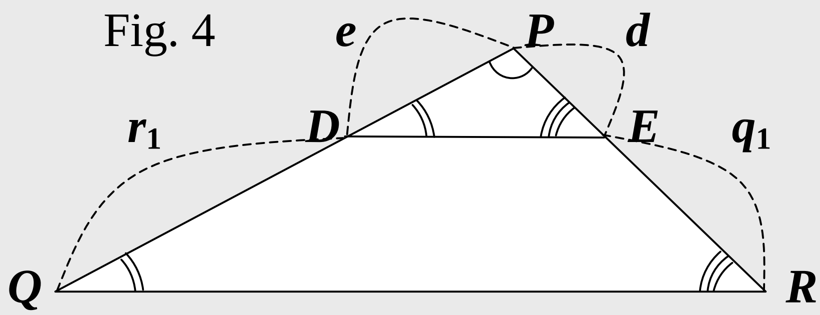


So the side CD matches the side PD, and CE matches PE.

By the way, indicating a side, we often use the labels at the vertices, too, and in that case, indicating the length of it, we use a pair of short vertical bars. For instance, we can indicate the length of the side PQ this way: $|PQ|$.

When similar triangles are nested the way shown in the figure below, we can come up with another equality that shows that two ratios are equal, but the two are a bit different from those ones we’ve covered earlier.

Fig. 4



And we can put them the way as follows.

$$\frac{|PD|}{|DQ|} = \frac{|PE|}{|ER|}\text{, that is, } \frac{e}{r_1} = \frac{d}{q_1}\text{, as well as this:}$$

$$\frac{|PD|}{|PQ|} = \frac{|PE|}{|PR|}\text{, that is, } \frac{e}{r} = \frac{d}{q}.$$

How?

We can add a line segment ***DS*** the way as follows.

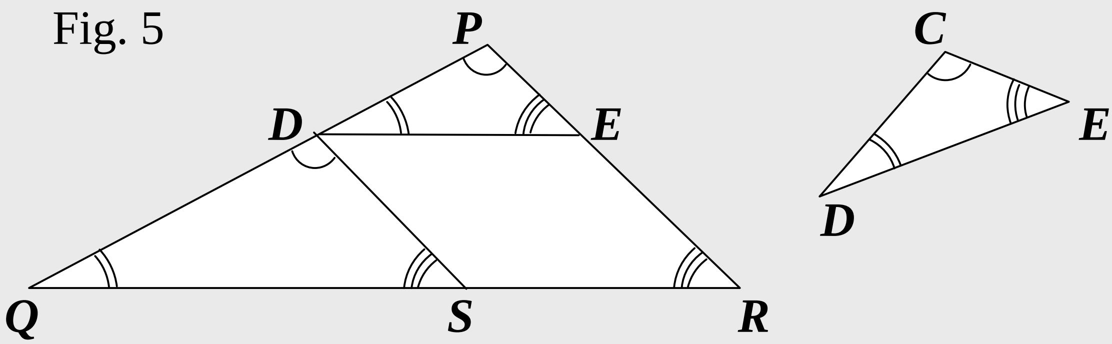


Fig. 5

In the figure above, the line segment ***DS*** is parallel to the line segment ***ER***. Then, we get this: |***DS***| = |***ER***|.

It's because ***DS*** is parallel to ***ER***, and since we have this: ∠***E*** = ∠***R***, the line segment ***DE*** is parallel to the line segment ***SR***, so the tetragon ***DERS*** is a parallelogram.

And more importantly, ***QR*** is the transversal to ***DS*** and ***PR***, and also, ***QP*** is the transversal to ***DS*** and ***PR***.   So?

We get these two: ∠***P*** = ∠***QDS***, and ∠***R*** = ∠***QSD***.

So what about it?

We get this: $\Delta \boldsymbol{PDE} \sim \Delta \boldsymbol{DQS}$. So we can have an equality as follows: $\dfrac{|\boldsymbol{PD}|}{|\boldsymbol{DQ}|} = \dfrac{|\boldsymbol{PE}|}{|\boldsymbol{ER}|}$. Why?

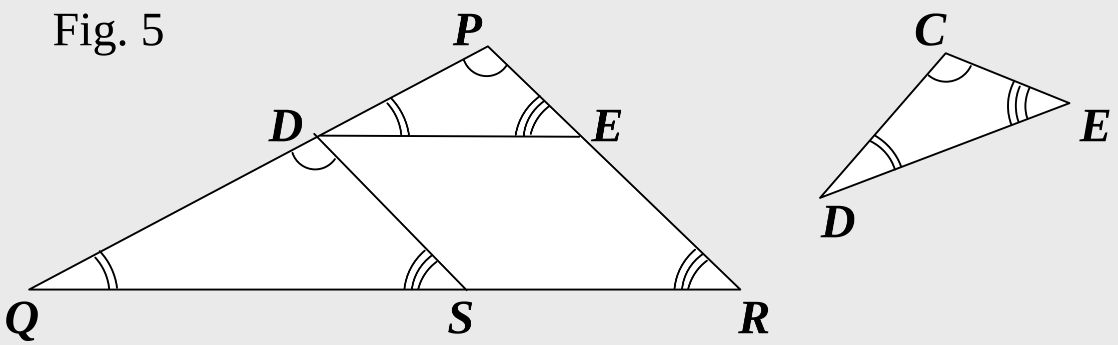


Fig. 5

It's because we have this: $\dfrac{|\boldsymbol{PD}|}{|\boldsymbol{DQ}|} = \dfrac{|\boldsymbol{PE}|}{|\boldsymbol{DS}|}$, and also, $|\boldsymbol{DS}|$ is the same as $|\boldsymbol{ER}|$.

And thus, we have not only this: $\dfrac{|\boldsymbol{PD}|}{|\boldsymbol{PQ}|} = \dfrac{|\boldsymbol{PE}|}{|\boldsymbol{PR}|}$ but this: $\dfrac{|\boldsymbol{PD}|}{|\boldsymbol{DQ}|} = \dfrac{|\boldsymbol{PE}|}{|\boldsymbol{ER}|}$, too.

