

# Triangle Basics B

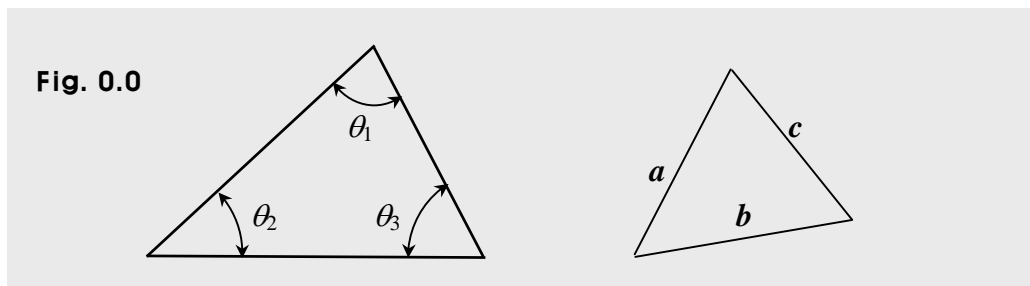
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# o. What is a triangle?

It is a three angle figure literally.

It's because tri means three, so literally, a triangle is made of three angles.

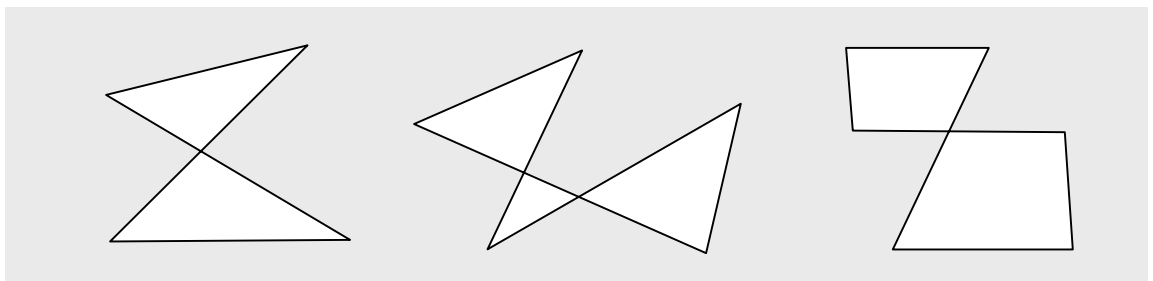


Next, a triangle is made of three sides, and is a polygon.

What is a polygon though?

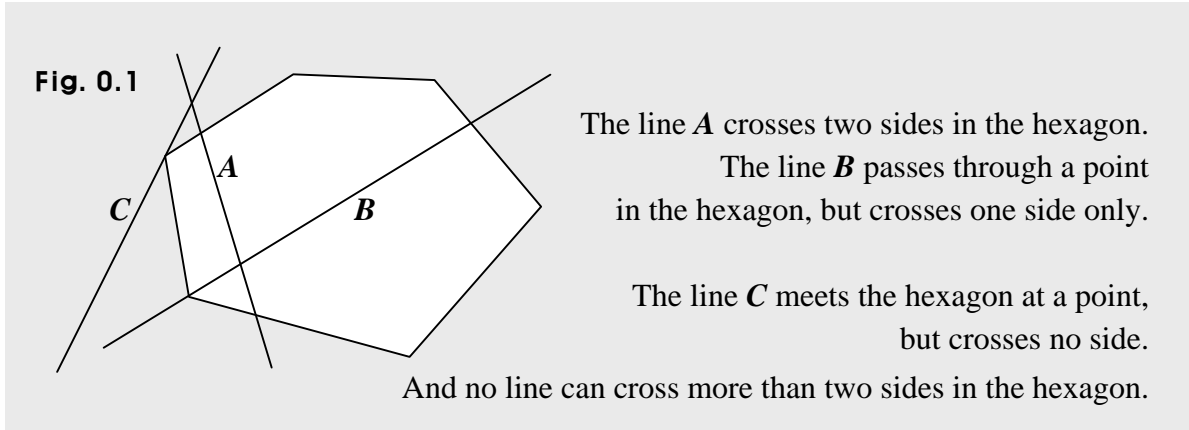
It is a 2-D object closed, and is made of three or more line segments. Usually though, we just call each line segment a side. And in basic math, we normally work with polygons said to be simple and convex. What then is such a polygon?

In each side in a polygon simple and convex, each endpoint is an endpoint of one of the other sides. So for instance, the polygons below are not simple and convex.

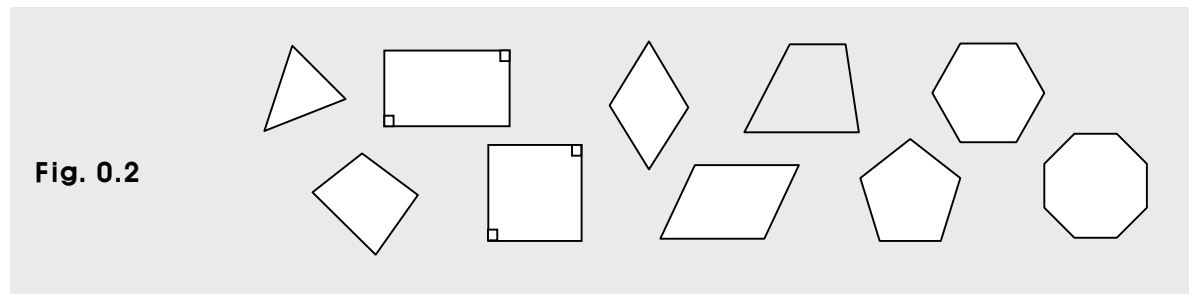


That's because in each polygon above, some side has an endpoint that belongs to more than one of the other sides.

And a line passing through a polygon simple and convex can cross up to two sides in the polygon, so it cannot cross three or more sides. How?



And among such polygons, we have triangles, quadrangles as rectangles, squares, rhombuses, parallelograms, and trapezoids, and others as pentagons, hexagons, etc.



So what can we say about triangles?

A triangle is a polygon made of three sides, in each of which, each endpoint is an endpoint of one of the other two sides. Thus, triangles are the simplest polygons.

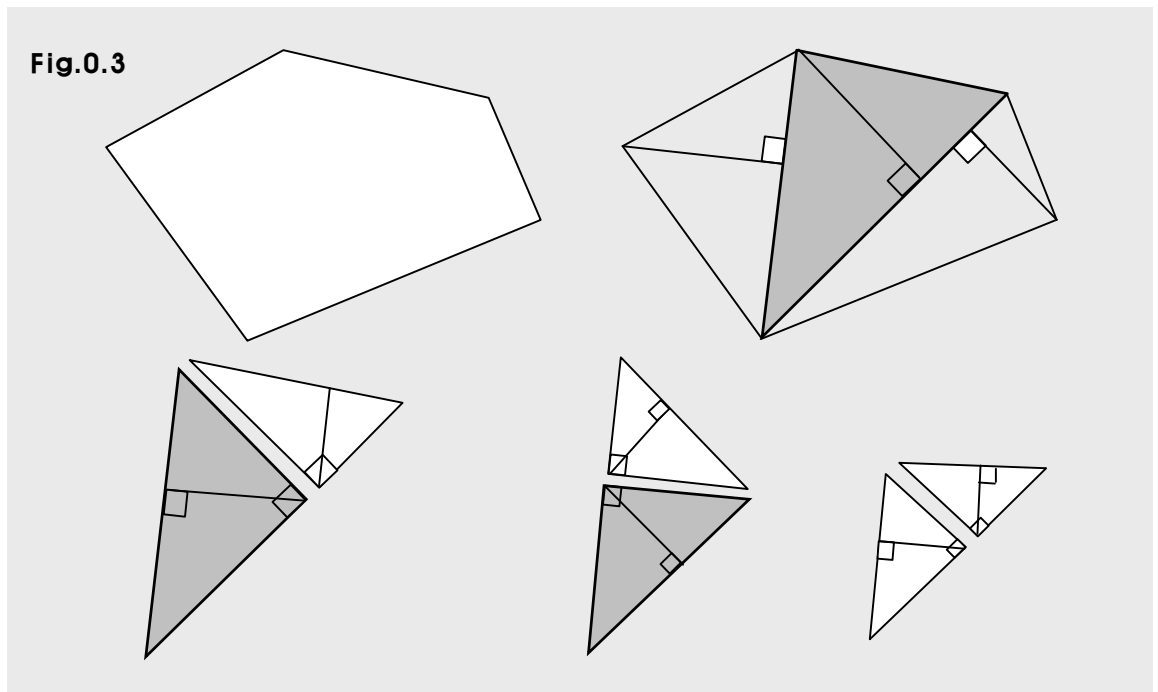
No matter what polygon it may be though, it can be said to be made of triangles. More specifically, it can be partitioned into triangles. Thus, triangles are not only the simplest but the most basic polygons, too.

And we can say also, a triangle is made of many other triangles, too, in each of which, one angle is  $90^\circ$ , called a right angle.  
So those other triangles are called right triangles.

More specifically, no matter what triangle it may be, it can be partitioned into two right triangles at a time.

And we can partition a right triangle into two other right triangles at a time, also.

Thus, it can be said that every polygon is made of right triangles.



So what can we say about right triangles?

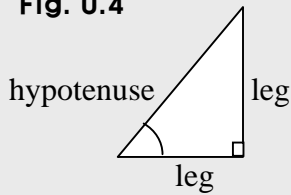
Right triangles are fundamental triangles, and thus, are important. And they are very much so. It is often the case we can't do much without right triangles solving problems.

And we can do a lot using right triangles.

Right triangles do much not only in geometry but in algebra, too. It is often vital that we use right triangles right.

Thus, we want to know them very well, and use them very well, too.

Fig. 0.4



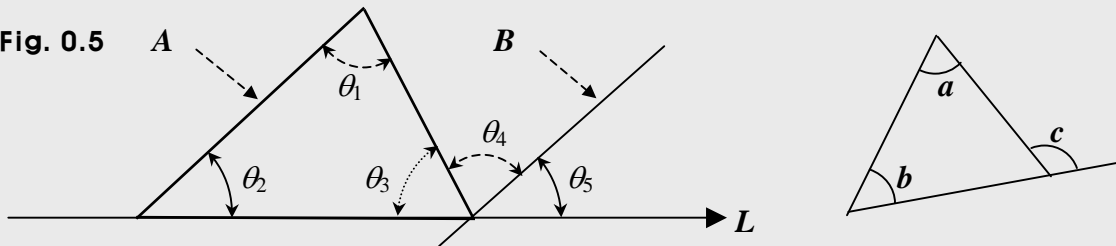
A right triangle is made of three sides, called a hypotenuse and two legs. And the two legs are perpendicular to each other, and thus, makes  $90^\circ$ .

So the hypotenuse is facing the angle  $90^\circ$ , and thus, is opposite of the right angle.

And the sum of the two angles adjacent to the hypotenuse is  $90^\circ$ , because the sum of all the three angles in a triangle is  $180^\circ$ . How is it  $180^\circ$  though?

To begin with, putting a triangle on a line called  $L$ , we can put it the way below.

Fig. 0.5



And assuming next, the line segment  $A$  is parallel to the line  $B$  above, we can say that the two angles  $\theta_2$  and  $\theta_5$  are corresponding angles, and thus, are equal, and the two angles  $\theta_1$  and  $\theta_4$  are alternate angles, and thus, are equal, too.

So we get  $\theta_1 + \theta_2 + \theta_3 = \theta_3 + \theta_4 + \theta_5$ , which is  $180^\circ$ .

That is to say that the sum of all the three (internal) angles in a triangle is  $180^\circ$ .

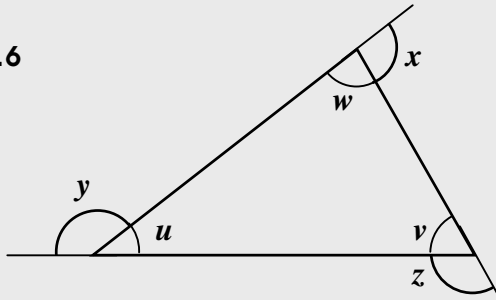
And we can notice that in the triangle on the right in Fig. 0.5 above, the sum of two angles  $a$  and  $b$  is the same as the angle  $c$ .

In a triangle, in fact, the sum of two internal angles is the same as the external angle *supplement* to the other internal angle. If two angles are *supplement* to each other, the sum of the two is  $180^\circ$ .

What then about the sum of all the three external angles?

We know if two line segments are in a line, the angle between the two is  $180^\circ$ .

Fig. 0.6



And we know  $u + v + w = 180^\circ$ .

So the sum is  $3 \cdot 180^\circ - 180^\circ = 360^\circ$ .

Thus, a triangle has three angles and three sides, and is a polygon with the smallest number of angles and sides. So it is a polygon the simplest and the most basic, and thus, is the most important of all polygons. The more basic, the more important.

And no matter what triangle it may be, what does its three angles add up to?

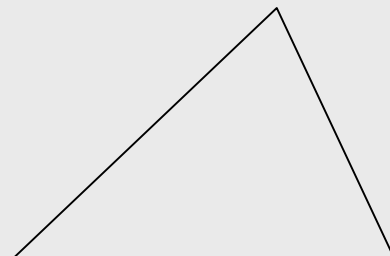
The three angles add up to  $180^\circ$ . And we can partition any triangle into two right triangles at a time. So note that the sum of all the three angles in a triangle is  $180^\circ$ , and every polygon can be made of right triangles.

And, we want to note also that a triangle is an idea and not a material object, and that saying just a triangle, we mean an object made of three line segments and not a triangular plate packed with points. So **nothing** inside a triangle is a part of the triangle, which is therefore, made of three line segments only, and is empty inside.

Fig. 0.7



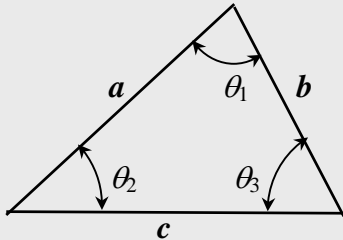
a triangular plate



a triangle

And in a triangle, if all the three angles are different, how are all the three sides? If in a triangle, all the three angles are different, all the three sides are different, too. And vice versa, so if all the three sides are different, all the three angle are different, too. And we call such a triangle a *scalene* triangle.

Fig. 0.8

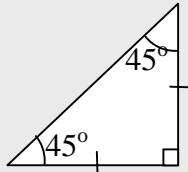


$$\theta_1 \neq \theta_2 \neq \theta_3 \Rightarrow a \neq b \neq c$$

Next, if in a triangle, two angles are the same, its two sides are the same, too, and vice versa. And it is called *isosceles*. So if a triangle is an isosceles, two sides and two angles are the same. Thus, in a right triangle isosceles, what are the three angles?

In a right triangle isosceles, two angles are  $45^\circ$  each, because the other angle is  $90^\circ$ , and all the three angles add up to  $180^\circ$ .

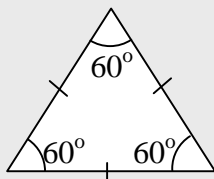
Fig. 0.9



And of course, if all the three angles are equal, all the three sides are equal, too, and vice versa. And we call it a *regular* or *equilateral* triangle. And we know the sum of the three angles is  $180^\circ$ . So what is each angle in a regular triangle?

Every angle in a regular triangle is  $60^\circ$ , because the sum of the three angles is  $180^\circ$ , and the three angles are equal.

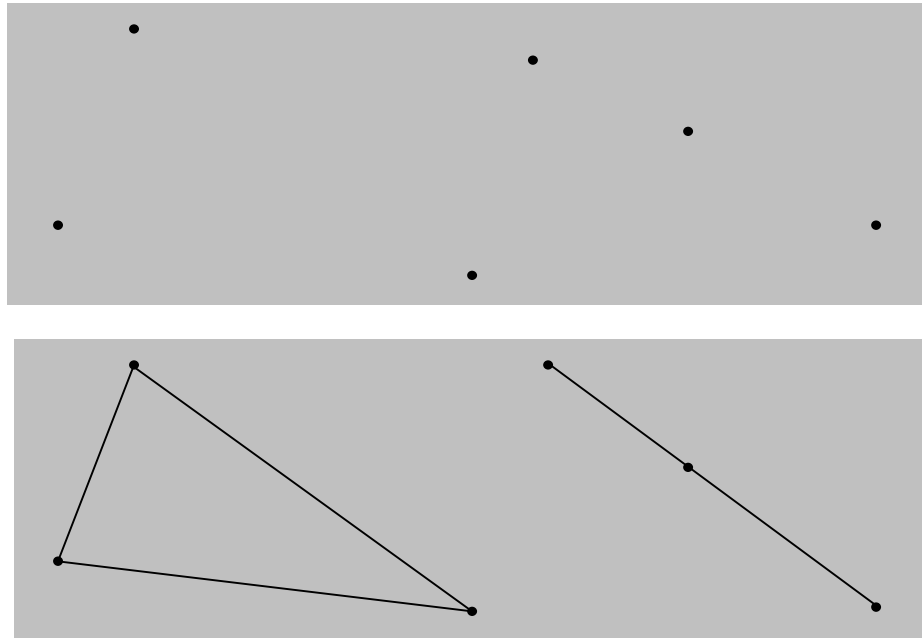
Fig. 0.A



So can we form a triangle, connecting three points, using three line segments, of course?