It's because of these: $\Delta \boldsymbol{PDE} \sim \Delta \boldsymbol{DQS}$ and $\Delta \boldsymbol{PDE} \sim \Delta \boldsymbol{PQR}$.

In short, it's because of this: $\Delta \boldsymbol{PDE} \sim \Delta \boldsymbol{DQS} \sim \Delta \boldsymbol{PQR}$.

Now, what's the key to the fact above?

The key is this: $\angle PDE = \angle CDE = \angle Q$.

And equivalently, it s this: $\angle PED = \angle CED = \angle R$.

What then is the idea behind, that is, the basics behind?

The basics behind is this: If the lines with corresponding angles are parallel, the corresponding angles are equal, and vice versa.

So applying the basics to this situation, we can use the facts as follows.

The two angles $\angle Q$ and $\angle PDE$ are corresponding angles that are the same, and so are the two angles $\angle R$ and $\angle PED$.

Similar Triangles 12

Taking the ratios, we don't just take them.
When we take the ratios, what matters is the correspondence relation between sides.

We need to keep in mind two things.

First off, taking the ratios, we need to keep a pattern, and the pattern is as follows.

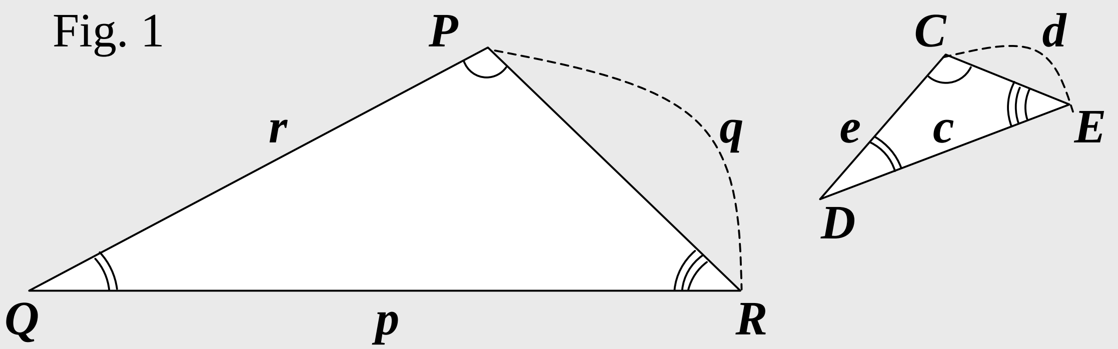
In all the ratios to be taken, all the denominators are the sides from one triangle only, and all the numerators are the sides from the other triangle only.

Next, the length order matters.

So the shortest side in one triangle corresponds to the shortest side in the other triangle.
The side of mid length in one triangle corresponds to the side of mid length in the other triangle.
And the longest side in one triangle corresponds to the longest side in the other triangle.

So looking at closely the triangles and the condition (1) shown below, you can notice the pattern that all the denominators are from ΔPQR only, and all the numerators are from ΔCDE only.

Fig. 1



(1) $$\frac{c}{p} = \frac{d}{q} = \frac{e}{r}$$

Also, you can notice that c is the longest side in ΔCDE and p is the longest side in ΔPQR, that d is the shortest in ΔCDE and q is the shortest in ΔPQR, and that e is of mid length in ΔCDE and r is of mid length in ΔPQR. So each ratio is made of corresponding sides in the pattern above.

Let's now take an example.

Suppose the side group in one triangle is (3, 5, 4), and the side group in the other triangle is (8, 6, 10). Then, we can check to see if they are similar in any of two ways as follows.

(1) $\frac{5}{10}, \frac{3}{6}$, and $\frac{4}{8}$ (2) $\frac{8}{4}, \frac{10}{5}$, and $\frac{6}{3}$

First off, notice that in all the ratios in each way, all the numerators are the sides from one triangle only, and all the denominators are the sides from the other triangle only.

Next, the length order matters. So the shortest side in one triangle corresponds to the shortest side in the other triangle. The mid length corresponds to the mid length. And the longest corresponds to the longest.

Now, beginning with (1), we can have this: $\frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{1}{2}$; that is, every ratio is equal, so we get the same scale factor, which is a half, which shows that each and every side in one triangle is half its corresponding side in the other triangle. And we can say that since every ratio between the corresponding sides is equal, the two triangles are similar.

What then about using (2) above?

We'll get the same conclusion, and work with the reciprocals of the ratios in (1).

So now, we can have this: $\frac{6}{3}=\frac{8}{4}=\frac{10}{5}=\frac{2}{1}=2$, that is, every ratio is the same, so we get the same scale factor, which is 2. Thus, the two triangles are similar. By the way, the scale factor shows that every side in one triangle is twice its corresponding side in the other triangle.

So reciprocal or not, if maintaining the correspondence right, we get the ratios right. If not, we get the ratios messed up.

For another instance, what if the side group in one triangle is (3, 2, 4), and the group in the other triangle is (3, 8, 6)?

> Note that in all the ratios, all the numerators are the sides from one triangle only, all the denominators are the sides from the other triangle only, and the length order matters.

So taking the ratios in accordance with the notice above, we can get this: $\frac{4}{8}=\frac{3}{6}\neq\frac{2}{3}$, or this: $\frac{8}{4}=\frac{6}{3}\neq\frac{3}{2}$, so either way, the triangles are not similar.

And that's not it. Putting together first, the conditions for similar triangles, we can put them the way as follows.

Fig. 1

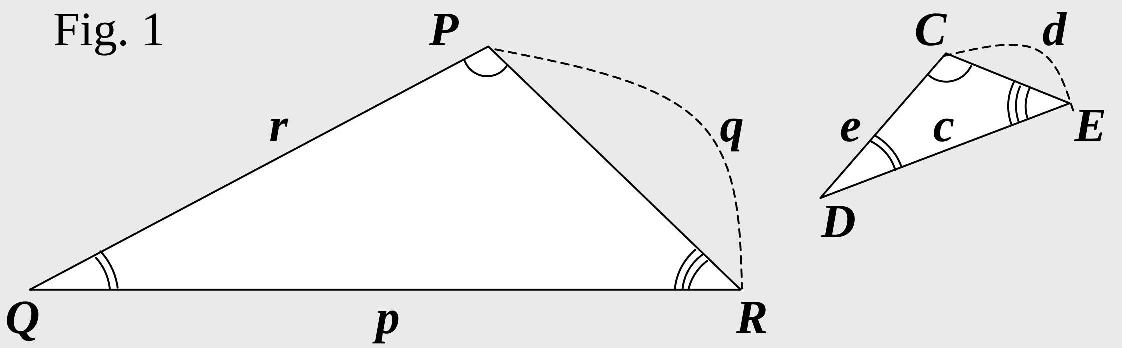
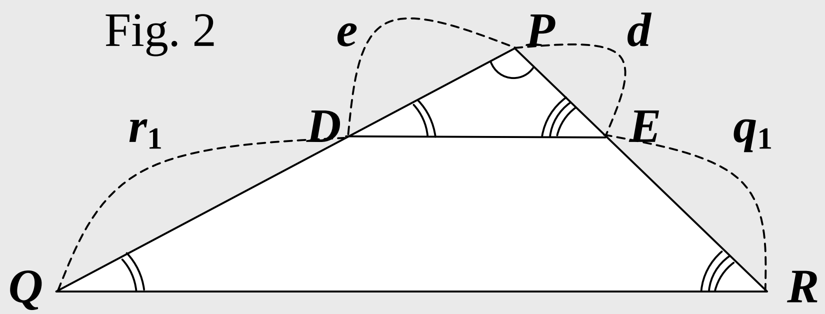


Fig. 2



(1) $\dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

(2) $\angle P = \angle C$, and $\angle Q = \angle D$

(3-1) $\angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$

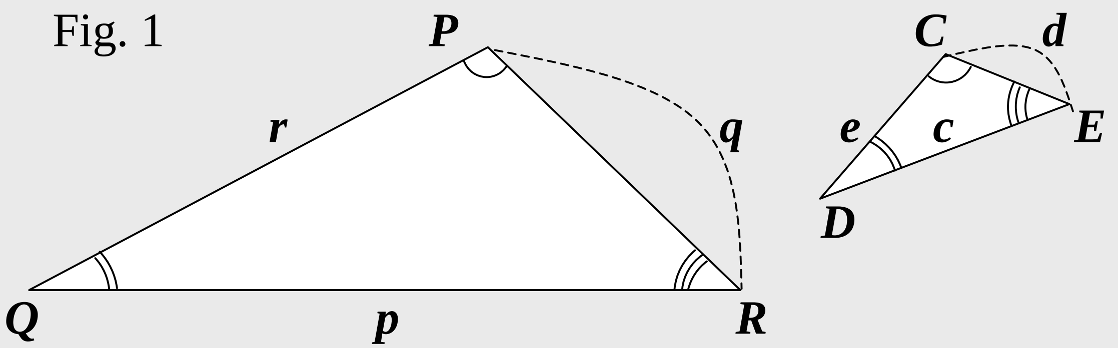
(3-2) $\dfrac{e}{r_1} = \dfrac{d}{q_1}$ in the case of Fig. 2.

And doing some algebra on the condition (1) $\dfrac{c}{p}=\dfrac{d}{q}=\dfrac{e}{r}$,

we can get equalities as follows.

$\dfrac{p}{q}=\dfrac{c}{d}$, $\dfrac{q}{r}=\dfrac{d}{e}$, and $\dfrac{r}{p}=\dfrac{e}{c}$ So what do they mean?

Fig. 1



If two triangles are similar, we can get three equalities, that is, three pairs of two same ratios the way as follows.

The longest side over the shortest side in one triangle is equal to the longest over the shortest in the other triangle.

The shortest over the mid length in one triangle is equal to the shortest over the mid length in the other triangle.

And the mid length over the longest in one triangle is equal to the mid length over the longest in the other triangle.

Also, if we get any two of the three equalities above, we can say that the two triangles are similar, because all the three sides are covered in each triangle.

And we can use them doing problems. How to get them?

The condition (1) is this: $\dfrac{c}{p}=\dfrac{d}{q}=\dfrac{e}{r}$.

First, taking the reciprocals in (1), we get this: $\dfrac{p}{c}=\dfrac{q}{d}=\dfrac{r}{e}$.

And dividing by q and multiplying by c both sides of this:

$\dfrac{p}{c}=\dfrac{q}{d}$, we get this: $\dfrac{p}{q}=\dfrac{c}{d}$.

And doing algebra similar manner, we can get the other two.

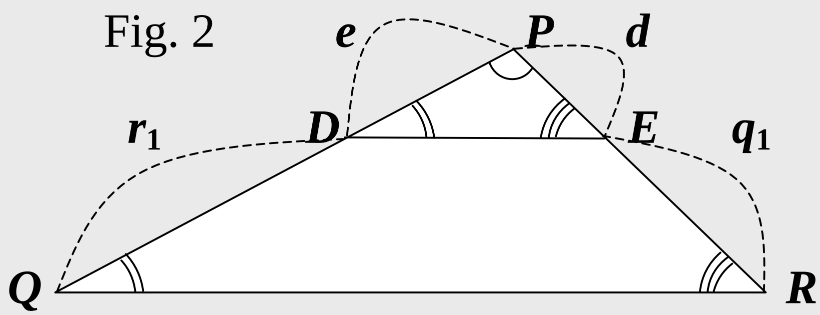
So next, dividing by r and multiplying by d both sides of this:

$\dfrac{q}{d}=\dfrac{r}{e}$, we get this: $\dfrac{q}{r}=\dfrac{d}{e}$.