We can form a triangle, connecting three points, only if the three points are *not* in a line. So given three points in a line, we cannot make a triangle connecting the three points.

Fig. 0.B



And when we work with a triangle, it's good idea to name the triangle. Naming a triangle, we often use the names of the three vertices in the triangle. What do we mean by the vertex though?

A vertex in a polygon is a point where two sides meet. So a triangle has three vertices.

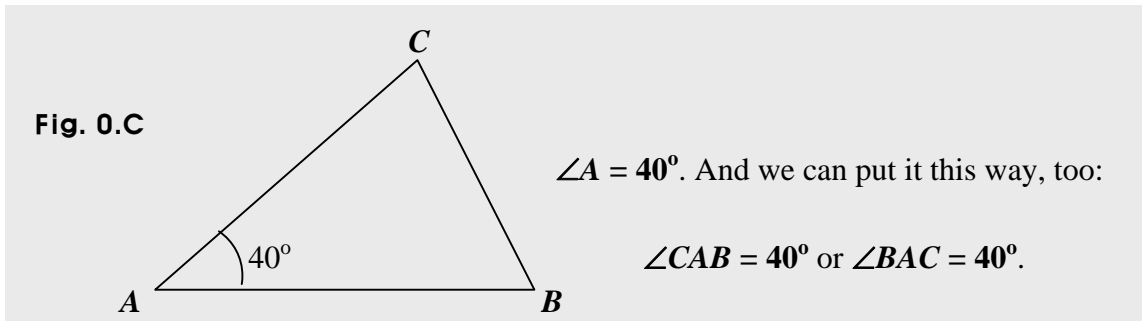
For instance, calling a triangle  $ABC$ , we mean  $A$ ,  $B$ , and  $C$  are its three vertices.

And specifying a triangle, we often use a symbol, together with the names of the vertices.

The symbol is  $\Delta$ , which is just a small triangle. So for instance,  $\Delta ABC$  means a triangle  $ABC$ . What then about naming each angle?

Naming an angle a triangle has, we usually use a capital letter as **A**, and the angle is an internal angle, of course. That is to say that naming an angle in a triangle, we use the name of the vertex that has the angle.

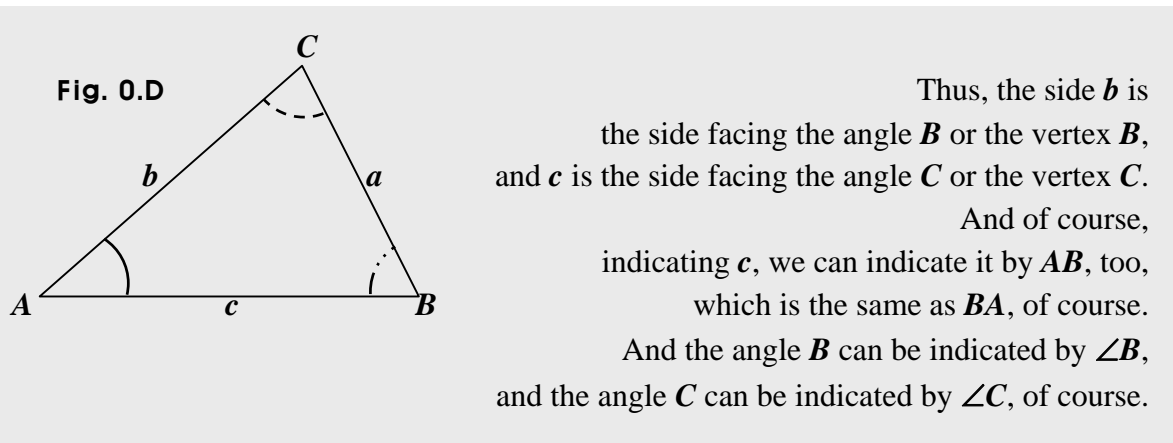
So for instance, saying the angle **A** in a triangle **ABC**, we mean the angle at the vertex **A**. And indicating such an angle, we often use a symbol called an angle symbol, which is  $\angle$ . So for instance,



What then about naming the sides in the triangle **ABC** above?

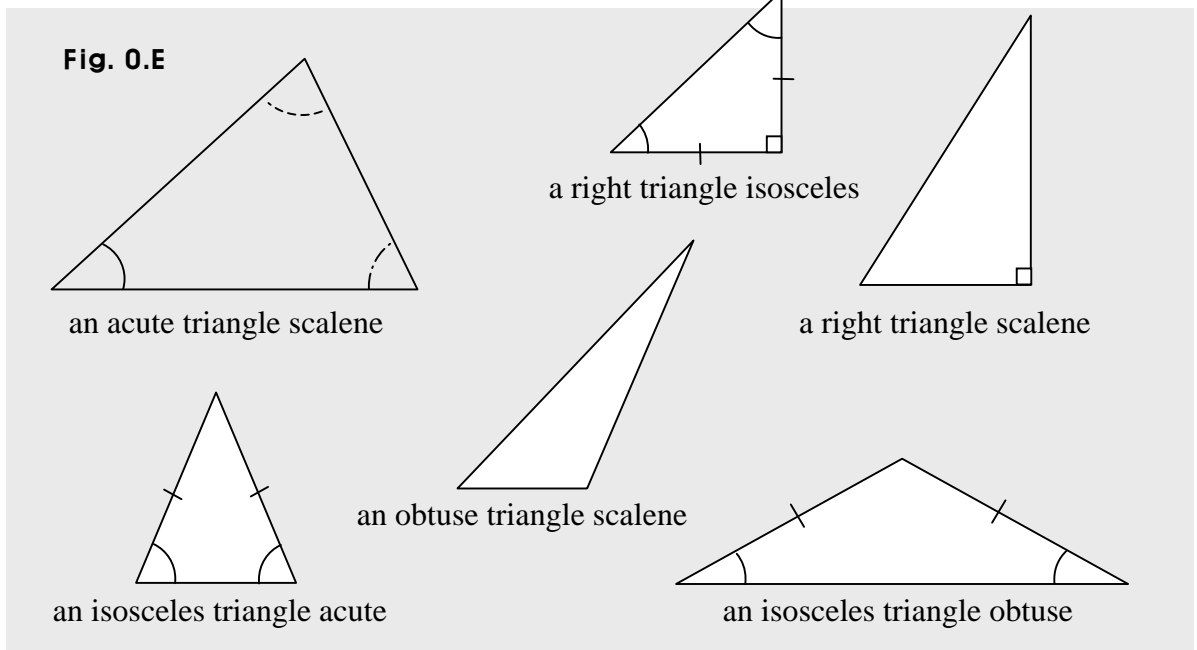
Naming each side in a triangle, we usually use a lowercase letter as **a**.

So for instance, in the triangle **ABC** above, we can use **a** as the side facing the angle **A** or the vertex **A**. That is, **a** is the side opposite of the angle **A** or the vertex **A**. So for instance, we can put the names of the angles, the vertices, and the sides the way below.



And we can put a triangle in one of three kinds: acute triangles, right triangles, and obtuse triangles. What are those three kinds though?

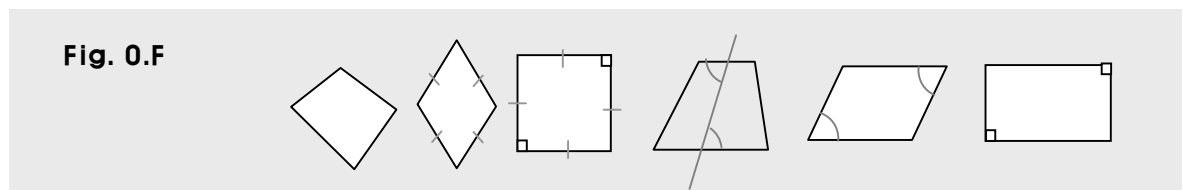
To begin with, in an acute triangle, all the three angles are between  $0$  and  $90^\circ$ . And such an angle is called an acute angle. For instance, a regular triangle is a triangle acute. Next, in a triangle obtuse, one angle is between  $90^\circ$  and  $180^\circ$ , and such an angle is called an obtuse angle. And as stated above, a right triangle has an angle of  $90^\circ$ , which is called a right angle.



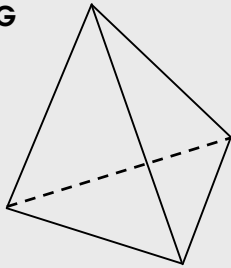
And a regular triangle can be taken as an isosceles triangle, too, because it has two sides equal, since all the three sides in it are the same.

So we have some kinds or categories in triangles, which are the most basic and the simplest of all polygons. And putting together triangles, we can form a polygon.

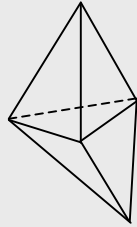
That is to say that we can partition a polygon into triangles. And among those polygons, we have tetragons (quadrangles), pentagons, hexagons, etc. Tetra or quad means four, so a tetragon is a four-sided polygon. And some examples of tetragons can be as follows.



Also, putting together triangles, we can form another kind of geometric object called triangle polyhedrons. In such a polyhedron, each face is a triangle. And for instance,

**Fig. 0.G**

tetrahedron

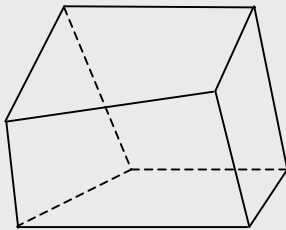


triangle hexahedron

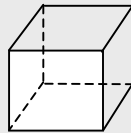


regular triangle hexahedron

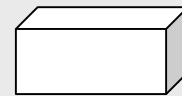
And if each face is a quadrangle in a hexahedron, we can call it a quadrangle hexahedron.

**Fig. 0.H**

quadrangle hexahedron



cube (square hexahedron)



rectangular hexahedron

Thus, solving problems with geometry, we can work with quite a few geometric objects. And it can be said that each of those begins with a triangle.

# 1. Triangle Inequality

So forming a triangle, can we just use three line segments? That is, given three line segments, can we make a triangle putting together the three line segments?

It is *not always* the case where we can make a triangle connecting three line segments.

If the sum of any two of the three is larger than the other, we can make one. That is to say that if any of the three is greater than the sum of the other two, we cannot make it.

So for instance, if  $a$ ,  $b$ , and  $c$  are three line segments in a triangle, we get either of the cases as follows:

- Assuming  $c$  is the biggest, we get:  $a + b > c$ . For instance,  $2 + 4 > 5$ .
- If  $a$  is the biggest, and  $b + c > a$ , three line segments  $a$ ,  $b$ , and  $c$  can form a triangle.
- If three line segments  $a$ ,  $b$ , and  $c$  form a triangle, and  $b$  is the longest, we get:  $c + a > b$ .

And if we don't know which of  $a$ ,  $b$ , and  $c$  is the biggest, we have to get:

$$a + b > c, b + c > a, \text{ and } c + a > b.$$

Note that it says not *or* but *and*.

So if  $a + b > c$ ,  $b + c > a$ , or  $c + a > b$ , it **can be** the case we get **no** triangle.

For instance, if  $a = 2$ ,  $b = 3$ , and  $c = 6$ , we don't get a triangle, because we don't get this:  $a + b > c$ , although we get these:  $b + c > a$ , and  $c + a > b$ .

For another instance, if  $a = 3$ ,  $b = 5$ , and  $c = 2$ , we don't get a triangle, because we don't get  $c + a > b$ , although we get  $a + b > c$ , and  $b + c > a$ .

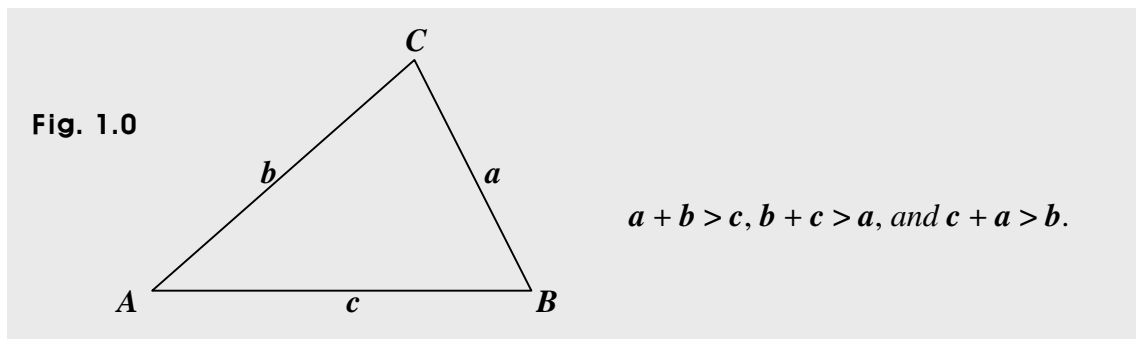
And for another instance, if  $a = 6$ ,  $b = 3$ , and  $c = 3$ , we don't get a triangle, because we don't get this:  $b + c > a$ , even though we get these two:  $a + b > c$ , and  $c + a > b$ .

Therefore, only if the three cases  $a + b > c$ ,  $b + c > a$ , and  $c + a > b$  are all true at the same time, we can get a triangle, that is,  $a$ ,  $b$ , and  $c$  form a triangle.

So a triangle is not just made of three sides, and if need to check to see if three line segment can form a triangle, we have to check all the three cases above if not knowing which of the three is the biggest.

If knowing the biggest, we can see if the three can form a triangle taking the sum of the other two and comparing the sum with the one biggest. If the sum is the larger, we get a triangle; otherwise, we don't.

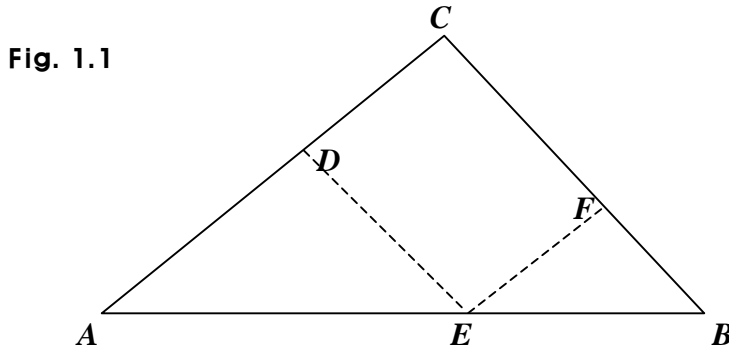
And the fact above is a property of a triangle, and we call it Triangle Inequality. And we can put the fact in a figure the way below:



So if  $a$ ,  $b$ , and  $c$  form a triangle, we get  $a + b > c$ ,  $b + c > a$ , and  $c + a > b$ .

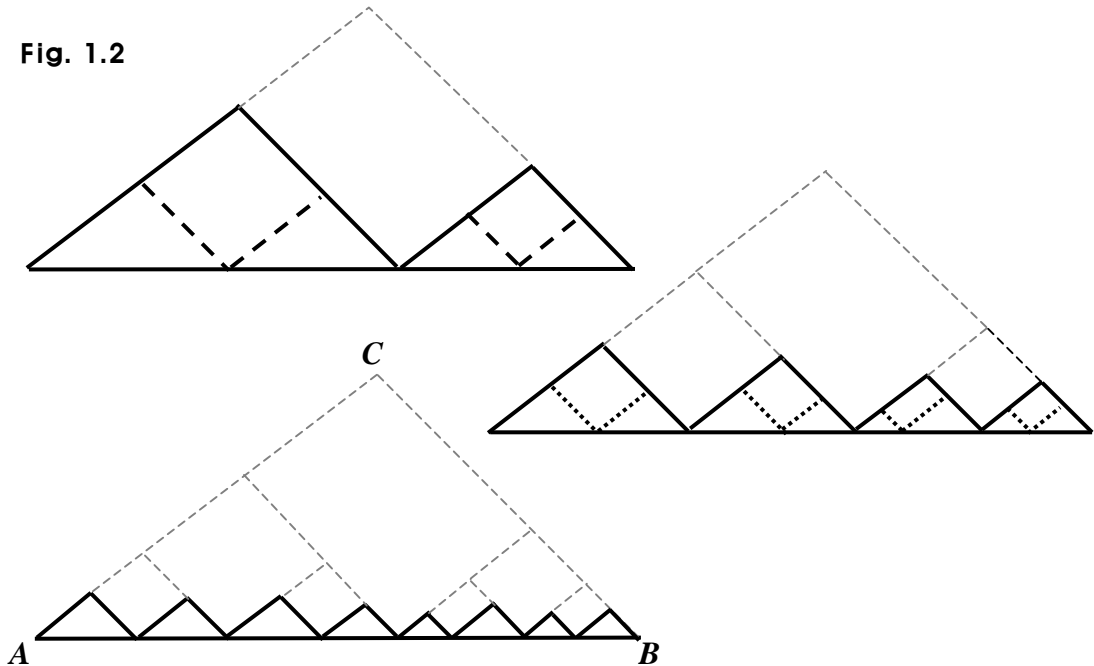
And using the fact above, we often find the solutions to problems not only in geometry but in other areas of math, too. So you may want to keep it in mind.