And next, dividing by p and multiplying by e both sides of this: $\dfrac{p}{c}=\dfrac{r}{e}$, we get this: $\dfrac{r}{p}=\dfrac{e}{c}$.

Also, we can put the condition (3-2) $\frac{e}{r_1} = \frac{d}{q_1}$ the way as follows: $\frac{q_1}{r_1} = \frac{d}{e}$, or this way: $\frac{r_1}{q_1} = \frac{e}{d}$.

Fig. 2



We can do so doing some algebra.

Dividing by e and multiplying by q_1 both sides of (3-2), we get this: $\frac{q_1}{r_1} = \frac{d}{e}$, and taking the reciprocal of both sides, we get this: $\frac{r_1}{q_1} = \frac{e}{d}$. And you can use those solving problems.

Doing algebra, you can change, modify, or alter expressions.

Let’s now take some examples.

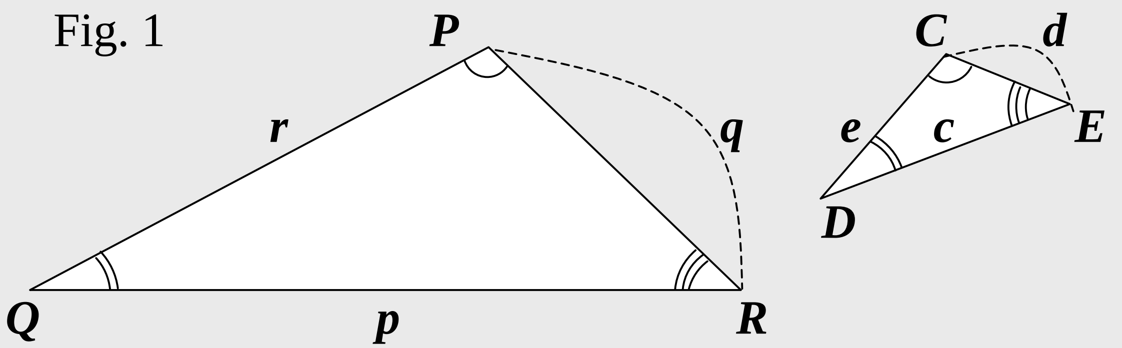
Suppose for instance, two triangles share 95^o, the two sides making 95^o in one triangle are 3 and 4, and the two sides making 95^o in the other triangle are 6 and 8.

Then, first, 95^o belongs to both triangles, that is, it is an angle common to both triangles.

So in this case, we can use the condition (3-1), which requires ***a common angle*** and ***two equal ratios***, and is as follows.

$$\text{(3-1)} \quad \angle \boldsymbol{P} = \angle \boldsymbol{C}, \text{ and } \frac{\boldsymbol{e}}{\boldsymbol{r}} = \frac{\boldsymbol{d}}{\boldsymbol{q}}$$

Fig. 1



Now that an angle is common, we have only to take two ratios, and see if the two are equal.

Taking the ratios, though, we don't just take them.

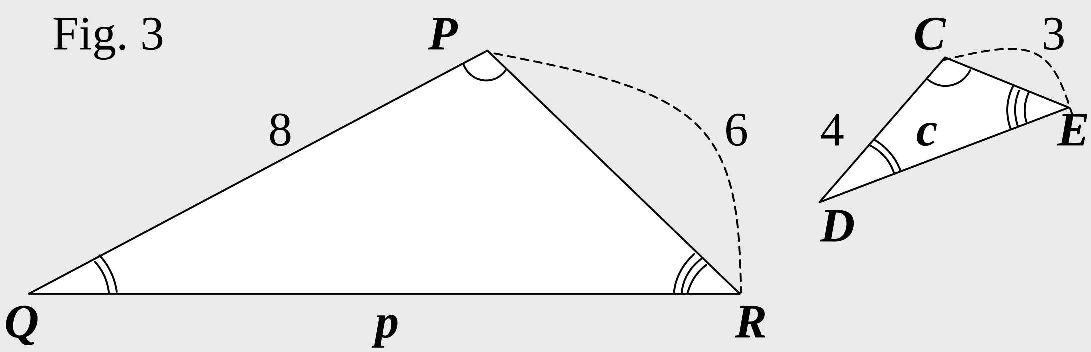
We need to maintain the correspondence.

In all the ratios, all the numerators are the sides from one triangle only, all the denominators are the sides from the other triangle only, and the length order matters.

In this case, though, we have only to get two ratios, because we use two sides making the common angle in each triangle.

So the shorter side in one triangle corresponds to the shorter side in the other triangle. And the longer side in one triangle corresponds to the longer side in the other triangle.

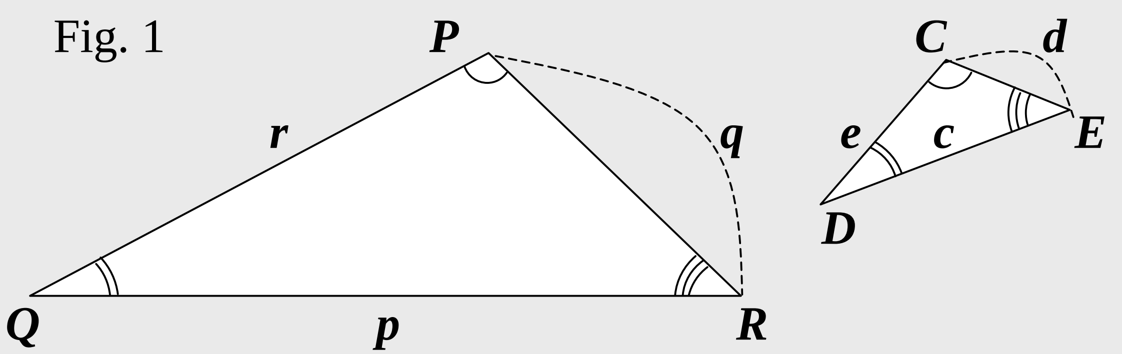
Fig. 3



So now, assuming $\angle P = \angle C = 95^{o}$ and taking the two ratios, we get $\frac{3}{6}$ and $\frac{4}{8}$. So we get $\frac{3}{6} = \frac{4}{8} = \frac{1}{2}$, and thus, similar. And that's not the only way.