Sometimes though, it might seem the sum of two sides can be the same as the other side. For instance, suppose this time, we cut a triangle the way below.



Suppose next,  $DE$  is parallel to  $CF$ , and  $EF$  is parallel to  $DC$ . Then, we get  $AC + CB = AD + DE + EF + FB$ .

Suppose now, that we keep making more of the parallel line segments the way above. Then, we get this:



Then, it seems if we keep doing the same process as above, the sum of the two sides  $AC$  and  $CB$  is getting close to the length of  $AB$ . Do we then eventually get  $AC + CB = AB$ ?

It is not the case, of course. That is to say that we still get  $AC + CB > AB$ .

No matter how many times we may do the same processes as above, the sum of all the small line segments will be the same as the sum of the two sides  $AC$  and  $CB$ .

That is to say that the sum of all the small line segments will not converge to  $AB$ .

That's because like any other object in math, a triangle is an idea, too.

Nothing in math is a material object. Working with geometric objects as lines, triangles, circles, etc., we can get fooled by their looks.

If for instance, we make an actual triangle using wooden rods, and keep cutting two of the three and putting the pieces together the way above, there will be losses in the lengths of the two because of the waste as sawdust.

In math however, we won't get any loss in length or whatsoever no matter how many cuttings we may do to the triangle, because a triangle in math is an idea. And the same is true, too, for any other object as lines, circles, rectangles, etc.

And we have a property in the real number system, and the property is quite close to the triangle inequality, and is as follows.

- Assuming  $P$  and  $Q$  are real numbers, we get:  $|P| + |Q| \geq |P + Q|$ .      How?

Assuming first,  $PQ < 0$ , in other words,  $P$  and  $Q$  have opposite signs as  $P > 0$  and  $Q < 0$ , we get  $|P| + |Q| > |P + Q|$ .

For instance, if  $P = 2$ , and  $Q = -1$ , we get  $|2| + |-1| = 2 + 1 = 3 > |2 + (-1)| = |2 - 1| = 1$ .

Assuming next,  $PQ \geq 0$ , in other words,  $P$  and  $Q$  have the same sign as  $P < 0$  and  $Q < 0$ , or  $P = Q = 0$ , we get  $|P| + |Q| = |P + Q|$ .

For instance, if  $P = -2$ , and  $Q = -1$ , we get  $|-2| + |-1| = 3 = |(-2) + (-1)| = |-3| = 3$ .

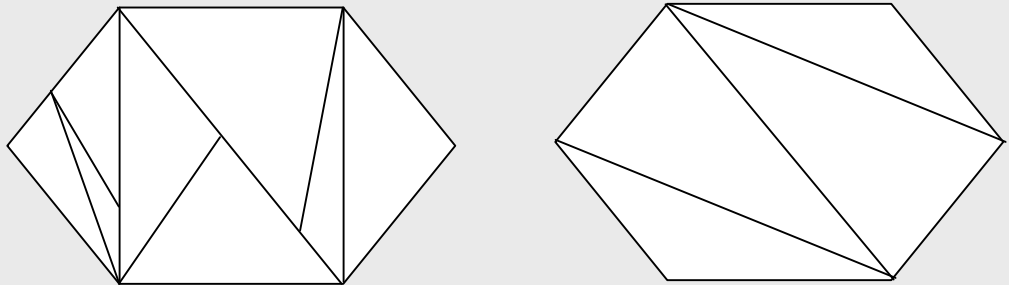
And of course, we can use the fact above solving many problems, and can use it doing algebra, too. So, together with the triangle inequality, we may want to keep in mind the property above.

## 2. Right Triangles

Triangles are like integers. And they are very much so if they are right triangles.

We use integers to form all other kinds of numbers as  $1.2$ ,  $1/3$ ,  $\sqrt{5}$ ,  $-\sqrt[4]{3}$ , etc. And the same is true for triangles, too. We can use triangles to build all kinds of structures as bridges. And we can take a polygon as a collection of triangles. That is to say that a polygon can be partitioned into many triangles.

**Fig. 2.0**



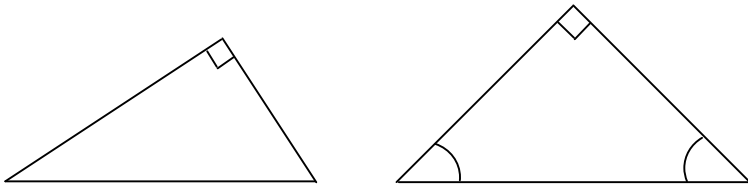
Also, we can take a triangle as a collection of many triangles called right triangles.

So a right triangle is a basic triangle, and has an angle called a right angle, which is  $90^\circ$ .

And the side facing the right angle is called the hypotenuse, and the other two sides perpendicular to each other are called the legs.

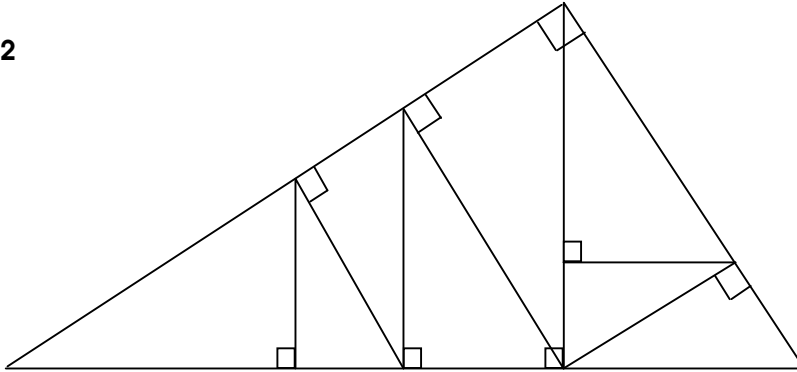
We call such a triangle a right triangle, since it has a right angle. So showing a right triangle, we want to show where the right angle is. Usually, we place a small rectangle or square at the vertex that has the right angle,  $90^\circ$ .

Fig. 2.1



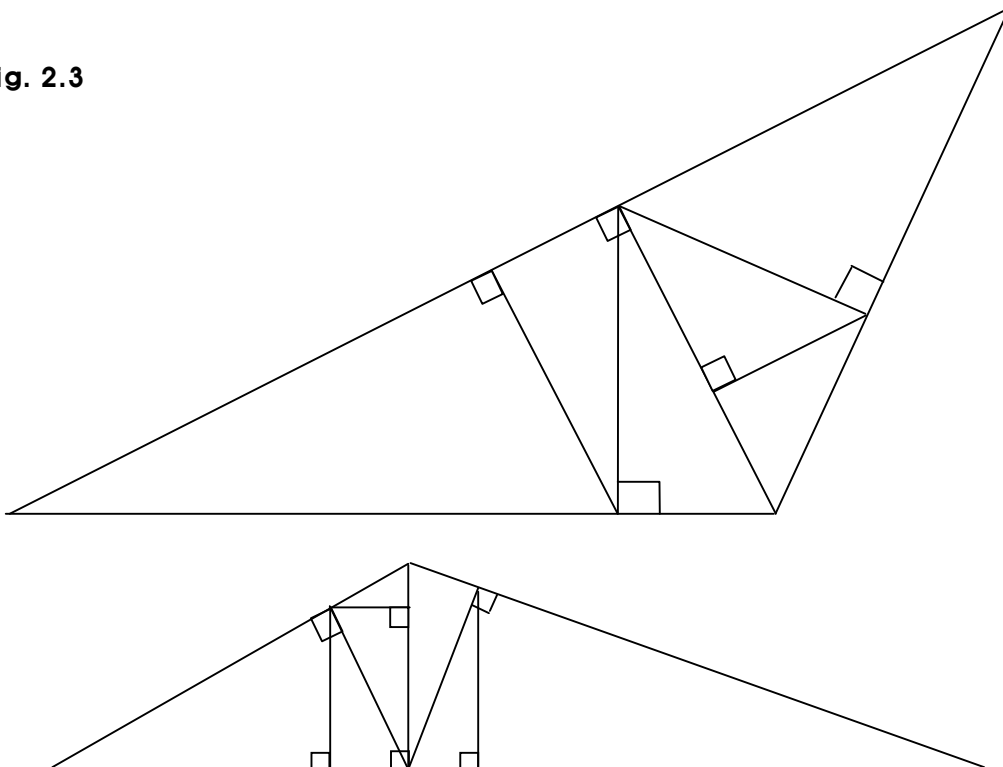
And we can partition a right triangle into many other right triangles, too.

Fig. 2.2



So we can generate as many right triangles as we want from a right triangle, and of course, construct as many right triangles as we want inside any triangle.

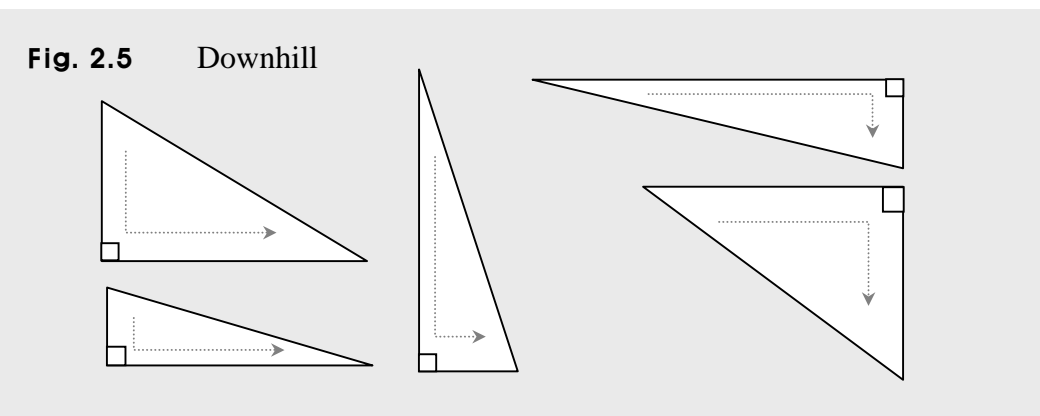
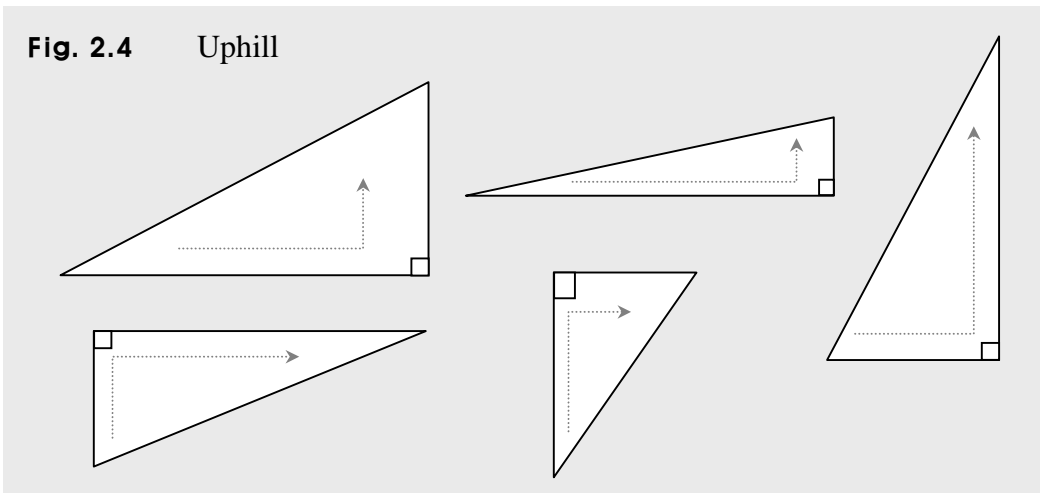
Fig. 2.3



And using a right triangle, we can define an important tool in math, and the tool is called a slope. What slope?

It indicates how a line or a line segment is slanted or inclined. That is, it shows the degree of inclination of a line or a line segment. And we can specify the degree, that is, the slope by means of the ratio between the legs of a right triangle. What ratio though?

The slope is the ratio of the vertical leg to the horizontal leg. That is to say that the ratio is the vertical side over the horizontal side, in a right triangle, of course.



So the ratio called the slope can show how the hypotenuse in a right triangle is slanted.

Thus, the slope specifies the steepness of the hypotenuse in a right triangle. What then is it for?

In the case of uphill, in each right triangle, we take the vertical side for a positive change. That is, we give a positive value to the vertical side.

In the case of downhill though, we take the vertical side for a negative change. That is, we give a negative value to the vertical side.

And in both cases, we take the horizontal side for a positive change. So we give a positive value to the horizontal side.

And we know that the slope is the vertical side over the horizontal side.

So in the case of uphill, the slope in each right triangle is positive.

In the case of downhill however, the slope in each right triangle is negative.

Thus, we can use the idea of the slope to explain how fast or slow things change as other things change. So in that case, we should call it a rate rather than just a ratio.

More specifically, we can call it *a rate of change*.

For instance, if a volume changes as time changes, we can call it a time rate of change in volume. So it shows how fast (or slow) the volume changes as time changes, and we can call it a speed. More precisely though, we call it a velocity, the magnitude of which is a speed. So it shows how fast or slow the volume increases or decreases as time increases.

And such an idea called the slope, that is, a rate of change is the fundamental idea in a very useful math called *calculus*.

And we know that the idea above is from right triangles. So right triangles are important. Thus, we need to know and use them very well.

It's not just important though. We cannot do much without it in math.  
Is that all then we can get from a right triangle?

We can get other crucial ideas from right triangles.

One of them is the distance formula, often called Pythagorean theorem, too.  
And the formula shows the fact below:

The square of the distance between the two endpoints of the hypotenuse is the sum of the square of one leg and the square of the other leg.

In short, the square of the hypotenuse is the sum of the squares of the two legs.

So assuming  $h$  is the hypotenuse,  $a$  and  $b$  are the two legs, we get  $h^2 = a^2 + b^2$ .

And some other ideas are trigonometric ratios, called trig ratios, for short.  
They are called the sine, the cosine, and the tangent.

The ratios are subject to an angle, which is between the hypotenuse and the leg called the adjacent. Why the adjacent?

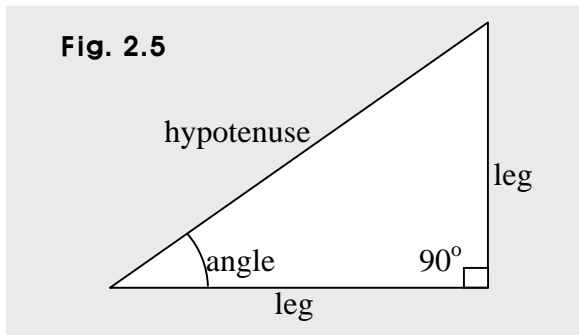
It's because the leg is adjacent to the angle. What then is the sine?

The sine is the ratio of a leg to the hypotenuse, and the leg is called the opposite, because the leg is opposite of the angle.

So if the sine is 0.5 and the hypotenuse is 3, the opposite is 1.5, which is the product of the sine and the hypotenuse, so we get  $1.5 = 0.5 \times 3$ .

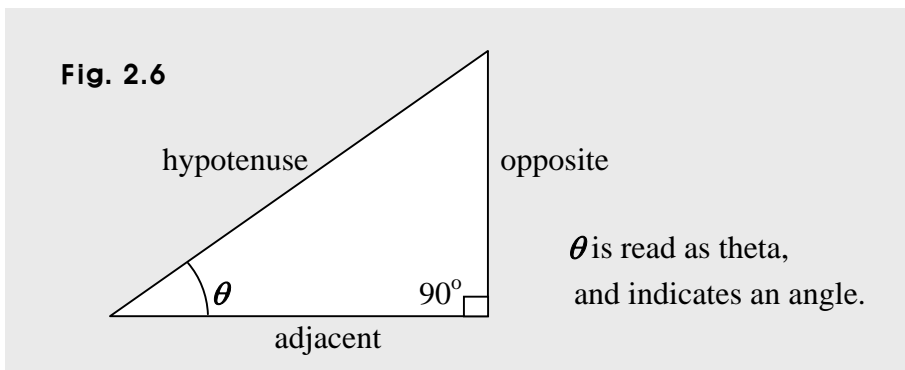
We can thus, put the leg called the opposite the way as follows.

The opposite is the product of the sine and the hypotenuse. That is, the opposite is the sine times the hypotenuse. So multiplying by the sine, we get the opposite, which is often the vertical leg, but not always. It depends on the angle we take the ratio for. So sometimes, the opposite can be the horizontal leg.



In a right triangle, an angle is  $90^\circ$ , called a right angle, and is between the two sides called legs. And showing a right angle, we use a small rectangle

And doing trigonometry, we name the sides in a right triangle the way as follows.



$\theta$  is read as theta, and indicates an angle.

As in Fig. 2.6, the side slanted and connecting the two legs is called ***the hypotenuse***,

the side adjacent to the angle  $\theta$  is called ***the adjacent*** for short,

so the angle  $\theta$  is between the hypotenuse and the adjacent, and the side opposite of the angle is just called ***the opposite***.

So what is the sine about?

The sine is about the leg, often just called ***the opposite***, which means the side opposite of the angle the hypotenuse makes with the leg called the adjacent.

The cosine is the ratio of the adjacent to the hypotenuse.

So if the cosine is 0.2, and the hypotenuse is 5, the adjacent is 1, which is the product of the cosine and the hypotenuse. So we get  $1 = 0.2 \times 5$ .

We can therefore, put the leg called the adjacent the way as follows:

The adjacent is the product of the cosine and the hypotenuse. That is, the adjacent is the cosine times the hypotenuse. So multiplying by the cosine, we get the adjacent. So what is the cosine about?

The cosine is about the leg often just called *the adjacent*.

And the tangent is the ratio of the leg called the opposite to the leg called the adjacent, which means the slope of the hypotenuse. So the tangent is the slope of the hypotenuse.

Also, the slope is often called the rise-over-run, since we often call the opposite the rise, and call the adjacent the run.

And since the tangent is the ratio of the opposite to the adjacent, we can say that the tangent is the opposite over the adjacent.

So given the tangent and the adjacent, we can get the opposite. How?

Multiplying the adjacent by the tangent, we get the opposite.

What if we are given the tangent and the opposite, and want to find the adjacent?

We can get the adjacent dividing the opposite by the tangent, or multiplying the opposite by the reciprocal of the tangent. And the reciprocal is called the cotangent.

And those trig ratios form a math area called trigonometry.

So right triangles are places where many important math tools get made.



### 3. Ratios from Right Triangles

We have another important fact about a right triangle, and the fact is as follows:

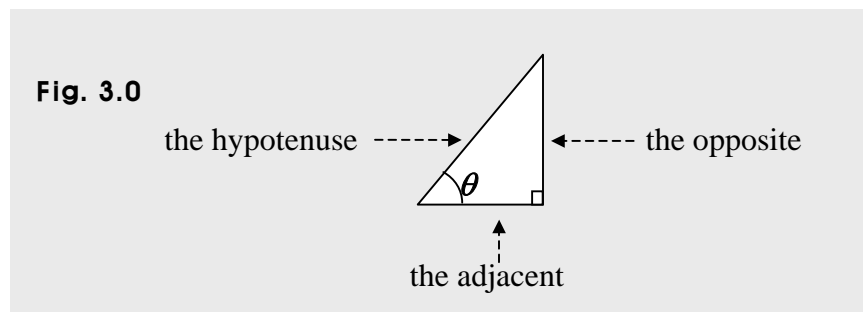
- A right triangle is the place where a special geometry called trigonometry begins.

From a right triangle, we can get another important tool called a *trigonometric ratio*, called a *trig ratio* for short. We can get in fact, six of those in kinds. Getting such a ratio, we use two of the three sides in a right triangle. So a trigonometric ratio is basically made of two sides in a right triangle. What are the two sides though?

To begin with, we use different names for the legs of a right triangle. And naming them, we consider a particular angle we refer to when getting a trig-ratio.

One of the two is called the adjacent side called briefly the adjacent, because it is adjacent to the particular angle stated above. And the other is called the opposite side called just the opposite, because it is opposite of the particular angle.

So in trigonometry, we put a right triangle the way below:



In Fig. 3.0,  $\theta$  read as theta is the particular angle in the case above. Depending on the particular angle  $\theta$ , we get a different ratio, so the angle  $\theta$  is crucial.

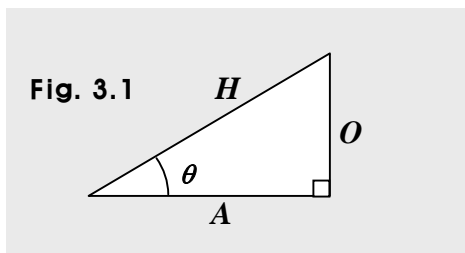
We have three basic trig ratios, each of which has its multiplicative inverse, often called the reciprocal, too. So multiplying each of the three and its reciprocal, we get 1.

We want to note however, that the inverse of a trig ratio is an angle, and that the multiplicative inverse or the reciprocal of a trig ratio is another trig ratio, which is a number. What are those three basic trig ratios, then?

Getting a trigonometric ratio, we pick two of the three sides in a right triangle, and then, take a ratio between the two. We don't just pick two sides though, and do not simply take a ratio between the two.

Two of the three sides make an important angle in a right triangle. And we may want to call the important angle the governing angle. That's because such an angle governs or determines the ratios. What sides are the two though?

Assuming first, in a right triangle,  $H$  is the hypotenuse,  $A$  is the adjacent, and  $O$  is the opposite, we can put the right triangle the way below.



Then, we call  $\theta$  (read as theta) the *governing angle*, because we have chosen the side  $A$  to be the adjacent, or we have chosen the side  $O$  to be the opposite. So the adjacent is the side adjacent to the governing angle, and the opposite is the side opposite of (or facing) the governing angle.

Thus, the two sides are the adjacent and the hypotenuse. So in a right triangle, the adjacent and the hypotenuse make the governing angle.

And in return, *the governing angle determines every trig ratio* in the right triangle.

Depending on the way we look at the right triangle though, either of the two sides other than the hypotenuse can be the adjacent. So using a wrong side as the adjacent, we get wrong ratios. Thus in short, *the wrong adjacent makes wrong ratios*.

So it is crucial to choose the adjacent correctly.

Now, we can get six trig-ratios in kind from a right triangle, and each is between two of the three sides. What then, are the six ratios?

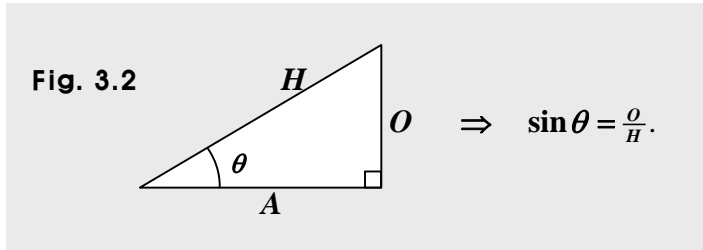
Two are between the opposite and the hypotenuse, another two are between the adjacent and the hypotenuse, and the other two are between the opposite and the adjacent.

Of the six though, three are in fact, the reciprocals of the other three, which are called therefore, three basic trig-ratios. And more specifically, of the three basic ratios:

- One is called *sine*, which is denoted by **sin**, and is the ratio of the opposite to the hypotenuse, that is, the opposite over the hypotenuse: the opposite / the hypotenuse.
- Another is called *cosine*, which is denoted by **cos**, and is the ratio of the adjacent to the hypotenuse, that is, the adjacent over the hypotenuse: the adjacent / the hypotenuse.
- And the other is called *tangent*, denoted by **tan**, which is the ratio of the opposite to the adjacent, that is, the opposite over the adjacent: the opposite / the adjacent. So the tangent can tell us the slope of the right triangle.

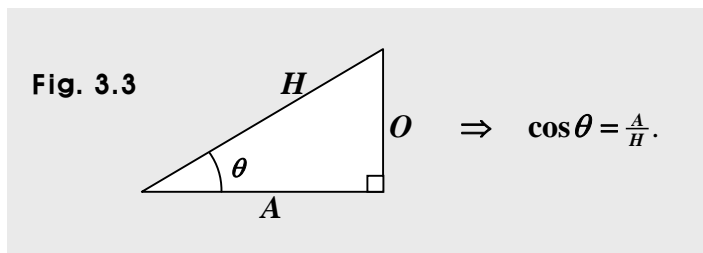
Suppose now, the governing angle is  $\theta$ , read as theta, which is the eighth letter of the Greek alphabet. Then, we put the three basic trig-ratios the way below.

- To begin with, the sine of the governing angle  $\theta$  is **sin**  $\theta$ , and is the ratio of the opposite to the hypotenuse, so **sin**  $\theta$  is the opposite over the hypotenuse.



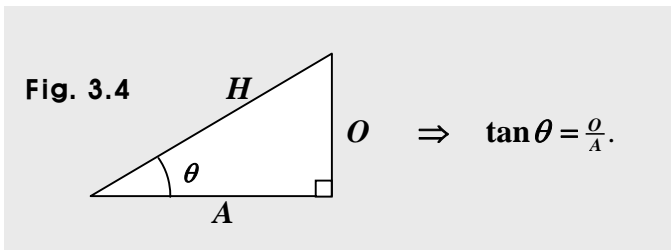
And we read **sin**  $\theta$  as the sine of  $\theta$  or just sine  $\theta$  for short. So for instance, we read **sin**  $30^\circ$  as the sine of  $30^\circ$  or just sine  $30^\circ$ , which equals **sin**  $\pi/6$ , since  $30^\circ$  is  $\pi/6$  in radians.

- Next, the cosine of the governing angle  $\theta$  is **cos**  $\theta$ , and is the ratio of the adjacent to the hypotenuse, so **cos**  $\theta$  is the adjacent over the hypotenuse.



And we read **cos**  $\theta$  as the cosine of  $\theta$  or just cosine  $\theta$  for short. So for instance, we read **cos**  $45^\circ$  as the cosine of  $45^\circ$  or just cosine  $45^\circ$ , which equals **cos**  $\pi/4$ .

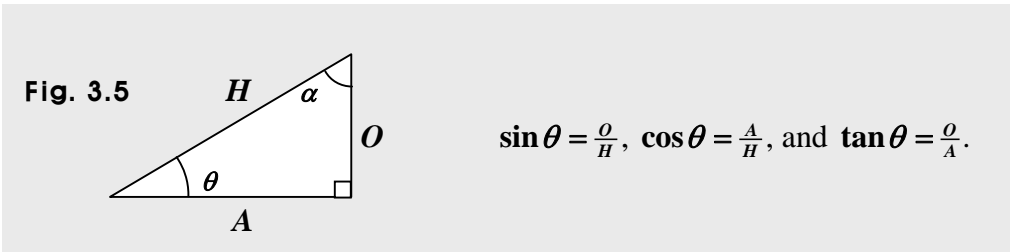
- And next, the tangent of the governing angle  $\theta$  is **tan**  $\theta$ , and is the ratio of the opposite to the adjacent, so **tan**  $\theta$  is the opposite over the adjacent.



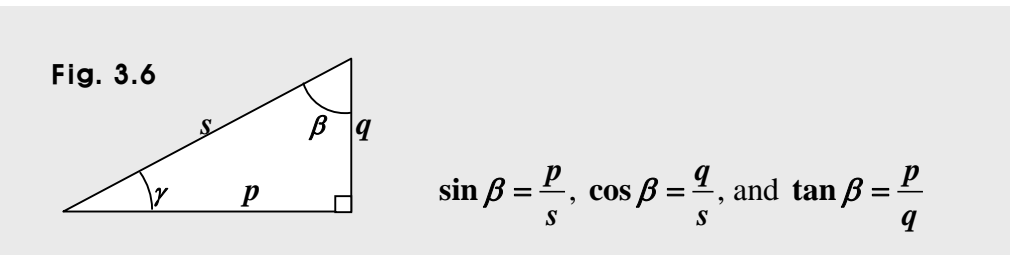
And we read **tan**  $\theta$  as: tangent of  $\theta$   
or just tangent  $\theta$  for short.

So for instance, we read **tan**  $60^\circ$  as the tangent of  $60^\circ$  or just tan of  $60^\circ$ , which is equal to **tan**  $\pi/3$ . And we can call the tangent the slope of the hypotenuse, because the tangent is the opposite over the adjacent.

Thus, putting threads together, and assuming  $A$  is the adjacent in the right triangle below, we have to use  $\theta$  as the governing angle, and can put the three basic trig ratios the way as follows.



And for another instance, assuming  $\beta$  is the governing angle in the right triangle in Fig. 3.6, we use  $q$  as the adjacent, and can put the three basic trig ratios the way as follows.

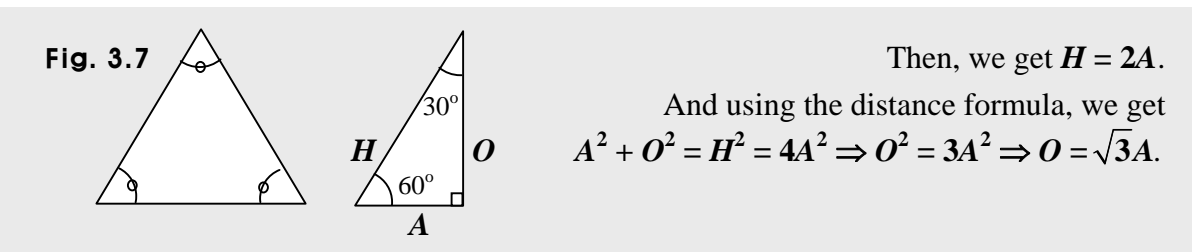


And we have three popular governing angles, which are  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$ .

We can easily get the trig ratios for those angles using two triangles isosceles.

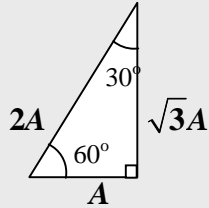
One is a regular (equilateral) triangle, where every angle is  $60^\circ$ , and the other is a right triangle isosceles, where two angles are equal, and thus, are  $45^\circ$  each.

So to begin with, cutting in half a regular triangle, we can get a right triangle as follows.



So we get this:

Fig. 3.8



Thus, we can readily get these:

$$\sin 60^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}A}{2A} = \frac{\sqrt{3}}{2}$$

$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{A}{2A} = \frac{1}{2}$$

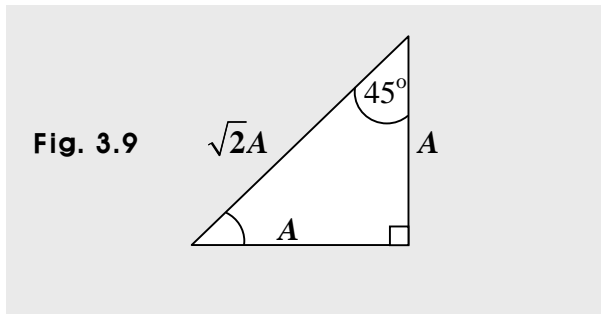
$$\cos 60^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A}{2A} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}A}{2A} = \frac{\sqrt{3}}{2}$$

$$\tan 60^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}A}{A} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\tan 30^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{A}{\sqrt{3}A} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

And next, assuming  $A$  is the adjacent of a right triangle isosceles, we can put the triangle the way below:



So we get these:

$$\sin 45^\circ = \text{the opposite / the hypotenuse} = \frac{A}{\sqrt{2}A} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \text{the adjacent / the hypotenuse} = \frac{A}{\sqrt{2}A} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \text{the opposite / the adjacent} = \frac{A}{A} = 1$$

Is it the case though, for instance, we get  $\sin 30^\circ = 1/2$  in every right triangle where the governing angle is  $30^\circ$ ?

Yes, it is. So for instance, we have  $\cos 60^\circ = 1/2$  in every right triangle where the governing angle is  $60^\circ$ . And the same is true for all the other trig-ratios, too.

And the full explanations will be covered in the book, **TRIGONOMETRY**.