We are now using this: (3-1) $\angle P = \angle C$, and $\frac{e}{r} = \frac{d}{q}$

Fig. 1



Doing algebra, we can convert this: $\frac{e}{r} = \frac{d}{q}$ to this: $\frac{q}{r} = \frac{d}{e}$, and vice versa. So we have this: $\frac{e}{r} = \frac{d}{q} \Leftrightarrow \frac{q}{r} = \frac{d}{e}$,

So we can check this: $\frac{q}{r} = \frac{d}{e}$ to see if the triangles are similar. We can work with the reciprocals, too, of course.

So we can check this, too: $\frac{r}{q} = \frac{e}{d}$.

And of course, taking the ratios, we don’t just take them.

We need to maintain the correspondence.

Unlike the case covered earlier, the correspondence in this case goes bit differently. We are now taking each ratio using two sides in one triangle, not one side from each triangle.

And mentioned earlier that if two triangles are similar, we can get three equalities, that is, three pairs of two same ratios the way as follows.

The longest side over the shortest side in one triangle is equal to the longest over the shortest in the other triangle.

The shortest over the mid length in one triangle is equal to the shortest over the mid length in the other triangle.

And the mid length over the longest in one triangle is equal to the mid length over the longest in the other triangle.

In this case, though, we use two sides in each triangle, not three. So we take the ratios the way as follows.

If one ratio is the longer side over the shorter side in one triangle, the other ratio has to be the longer over the shorter in the other triangle, too. And the same is true of the reciprocals, too.

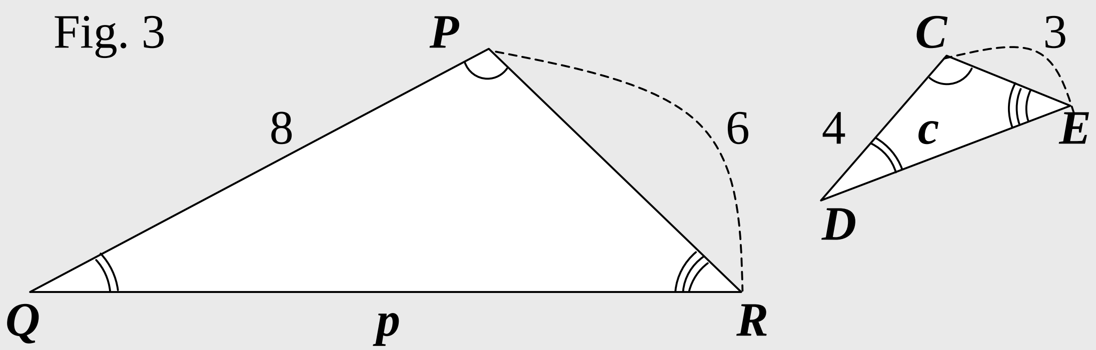
So if one ratio is the shorter side over the longer side in one triangle, the other ratio has to be the shorter over the longer in the other triangle, too.

And of course, if the two ratios are equal, we get the same scale factor, that is, the triangles are similar.

We have used this: (3-1) $\angle \boldsymbol{P} = \angle \boldsymbol{C}$, and $\dfrac{\boldsymbol{e}}{\boldsymbol{r}} = \dfrac{\boldsymbol{d}}{\boldsymbol{q}}$

So let's this time, use this: (3-2) $\angle \boldsymbol{P} = \angle \boldsymbol{C}$, and $\dfrac{\boldsymbol{q}}{\boldsymbol{r}} = \dfrac{\boldsymbol{d}}{\boldsymbol{e}}$

Fig. 3



So taking the ratio using (3-2), we can get these: $\frac{6}{8}$ and $\frac{3}{4}$

So we get this: $\frac{6}{8}=\frac{3}{4}$, and thus, similar.

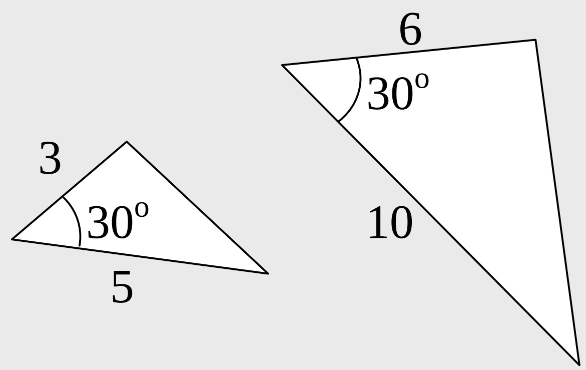
And of course, we can take the reciprocal, too.

Then, we can get these: $\frac{8}{6}$ and $\frac{4}{3}$.

So we get this: $\frac{8}{6}=\frac{4}{3}$, and thus, similar.

Suppose for another instance, in one triangle, the angle between two sides of lengths 3 and 5 is 30^o, and also, in the other triangle, the angle between two sides of lengths 6 and 10 is 30^o.

Fig. 4



Then, taking the ratios, we get $\frac{3}{6}$ and $\frac{5}{10}$, so both are $\frac{1}{2}$.

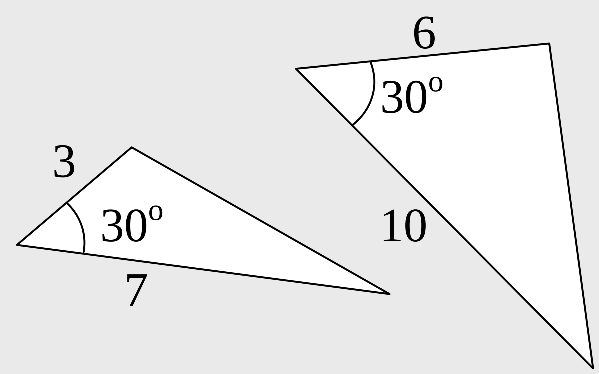
Thus, the two triangles are similar.