## 4. Identical or Similar?

To begin with, among triangles, we have three in kinds.

One is scalene, another is isosceles, and the other is regular or equilateral.

And of the three above, two can be called symmetric.

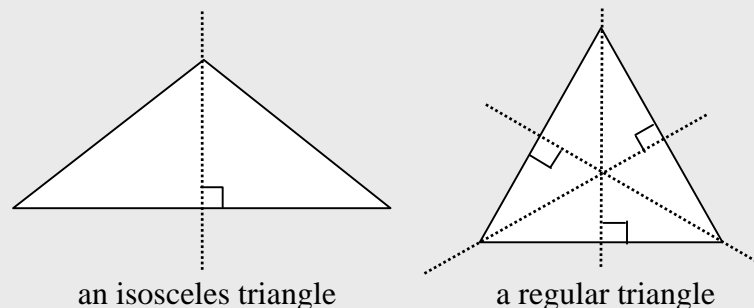
What then are the two?

The two kinds are isosceles and regular. So triangles isosceles or regular are symmetric. We know however, a regular triangle can be called an isosceles triangle, because an isosceles triangle has two sides equal, and a regular triangle has three sides equal, so anyway, two sides are equal in a regular triangle. Thus, triangles isosceles include regular triangles.

So if a triangle has two sides equal, it is isosceles, and is symmetric. And we know if a triangle is isosceles, it has two angles equal, too. So if a triangle has two angles equal, it is isosceles, and is symmetric, also. Symmetric about what though?

If a triangle is isosceles, it is symmetric about a line that passes through a vertex and is perpendicular to the side connecting the two sides equal. And we call the line the *axis of symmetry*.

**Fig. 4.0**



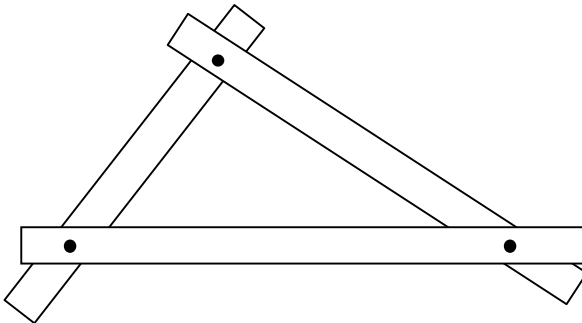
So we can see that cutting an isosceles or regular triangle along the axis of symmetry, we get two right triangles identical.

Suppose next, three line segments can make a triangle.  
Then, how many different triangles can we make using the three?

We can make one triangle only. That is to say that if two triangles share the same set of three sides, the two triangles are identical. How?

Suppose we made a triangle connecting three rods, and connecting them, we used pins so that the connections are loose, that is, no friction exists between each pin and each rod.

**Fig. 4.1**

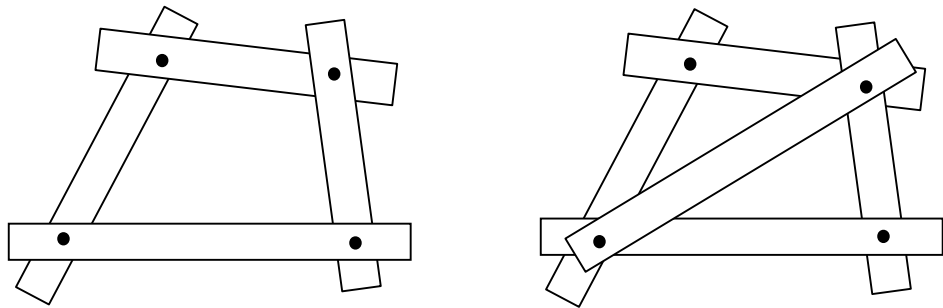


Can we then, get the triangle changed, without moving any of the positions of the pins, of course?

None of the rods can move as if they were riveted or glued together.  
In other words, we cannot get any of the rods moved once the rods have been pinned together making a triangle. So we cannot make different triangles using a set of three line segments.

What then about making a tetragon connecting four rods, using pins, of course?

Fig. 4.2



Making a tetragon like the one on the left in the figure above, we can get it changed even if we push slightly any of the rods. So we can make many different tetragons using a set of four line segments. Making however, an object like the one on the right in the figure above, we cannot get it changed if we do not move any of the positions of the pins.

That's because it is partitioned into triangles, and if three line segments can make a triangle, we can make one triangle only. That is to say that if two triangles share the same set of three sides, the two triangles are identical.

What then about using three angles, if they can make a triangle, of course? For instance, using three angles  $30^\circ$ ,  $60^\circ$ , and  $90^\circ$ , can we make only one triangle?

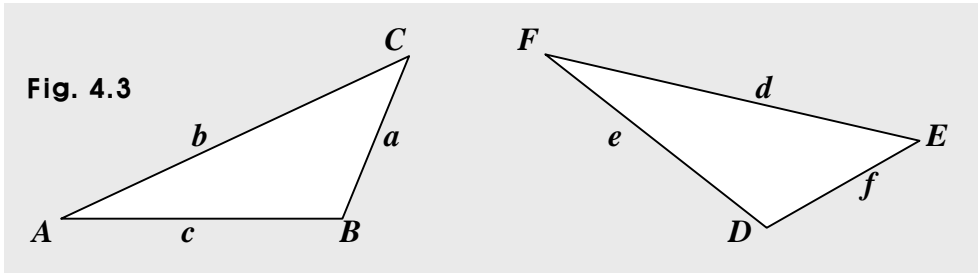
We can make as many triangles as we want. In each of the triangles, the three angles are  $30^\circ$ ,  $60^\circ$ , and  $90^\circ$ , of course.

Those triangles do not share though, the same set of three sides. And we call all those triangles similar triangles.

So similar triangles share the same set of three angles, but do not share the same set of three sides. For instance, if two triangles are similar, and one is made of  $35^\circ$ ,  $60^\circ$ , and  $85^\circ$ , the other has to be made of  $35^\circ$ ,  $60^\circ$ , and  $85^\circ$ , too. Thus, putting threads together, we can say that

- if two triangles are *identical*, they share the same set of three sides, and of course, share the same set of three angles, too,
- and that if two triangles are *similar*, they share the same set of three angles, but do *not* share the same set of three sides.

Suppose for instance, the two triangles below are identical, so  $\triangle ABC \equiv \triangle DEF$ .



Then, we get  $b = d$ ,  $c = e$ , and  $a = f$ .

And of course, we get this, too:  $\angle A = \angle F$ ,  $\angle B = \angle D$ , and  $\angle C = \angle E$ .

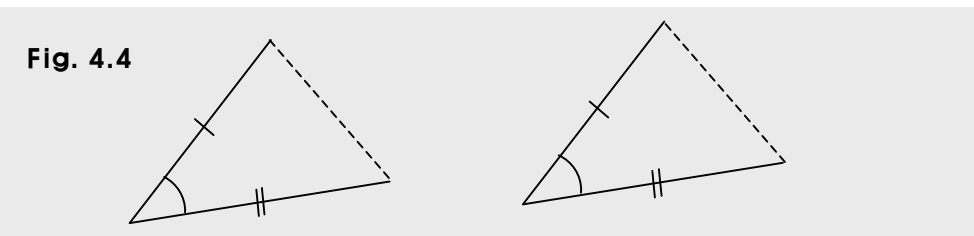
So checking to see if two triangles are the same, that is, identical, do we have to actually compare all the six sides?

Not necessarily.

If two triangles share the same set of two sides, and the angle between the two sides in one triangle is the same as the angle between the two sides in the other triangle, then the two triangles are identical.

For instance, in the two triangles  $\triangle ABC$  and  $\triangle DEF$  above, if  $b = d$ ,  $c = e$ , and  $\angle A = \angle F$ , we get  $\triangle ABC \equiv \triangle DEF$ .

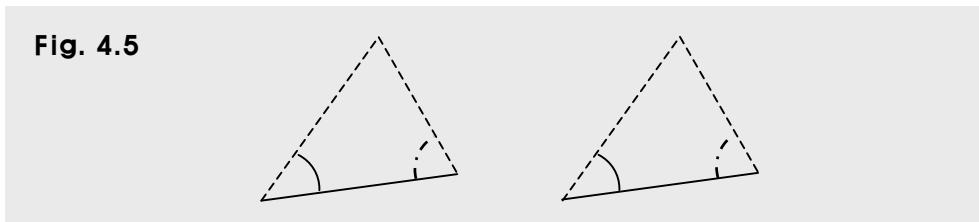
Thus for instance, the two triangles below are identical:



And we can check to see if two triangles are the same the way as follows, too:

If two triangles share the same set of two angles, and the side between the two angles in one triangle is the same as the side between the two angles in the other triangle, then the two triangles are identical.

For instance, in the two triangles  $\triangle ABC$  and  $\triangle DEF$  in Fig. 4.3 above, if  $\angle A = \angle F$ ,  $\angle B = \angle D$ , and  $c = e$ , we get  $\triangle ABC \cong \triangle DEF$ .



- And checking to see if two triangles are similar, we do not have to actually compare all the six angles.

If two triangles share the same set of two angles, they have to share the same set of three angles, and thus, are similar.

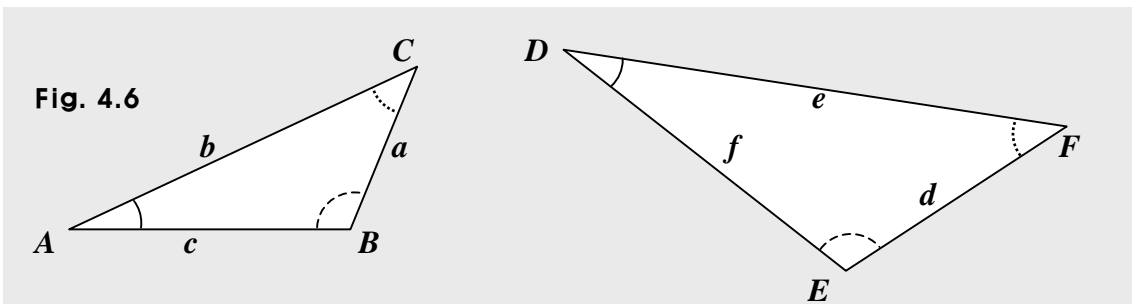
That's because the sum of the three angles in every triangle is  $180^\circ$ , which means, the sum is the same. For instance, if two angles in a triangle are  $20^\circ$  and  $60^\circ$ , and another triangle has the two angles, too, the two triangles are similar, because both triangles have to have  $100^\circ$  each, too.

- And technically, identical triangles are similar triangles, too, because identical triangles share the same set of three angles. Normally though, similar triangles are different triangles. Saying therefore, similar triangles, we mean different triangles. What do we mean by though, similar triangles?

They don't just look similar, of course. Though similar triangles are different triangles, they are not just randomly different. In fact, they have a couple of things the same.

First, they have to share the same set of three angles. And next, if two triangles are similar, the ratio between corresponding sides has to be the same. That is to say that all the three pairs of corresponding sides share the same ratio. What do we mean by though, corresponding sides?

They are the sides facing the same angle. Suppose for instance, in the two triangles below,  $\angle A = \angle D$ ,  $\angle B = \angle E$ , and  $\angle C = \angle F$ .

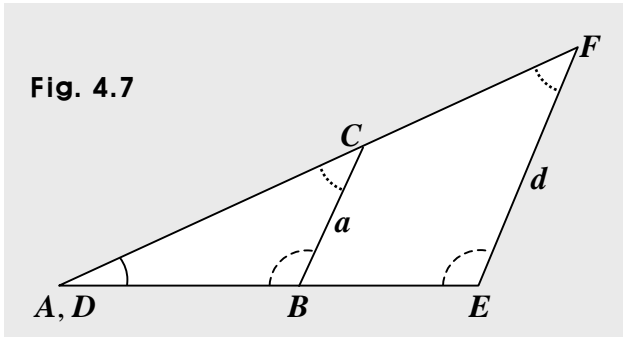


Then, we get  $\triangle ABC \approx \triangle DEF$ , which means the two triangles are similar. And we can say that  $b$  corresponds to  $e$ ,  $c$  corresponds to  $f$ , and  $a$  corresponds to  $d$ .

So we get  $b : e = c : f = a : d$ . That is to say that we get  $\frac{b}{e} = \frac{c}{f} = \frac{a}{d}$ .

So for instance, if  $b : e = 2 : 3$ , we get  $\frac{b}{e} = \frac{c}{f} = \frac{a}{d} = \frac{2}{3}$ .

In fact, we can put the two triangles  $\triangle ABC$  and  $\triangle DEF$  above the way below, too.



Then, of course,  $a$  is parallel to  $d$ .  
 So both triangles share  
 the same set of three angles.

And for instance, in the figure above, assuming  $a = 5$ ,  $AC = 10$ , and  $d = 8$ , and finding the length of  $CF$ , we can get it the way as follows.

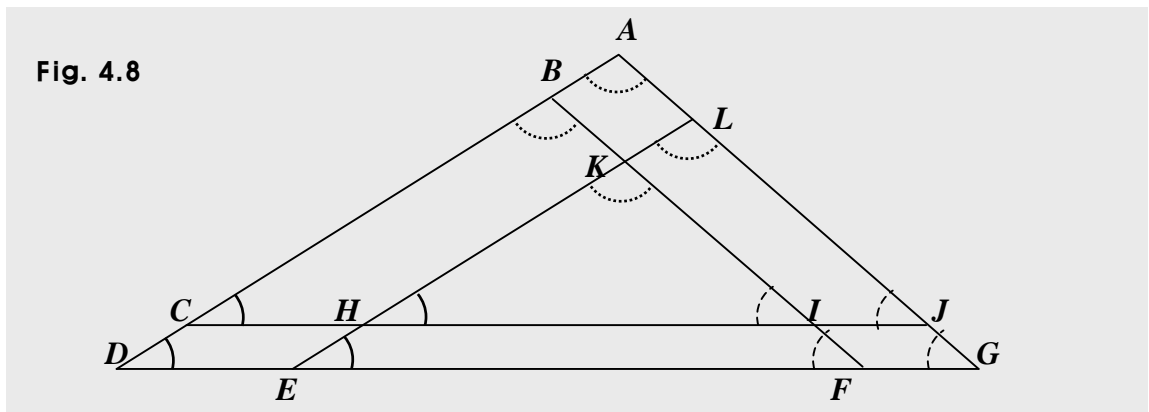
Finding  $AF$  first, we can get  $CF$ , because  $CF = AF - AC$ .  
 So finding  $AF$  now, we get

$$\frac{AC}{AF} = \frac{a}{d} \Rightarrow \frac{10}{AF} = \frac{5}{8} \Rightarrow 10 = AF \cdot \frac{5}{8} \Rightarrow AF = 10 \cdot \frac{8}{5} = 16.$$

Thus, we get  $CF = AF - AC = 6$ .

And for another instance, suppose in the figure below, we have this:

$AD \parallel LE$ ,  $AG \parallel BF$ , and  $CJ \parallel DG$ . (Note that  $AD \parallel LE$  means  $AD$  is parallel to  $LE$ .)



Then,  $\triangle ADG$ ,  $\triangle BDF$ ,  $\triangle LEG$ ,  $\triangle LHJ$ ,  $\triangle KEF$ ,  $\triangle KHI \dots$  are all similar triangles.

Thus, all the corresponding angles are the same. That is to say that we get these:

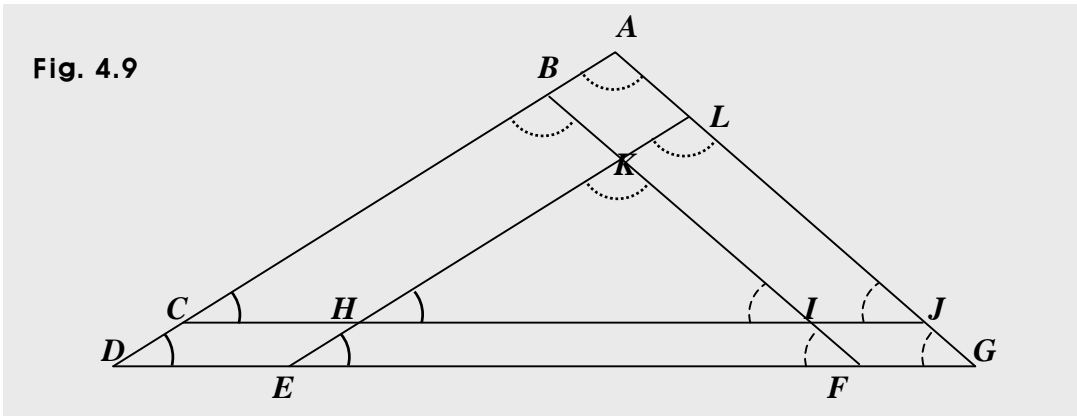
$$\angle BCH = \angle CDE = \angle HEF = \angle KHI.$$

$$\angle CBK = \angle BAL = \angle HKI = \angle KLJ.$$

$$\angle KIH = \angle IFE = \angle JGF = \angle LJI.$$

And the ratio between each pair of corresponding sides is the same, too.

So for instance, considering  $\triangle ADG$  and  $\triangle KHI$ , we can get  $\frac{KH}{AD} = \frac{HI}{DG} = \frac{KI}{AG}$



Next, looking at  $\triangle ADG$  and  $\triangle KEF$ , we can get  $\frac{AD}{KE} = \frac{DG}{EF} = \frac{AG}{KF}$ .

And next, looking at  $\triangle BCI$  and  $\triangle LEG$ , we can get  $\frac{BC}{LE} = \frac{CI}{EG} = \frac{BI}{LG}$ .

- Also, the ratio between two sides in one triangle is the same as the ratio between the corresponding two sides in another triangle similar to the one.

For instance, in the two triangles  $\triangle ADG$  and  $\triangle KHI$ , the ratio of  $AD$  to  $DG$  is the same as the ratio of  $KH$  to  $HI$ . That is, we get  $\frac{AD}{DG} = \frac{KH}{HI}$ . How?

Suppose the length of  $AD$  is twice the length of  $KH$ .

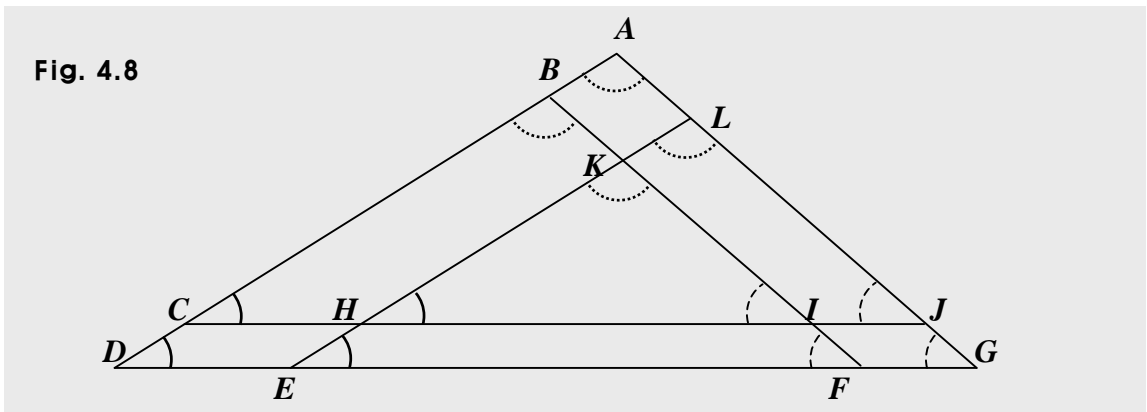
Then, since  $\triangle ADG$  and  $\triangle KHI$  are similar, the length of  $DG$  is twice the length of  $HI$ , too, and also, the length of  $AG$  is twice the length of  $KI$ .

Thus, we get  $\frac{AD}{DG} = \frac{2KH}{2HI} = \frac{KH}{HI}$ . And the same is true for the other pairs, too.

That is, we get  $\frac{AD}{AG} = \frac{KH}{KI}$  and  $\frac{AG}{GD} = \frac{KI}{IH}$ .

And of course, the same is true, too, for all the other similar triangles.

So let's, for instance, find the length of  $EG$  in the figure below assuming  $BC = 2$ ,  $CI = 3$ , and  $LE = 4$ .



Then,  $\triangle BCI$  and  $\triangle LEG$  are similar triangles.

So we can get  $\frac{BC}{CI} = \frac{LE}{EG} \Rightarrow \frac{2}{3} = \frac{4}{EG} \Rightarrow 2EG = 12 \Rightarrow EG = 6$ .



## 5. Areas of Triangles

Taking the magnitude of a line segment, we take the length of it.

And taking the magnitude of a triangle, we take the area of it.

How then can we take the area of a triangle?

Taking, for instance, the area of a rectangle, we measure first, its two dimensions. And we call the two dimensions the length and the width or the base and the height. And then, taking the product of the two, we get the area. So given its dimensions, we can get the area taking the product of the dimensions. What then about the area of a triangle?

We can make a polygon putting together triangles, and can make a rectangle putting together two same right triangles. What then, are the two legs in each right triangle?

The two legs are the base and the height, that is, the two dimensions of the rectangle.

And we know that the area of the rectangle is the product of its two dimensions. So taking the half of the product, we get the area of the right triangle.

Thus, taking the area of a right triangle, we take the product of the two legs, and then, divide it by 2, that is, multiply it by one half,  $1/2$ .

What if however, the triangle is not a right triangle?

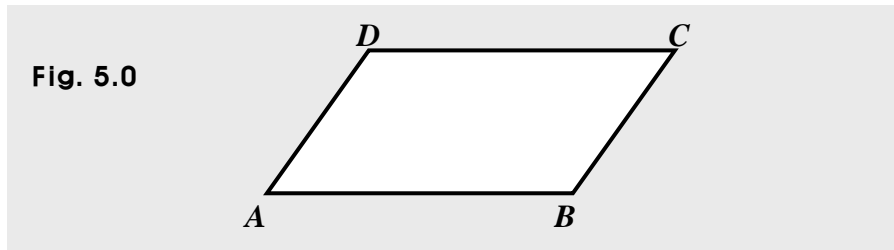
We can get the area of such a triangle, too, using the way we used above. This time though, we use the way we use getting the area of a parallelogram.

Taking the area of a parallelogram, we measure first, its two dimensions, which are the base and the height. And then, taking the product of the two, we get the area. So given its base and height, we can get the area taking the product of the base and the height.

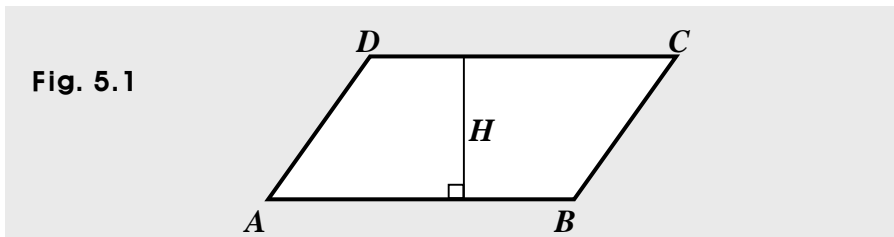
What do we mean by though, the base and the height in a parallelogram?

We can in fact, take as the base any of the four sides. Then, the height is the distance between the base and the side facing the base.

So for instance, putting a parallelogram  $ABCD$  the way below, we can take the side  $AB$  as the base, and can take as the height the distance between the two sides  $AB$  and  $CD$ .



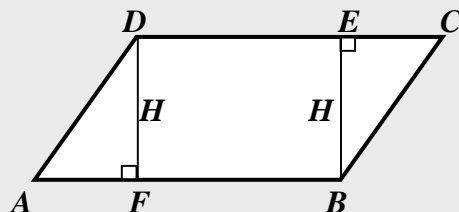
And indicating a parallelogram, we often use a symbol, which is a small parallelogram. So calling the height is  $H$ , we can put  $\square ABCD$  the way below.



How then is the product of the base and the height the area?

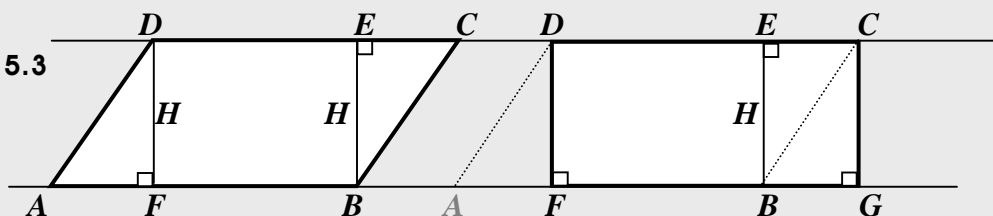
We can partition the parallelogram above so that we get a rectangle and two same right triangles the way below.

Fig. 5.2



Then, moving the right triangle on the left next to the one on the right, we get a rectangle.

Fig. 5.3



Then, we get  $AB = FG$ .

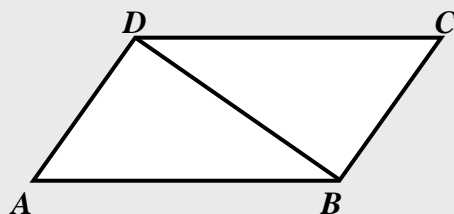
So the area of  $\square ABCD$  is the same as the area of the rectangle  $FGCD$ .

Thus, the area of  $\square ABCD$  is the product of the base  $AB$  and the height  $H$ .

What then about the area of a triangle?

We can make a parallelogram putting together two same triangles.

Fig. 5.4

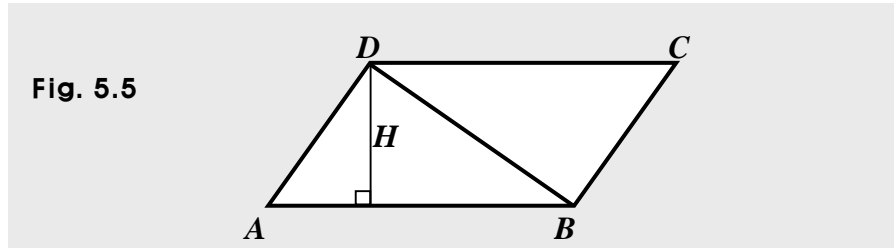


The two triangles  $ABD$  and  $CDB$  are the same. That is, we have  $\triangle ABD \cong \triangle CDB$ .

So the area of  $\square ABCD$  is twice the area of  $\triangle ABD$ .

And we know that the area of a parallelogram is the product of the base and the height, and the height in the case above is the distance between  $AB$  and  $CD$ .

So taking the half of the product, we get the area of the triangle. And assuming the height is  $H$ , we can put  $\square ABCD$  the way below.



We know  $D$  is the vertex facing the side  $AB$  in  $\triangle ABD$ . So taking the area of a triangle, we take the product of one of its side and the distance from the side to the vertex facing the side, and then, divide the product by 2, that is, multiply it by one half,  $1/2$ .

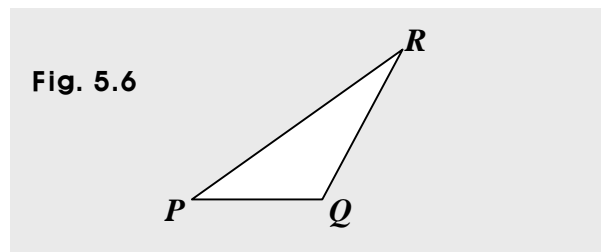
And the side chosen is called the *base* of the triangle, and the distance stated above is called the *height* of the triangle.

Thus, when taking the area of a triangle, we need to choose a side and take the side as the base, and we need to take the distance from the base to the vertex facing the base, and take the distance as the height.

And then, taking the product of the base and the height, and dividing the product by 2, we get the area of the triangle.

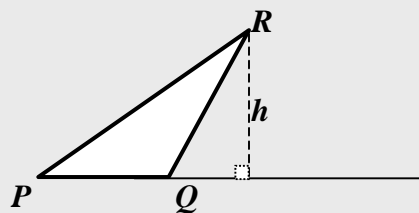
So in short, *the area of a triangle is half the base times the height.*

What if we want to take  $PQ$  as the base in the triangle  $PQR$  below?



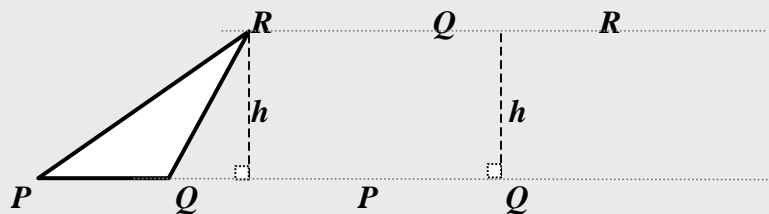
We know that the height is the distance from the base to the vertex facing the base. So assuming  $h$  is the height, we can take the height the way below:

Fig. 5.7



And putting together two of the same triangles as the one above, we can get this:

Fig. 5.8



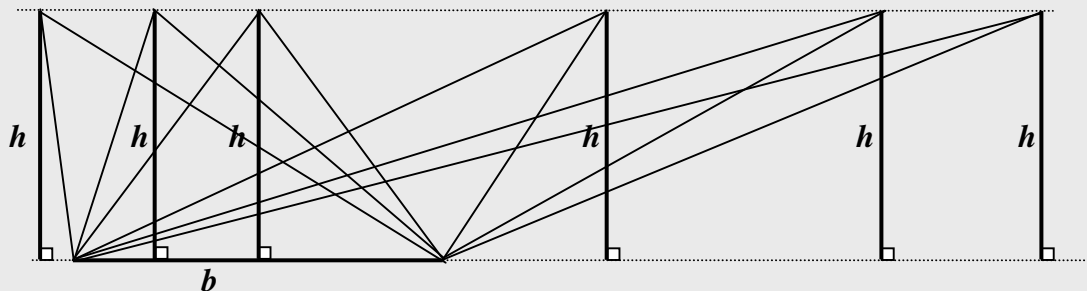
Thus, assuming the height is  $h$ , the base  $PQ$  is  $b$ , and the area is  $A$ ,

we get  $A = \frac{bh}{2}$ .

Can we have though, different triangles sharing the same base and the same height?

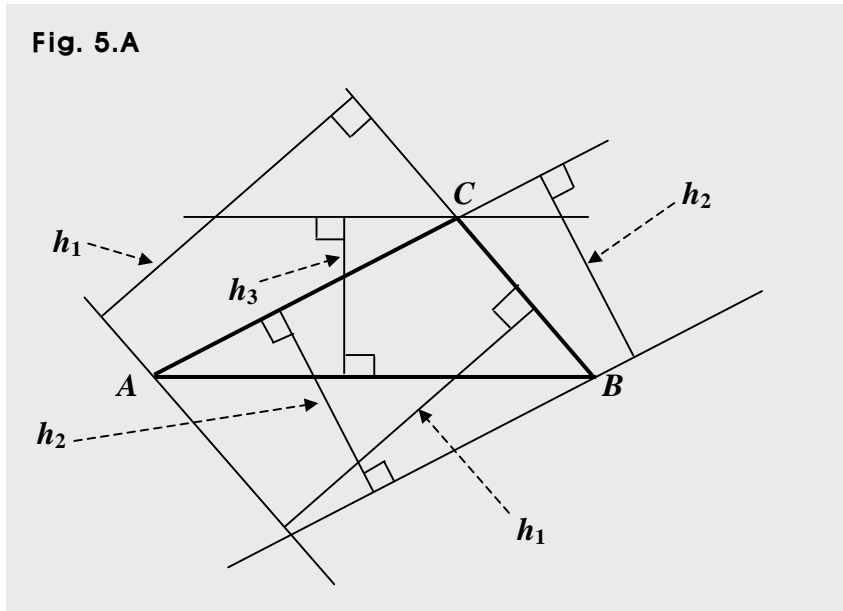
Yes, we can. And we can say that different triangles sharing the same base and the same height have the same area. So for instance, all the triangles below have the same area.

Fig. 5.9



And we often use a symbol  $\perp$  to indicate objects are perpendicular to each other. So in the figure above, we have this:  $b \perp h$ .

And taking the area of a triangle, we can take as the base any side of the triangle. Then, we can take as the height the distance from the base to the vertex facing the base. So for instance, we can take the base and its corresponding height any of the ways as follows:



In the figure above, we have  $AB \perp h_3$ ,  $BC \perp h_1$ , and  $CA \perp h_2$ . That is,

The height  $h_3$  corresponds to the base  $AB$ .

The base  $BC$  corresponds to the height  $h_1$ .

And the height  $h_2$  corresponds to the base  $CA$ .

So assuming  $AB = c$ ,  $BC = a$ ,  $CA = b$ , and  $S$  is the area of  $\triangle ABC$  above, we can put  $S$  the way below.

$$S = \frac{ah_1}{2} = \frac{bh_2}{2} = \frac{ch_3}{2}.$$

## 6. The Distance Formula

What is the distance formula?