If taking the ratios wrong way, that is, the way the correspondence is not maintained, we could get this:

$\frac{6}{3} = 2 \neq \frac{5}{10} = \frac{1}{2}$, which shows that it's not the case that the two ratios between the corresponding sides are equal, and thus, doesn't show that the two triangles are similar.

Suppose for another instance, in one triangle, two sides of lengths 3 and 7 make 30^{o}, and also, in the other triangle, the angle between two sides of lengths 6 and 10 is 30^{o}.

Fig. 5

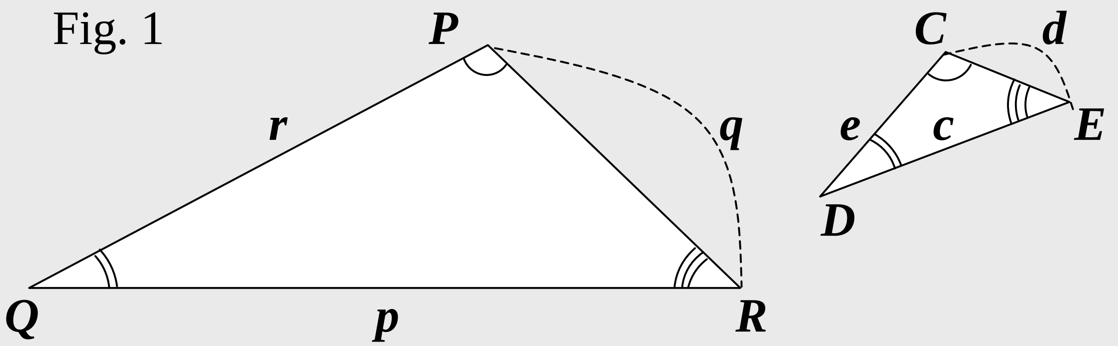


Then, taking the ratios, we can get these: $\frac{3}{6}$ and $\frac{7}{10}$, which are not equal, so the two triangles are not similar.

Now, summing up, if triangles are not nested, we can put together all the versions of the definition the way as follows.

By definition, $\Delta PQR \sim \Delta CDE$ iff we get any of these: (1), (2), and (3) as shown below.

Fig. 1



(1) $\dfrac{c}{p} = \dfrac{d}{q} = \dfrac{e}{r}$

(2) $\angle P = \angle C$, and $\angle Q = \angle D$

(3) $\angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$

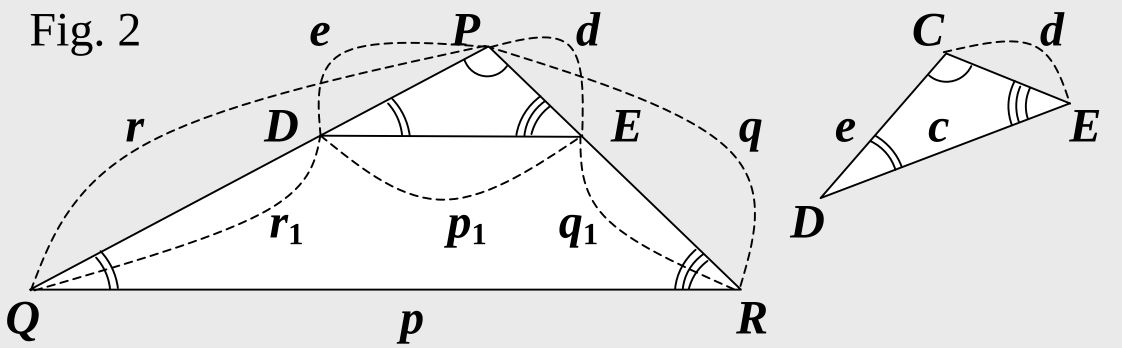
And again, each of the three above is a condition for similar triangles. Very nice and convenient tool. And from (1), we can get these, too: $\dfrac{p}{q} = \dfrac{c}{d}$, $\dfrac{q}{r} = \dfrac{d}{e}$, and $\dfrac{r}{p} = \dfrac{e}{c}$.

What if one triangle is nested inside the other?

If ΔCDE is nested, that is, placed inside ΔPQR so that it becomes ΔPDE as shown in Fig. 2 below, we can put the definition the way as follows.

By definition, $\Delta PQR \sim \Delta PDE \quad \Leftrightarrow \quad \frac{e}{r_1} = \frac{d}{q_1}$, since both triangles share $\angle P$, that is, an angle is common.

Fig. 2



And of course, we have this, too:

By definition, $\Delta PQR \sim \Delta PDE \quad \Leftrightarrow \quad \frac{e}{r} = \frac{d}{q}$

Also, we can get this: $\frac{q_1}{r_1} = \frac{d}{e}$ from this: $\frac{e}{r_1} = \frac{d}{q_1}$.

And putting together the two cases, nested and not nested, we can put the conditions for similar triangles the way below.