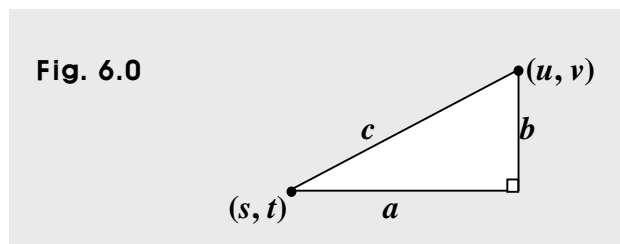
We can use it to find the distance between two points.

And the distance is the length of the line segment connecting the two points, which are placed in a coordinate plane as the  $x$ - $y$  plane or in a coordinate space as the  $x$ - $y$ - $z$  space.

And the formula is from a right triangle. For instance, assuming  $a$  and  $b$  are the two legs in a right triangle, and  $c$  is the hypotenuse, and putting the three sides in the distance formula, we get  $a^2 + b^2 = c^2$ .

How though, is the formula for the distance between two points?

Putting the right triangle above in the  $x$ - $y$  plane, and assuming a point  $(u, v)$  is an end point of the hypotenuse, and another point  $(s, t)$  is at the other end, we can put the right triangle and the two points the way below:



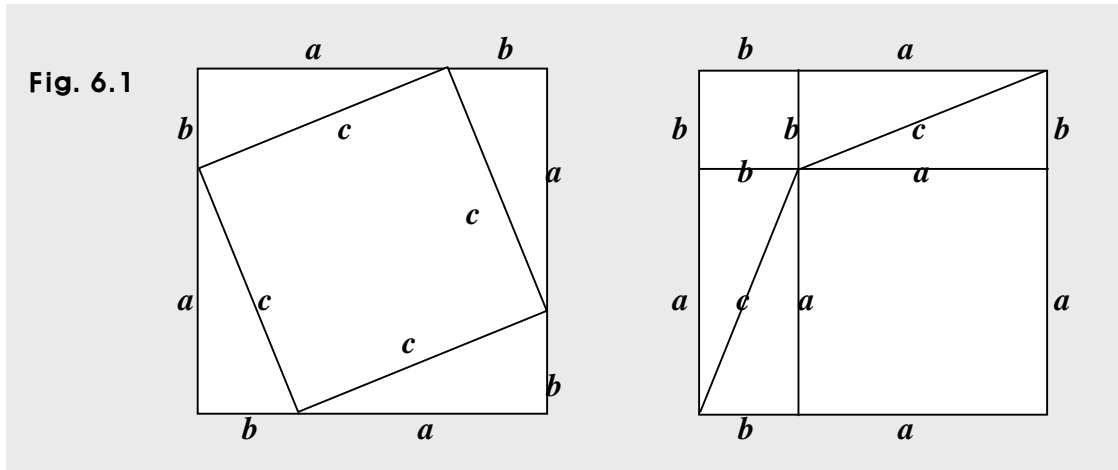
Then, the distance is the hypotenuse  $c$ .

And we often call it Pythagorean theorem, too.

The distance formula is one of the tools the most frequently used when we do math. So we cannot do much without it doing math.

How do we know if it is the case though?

We can partition two same squares the way below:



Each of the two squares has 4 right triangles, in each of which,  $a$  and  $b$  are the two legs, and  $c$  is the hypotenuse. And the square on the left has a smaller square  $c$  by  $c$ , and the square on the right has two smaller squares, one is  $b$  by  $b$ , and the other is  $a$  by  $a$ .

So the area of the square  $c$  by  $c$  is the same as the sum of the areas of the two squares, one of which is  $b$  by  $b$ , and the other is  $a$  by  $a$ . Thus, we get  $a^2 + b^2 = c^2$ .

And we can put the idea above this way, too:

The area of the large square is  $(a + b)$  by  $(a + b)$ , and thus, is  $(a + b)^2$ .

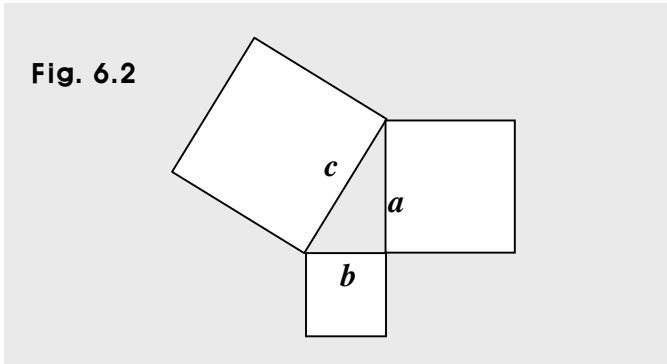
And the sum of the areas of the four right triangles is  $4 \cdot (ab/2) = 2ab$ .

So subtracting the sum from the area of the large square, we get the area of the smaller square, which is  $c$  by  $c$ , that is,  $c^2$ .

Thus, we get  $(a + b)^2 - 2ab = a^2 + 2ab + b^2 - 2ab = a^2 + b^2 = c^2$ .

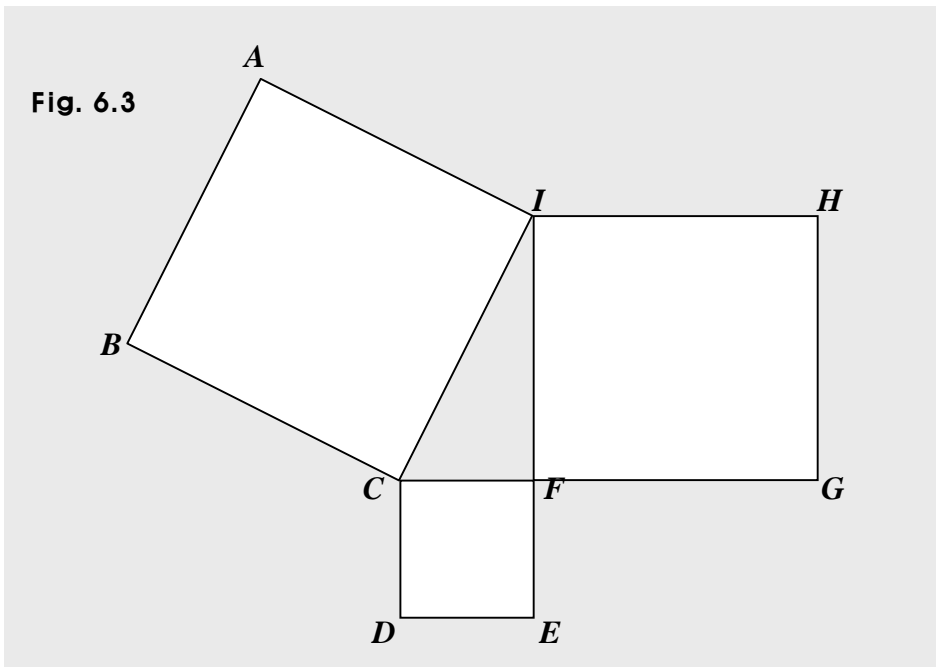
And we can put the same idea the way below, too.

To begin with, we can put three squares and a right triangle the way below.

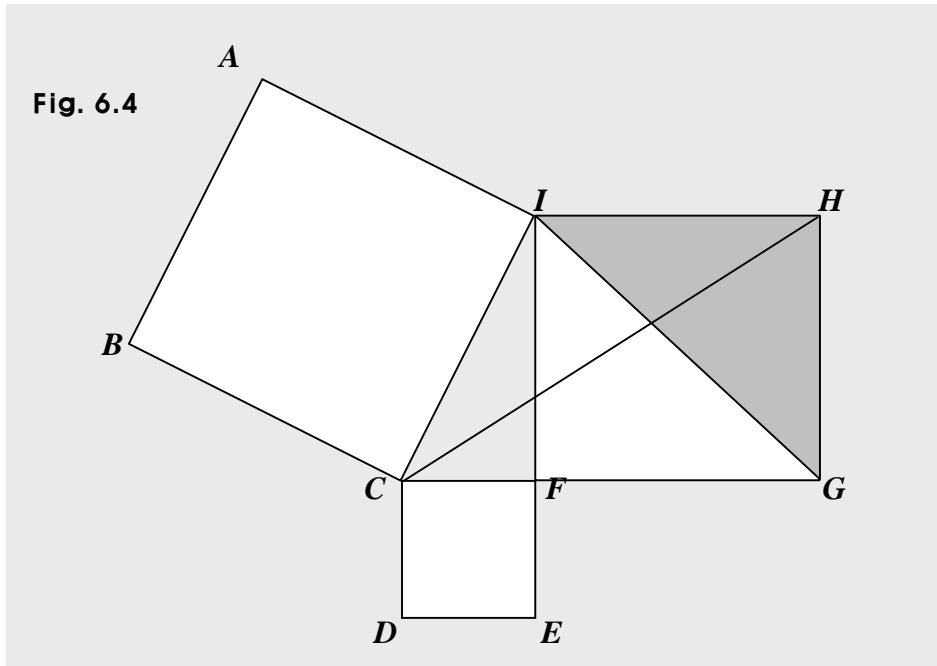


Suppose next, the square  $c$  by  $c$  is a square  $ABCI$ , the square  $a$  by  $a$  is a square  $IFGH$ , and the square  $b$  by  $b$  is a square  $DEFC$ .

Then, we want to show that the area of the square  $ABCI$  is the same as the sum of the areas of the two squares  $CDEF$  and  $IFGH$ .

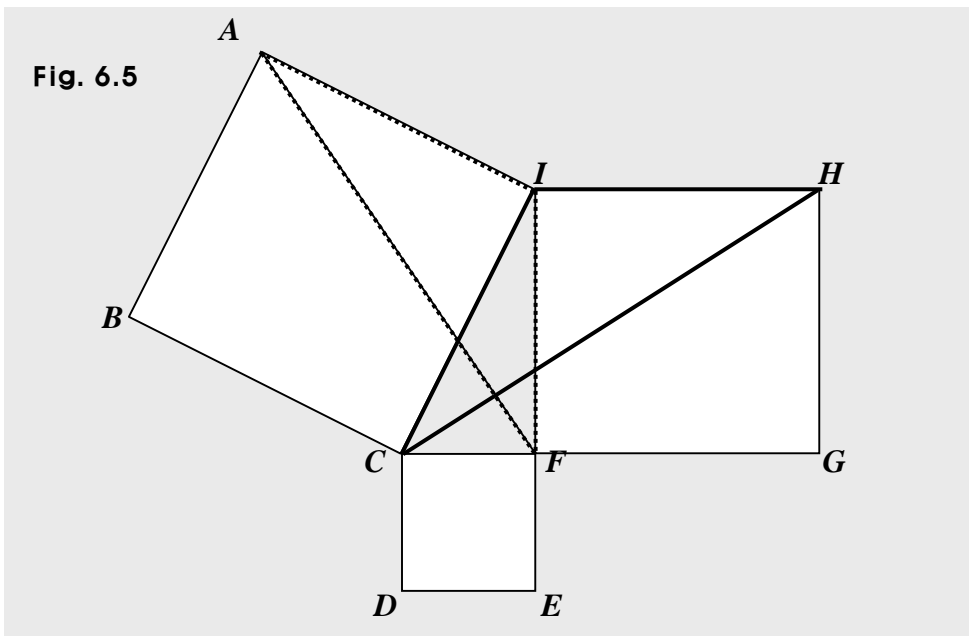


First, we can say that the area of  $\triangle ICH$  is the same as the area of  $\triangle IGH$ , which is the half the area of the square  $IFGH$ .



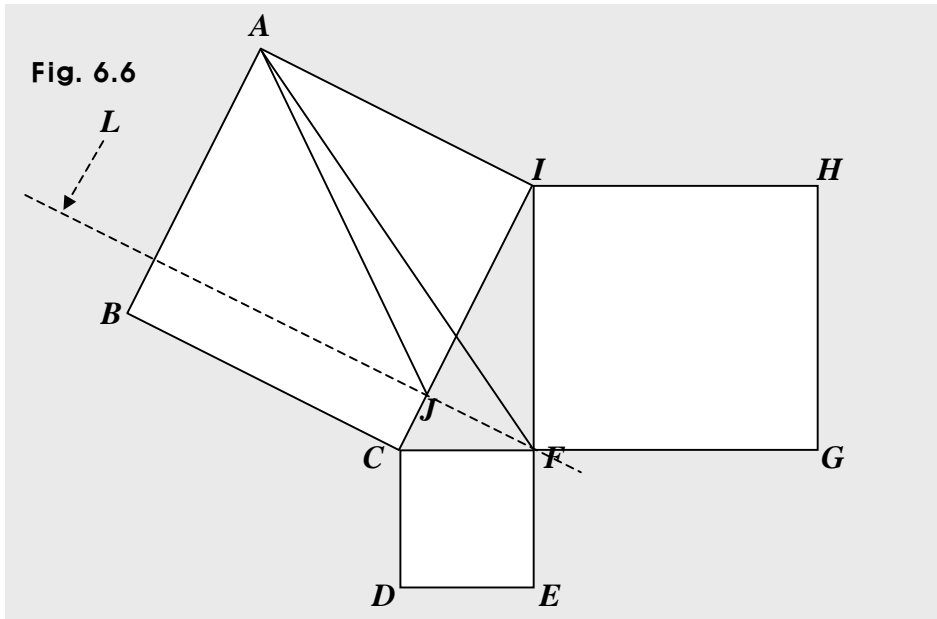
That's because assuming  $IH$  is the base, we can say that  $HG$  is the height of  $\triangle ICH$ , and also, is the height of  $\triangle IGH$ .

Next, we can say that  $\triangle ICH$  is the same as  $\triangle IAF$ . That is, both are identical.

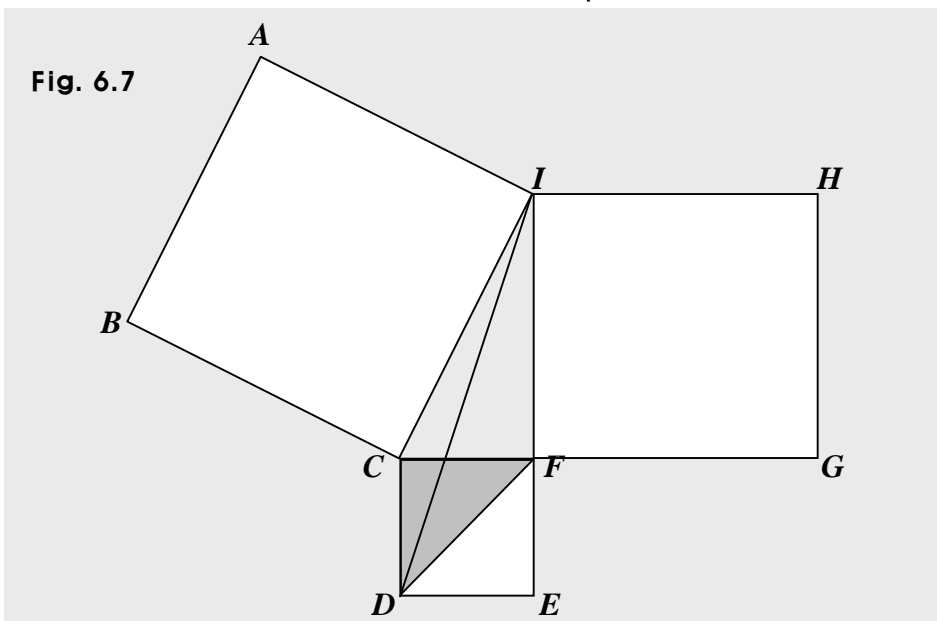


That's because  $IH$  is the same as  $IF$ ,  $IC$  is the same as  $IA$ , and  $\angle HIC$  is the same as  $\angle AIF$ , and thus,  $AF$  is the same as  $CH$ . So we get  $\triangle ICH \equiv \triangle IAF$ .

Next, assuming a line  $L$  is parallel to the side  $AI$ , meets the side  $IC$  at  $J$ , and passes through the vertex  $F$ , we can say that the area of  $\triangle IAF$  is the same as the area of  $\triangle IAJ$ .