Fig. 1

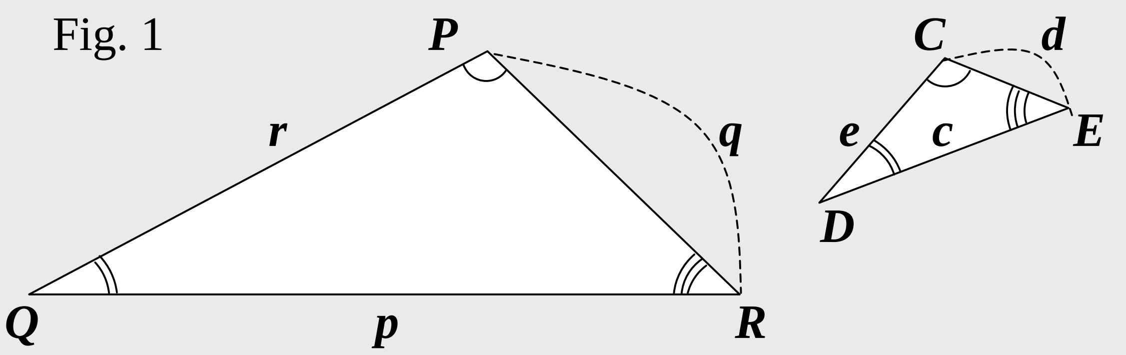
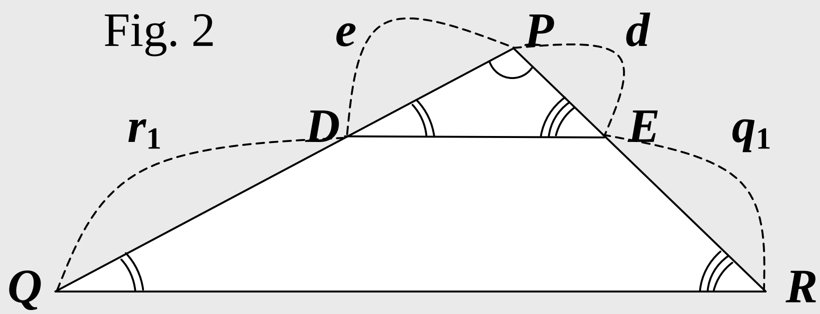


Fig. 2



(1) $$\frac{c}{p} = \frac{d}{q} = \frac{e}{r}$$

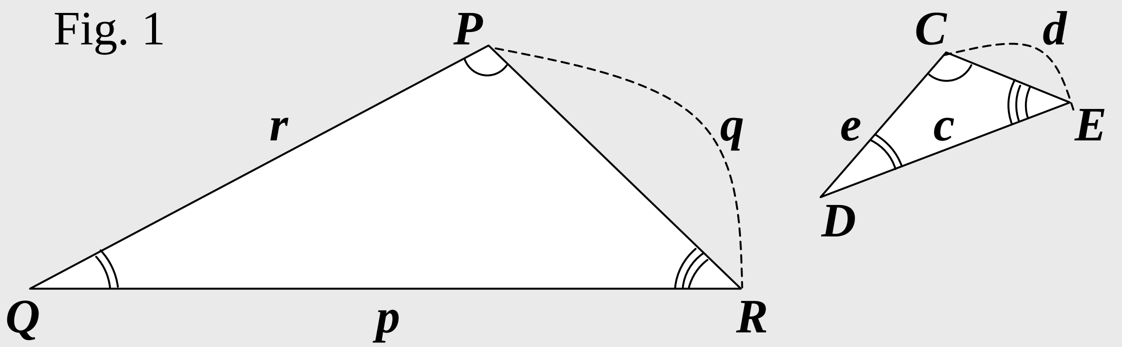
(2) $\angle P = \angle C$, and $\angle Q = \angle D$

(3-1) $\angle P = \angle C$, and $\dfrac{e}{r} = \dfrac{d}{q}$

(3-2) $\dfrac{e}{r_1} = \dfrac{d}{q_1}$ in the case of Fig. 2.

Thus, you have now math basics called ***definitions for similar triangles***, which are in two groups. Each group is made of three versions, and one group is as follows.

Fig. 1



(1) $\Delta PQR \sim \Delta CDE \iff \frac{c}{p} = \frac{d}{q} = \frac{e}{r}$

(2) $\Delta PQR \sim \Delta CDE \iff \angle P = \angle C$, and $\angle Q = \angle D$

(3) $\Delta PQR \sim \Delta CDE \iff \angle P = \angle C$, and $\frac{e}{r} = \frac{d}{q}$

And if any of (2) and (3) above is true, we get these:

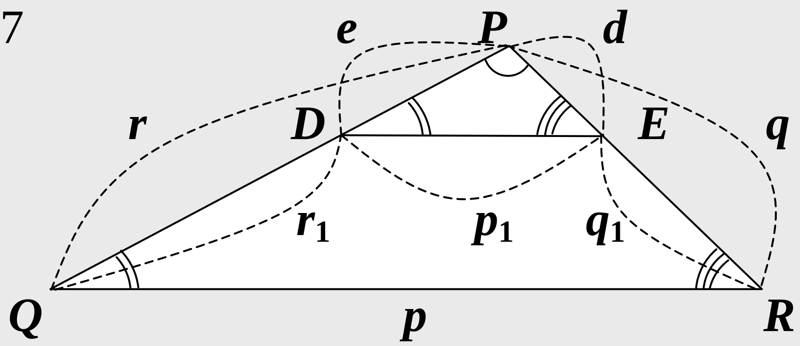
(a) $\frac{c}{p} = \frac{d}{q} = \frac{e}{r}$, three equal ratios, the same scale factor

(b) $\angle P = \angle C$, $\angle Q = \angle D$, and $\angle R = \angle E$,

i.e., both triangles have the same angle groups.

And the other group is as follows. Note that ΔPDE is nested inside ΔPQR, and ***very particularly*** that $\angle P$ is common.

Fig. 7



(1-1) $\Delta PQR \sim \Delta PDE \quad \Leftrightarrow \quad \dfrac{d}{q} = \dfrac{e}{r}$

(2-1) $\Delta PQR \sim \Delta PDE \quad \Leftrightarrow \quad \angle Q = \angle D$

(3-1) $\Delta PQR \sim \Delta PDE \quad \Leftrightarrow \quad \dfrac{e}{r_1} = \dfrac{d}{q_1}$

And if any of the three above is true, we get these:

(a-1) $\dfrac{p_1}{p} = \dfrac{d}{q} = \dfrac{e}{r}$, three equal ratios, the same scale factor

(b-1) $\angle Q = \angle D$, and $\angle R = \angle E$,

i.e., both triangles have the same angle groups.

And using the plane language, we can put the conditions the way as follows.

Equal Ratios (the same scale factor)

The Same Angle Groups

A Common Angle with Two Equal Ratios

What is the use of those?

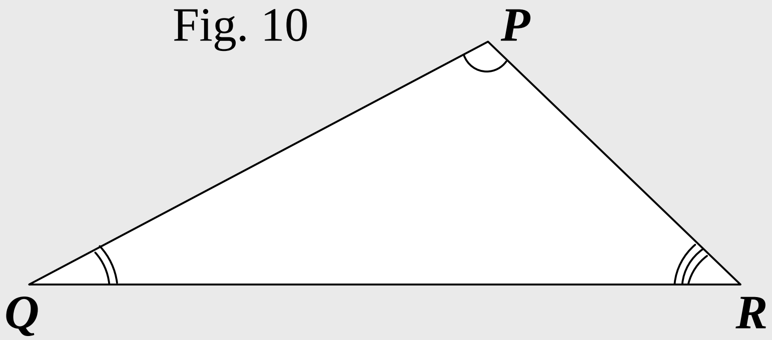
There are many uses, and the use depends on the fluency of the concept you get, the concept of similar triangles, together with, of course, the proper and practical understanding of the definitions for similar triangles.

For instance, using the equal ratio, that is, the same scale factor, we can find the side or angle in question in another triangle that is similar.

Also, we can find the area of a triangle without the information on the sides or the height. We can find the area using the same scale factor.

Assuming for instance, in Fig. 8, $\Delta PQR \sim \Delta CDE$, let's find the area of the ΔCDE if the area of ΔPQR is 8, $|PR| = 4$, and $|CE| = 2$.

Fig. 10



If two polygons are similar, the ratio between the areas is the square of the ratio between the sides. And the same is true for circles, too. Thus, the ratio between the areas is the square of the ratio between the radii.

So since the two triangles are similar, and the ratio of the side PR to the corresponding side CE is $\frac{4}{2} = 2$, the ratio of the area of ΔPQR to the area of ΔCDE is 2^2, which is 4.

Thus, the area of ΔPQR is 4 times the area of ΔCDE.
So since the area of the ΔPQR is 8, the area of the ΔCDE is 2.

Let's now take another example.

Fig. 11

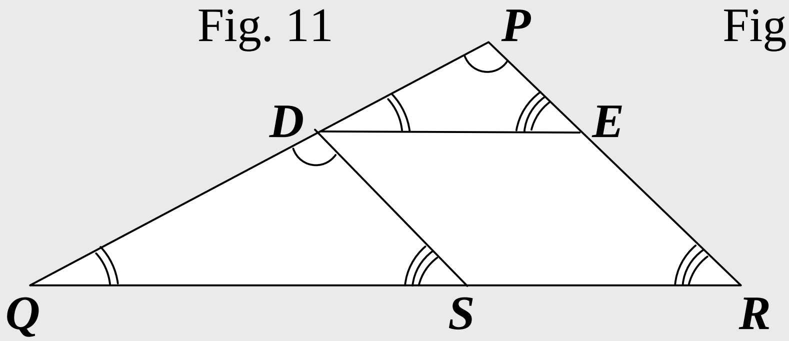
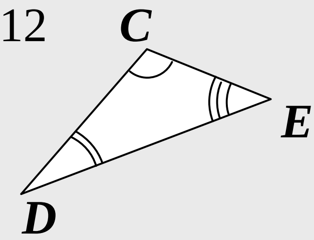


Fig.12



In this case, we refer to Fig. 11 where a similar triangle is nested. So we have $\Delta PDE \sim \Delta PQR$.

Then, from this: $\dfrac{|PD|}{|PQ|} = \dfrac{|PE|}{|PR|} = \dfrac{|DE|}{|QR|}$, we can have this, too: $\dfrac{|PD|}{|DQ|} = \dfrac{|PE|}{|ER|}$.

Also, since the two triangles ΔPQR and ΔCDE are similar, we get the properties of similar triangles as follows.

Fig. 11

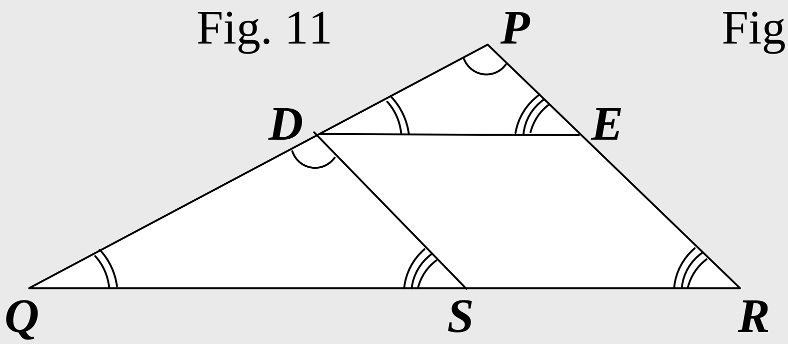
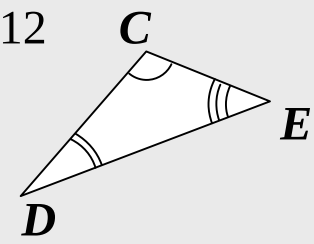


Fig.12



The three *ratios* between the corresponding sides are *equal*,

so we get $\dfrac{|CD|}{|PQ|} = \dfrac{|CE|}{|PR|} = \dfrac{|DE|}{|QR|}$.

And we get this, too: $\dfrac{|PD|}{|DQ|} = \dfrac{|PE|}{|ER|}$.

What if we get one of the two as follows?

$$\frac{|CD|}{|PQ|} = \frac{|CE|}{|PR|} = \frac{|DE|}{|QR|} = 1$$

$$\angle P = \angle C, \text{ and } \frac{|CE|}{|PR|} = \frac{|CD|}{|PQ|} = 1$$

Each of the two is the condition for identical triangles.