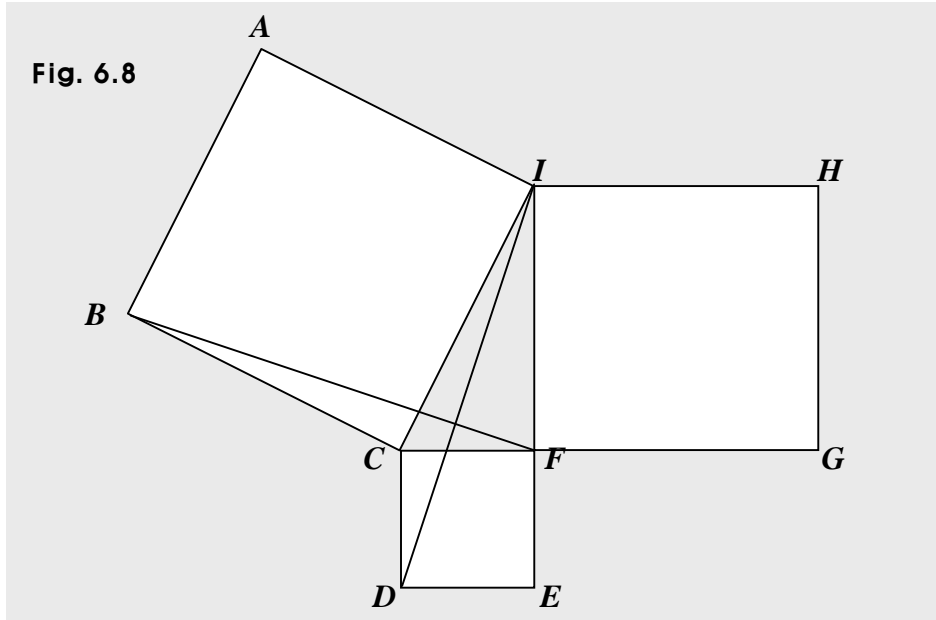


It's because assuming  $IA$  is the base, we can say  $IJ$  is the height of both  $\triangle IAF$  and  $\triangle IAJ$ . Next, we can say that the area of  $\triangle ICD$  is the same as the area of  $\triangle FCD$ , which is the half the area of the square  $CDEF$ .



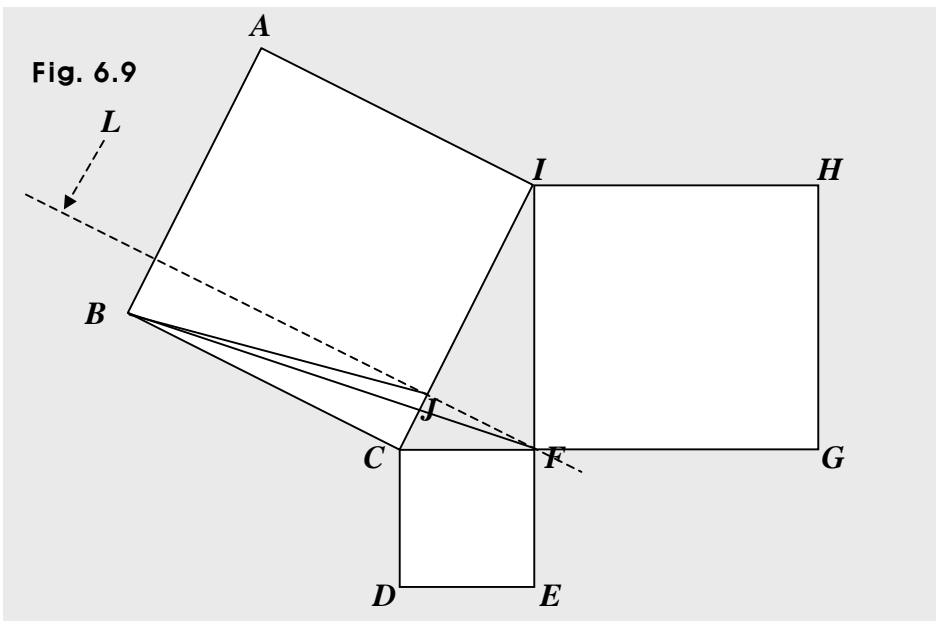
That's because assuming  $CD$  is the base, we can say that  $CF$  is the height of  $\triangle ICD$ , and is the height of  $\triangle FCD$ , too.

Next, we can say that  $\triangle ICD$  is the same as  $\triangle BCF$ , and thus, both are identical.

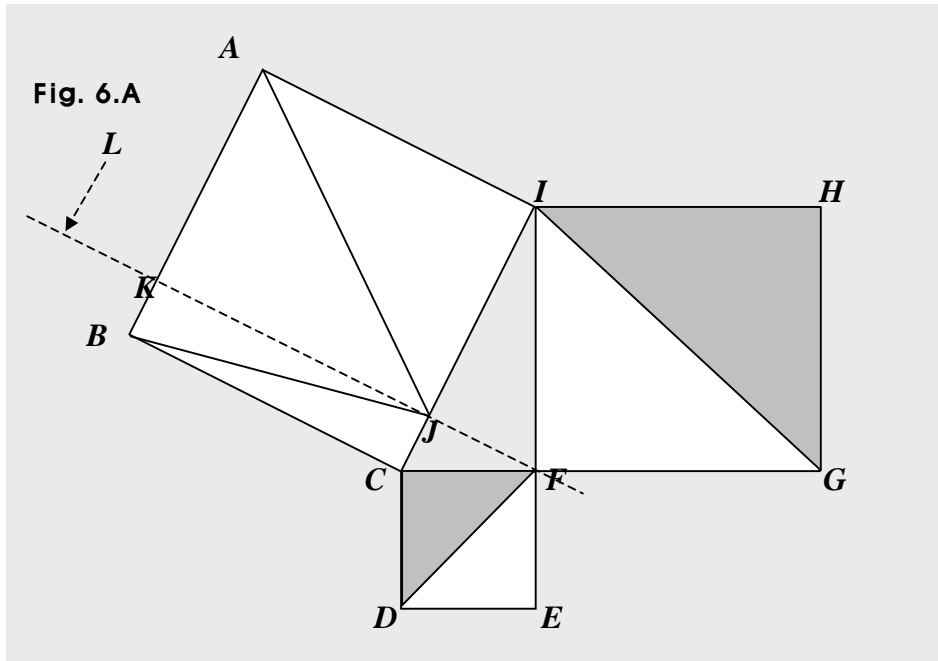


That's because  $IC$  is the same as  $BC$ ,  $CD$  is the same as  $CF$ , and  $\angle BCF$  is the same as  $\angle ICD$ , and thus,  $BF$  is the same as  $ID$ . So we get  $\triangle ICD \cong \triangle BCF$ .

Next, the line  $L$  is parallel to the side  $AI$ , meets the side  $IC$  at  $J$ , and passes through the vertex  $F$ . So the area of  $\triangle BCF$  is the same as the area of  $\triangle BCJ$ .



That's because assuming  $BC$  is the base, we can say that  $JC$  is the height of  $\triangle BCF$ , and is the height of  $\triangle BCJ$ , too. Now, putting threads together, we have



Then, to begin with, we know that the area of  $\triangle IAJ$  is the same as the area of  $\triangle IGH$ , which is half the area of the square  $IFGH$ .

And assuming the line  $L$  meets the side  $AB$  at  $K$ , we can say that the area of  $\triangle IAJ$  is half the area of the rectangle  $AKJI$ .

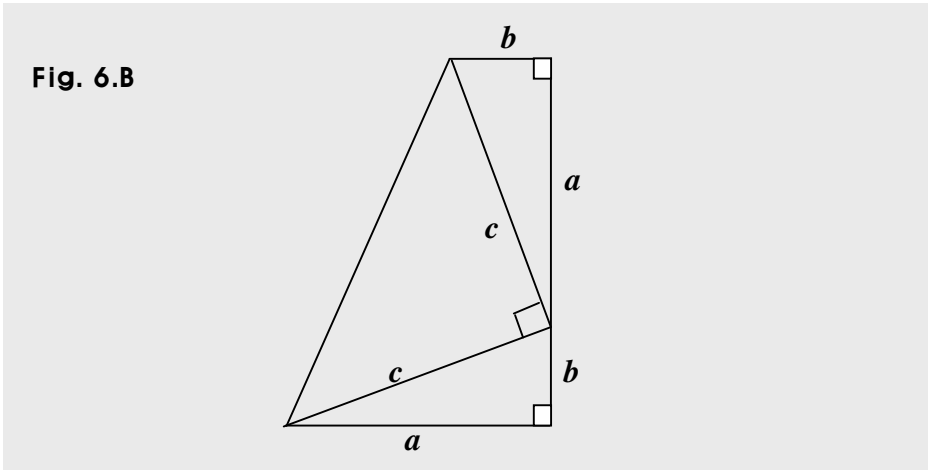
And next, we know that the area of  $\triangle BCJ$  is the same as the area of  $\triangle FCD$ , which is half the area of the square  $CDEF$ .

And since the line  $L$  meets the side  $AB$  at  $K$ , we can say that the area of  $\triangle BCJ$  is half the area of the rectangle  $KBCJ$ .

So we can say that the area of the square  $ABCI$  is the same as the sum of the areas of the two squares  $CDEF$  and  $IFGH$ .

And of course, there can be many other ways we can show the formula can hold.

And in fact, the former US President, James A. Garfield (the 20th) showed that it can hold, too. He used the idea of the area of a trapezoid.



The trapezoid above is made of three right triangles. One is  $c$  by  $c$ , and the other two are  $a$  by  $b$  each. So the area of the trapezoid is the same as the sum of the areas of the three right triangles.

To begin with, the area of the trapezoid is half the product of the height and the sum of the two sides parallel to each other.

And the height is the distance between the two sides parallel to each other. Of the two sides parallel, the lower is often called the base, and the upper is called the top.

So in sum, the area of a trapezoid is half the product of the height and the sum of the base and the top.

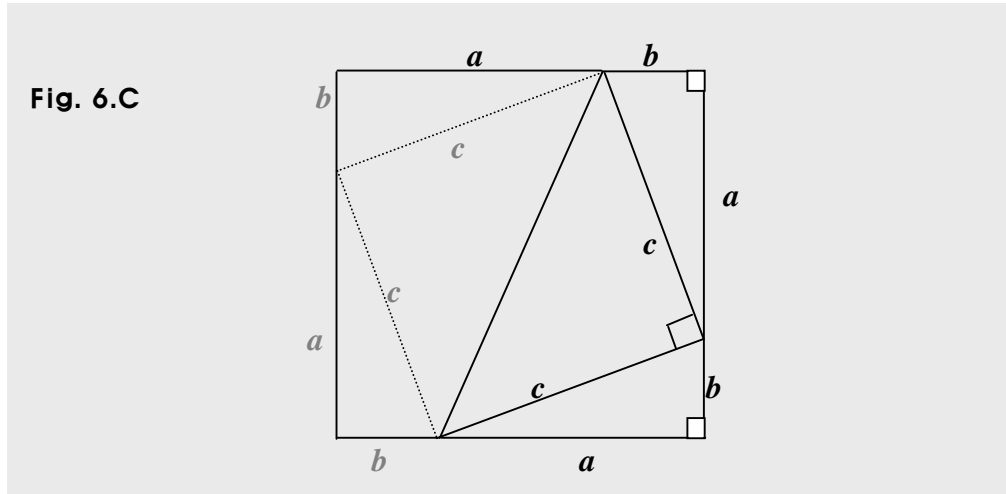
Thus, the area of the trapezoid above is  $\frac{(a+b)(a+b)}{2}$ , which is  $\frac{(a+b)^2}{2}$ , because the sum of the two sides parallel is  $a + b$ , and the height is  $a + b$ , too.

Next, the sum of the areas of the three right triangles is  $\frac{c^2}{2} + \frac{ab}{2} + \frac{ab}{2} = \frac{c^2}{2} + ab$ .

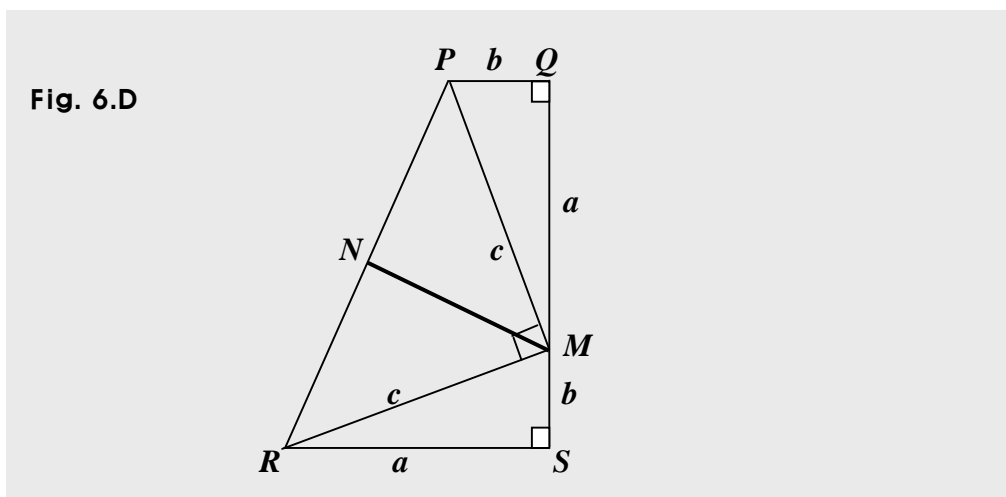
Thus, we need to have:  $\frac{(a+b)^2}{2} = \frac{c^2}{2} + ab$ .

So we get  $(a + b)^2 = c^2 + 2ab$ . And we have  $(a + b)^2 = a^2 + 2ab + b^2$ .

Thus, we get  $a^2 + 2ab + b^2 = c^2 + 2ab \Rightarrow a^2 + b^2 = c^2$ , which is the formula. And in Fig. 6.c, we can see the reason that the area of a trapezoid is half the product of the height and the sum of the base and the top. Putting together two identical trapezoids as the one above, we get a square,  $a + b$  by  $a + b$ , and is as follows.



And the area of the square above is  $a + b$  by  $a + b$ , and thus, is  $(a + b)^2$ . So the area of the trapezoid is half the area above, and thus, is  $(a + b)^2/2$ . And we can notice that we can make two identical quadrangles applying one cut to the trapezoid above. Cutting along a line perpendicular to the side on the left in the trapezoid below and passing through  $M$ , we get this:



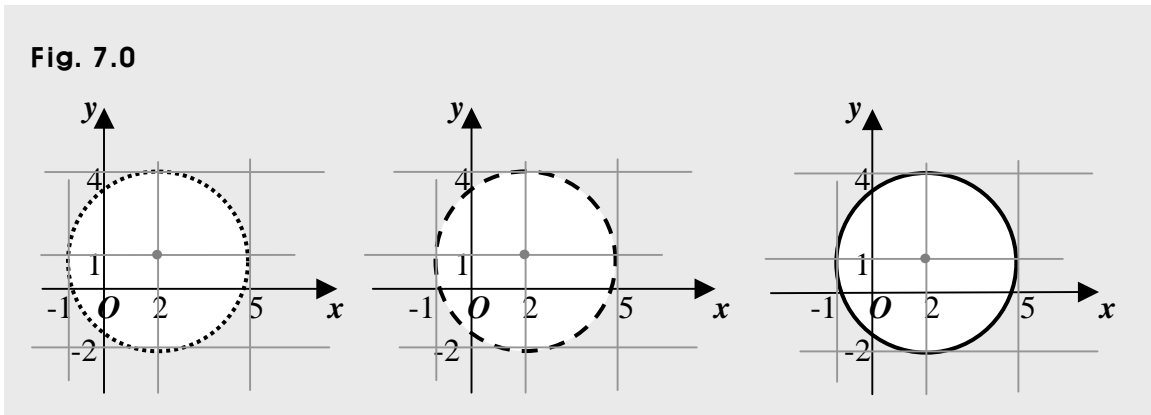
Then, we can see that the quadrangle  $PQMN$  is identical to the quadrangle  $MSRN$ .



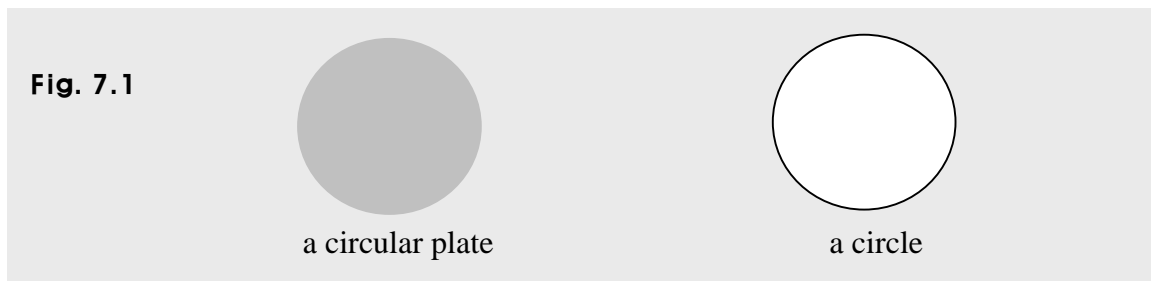
# 7. What is a circle?

A circle is in a plane, and is a collection of all the points that share an equal distance called the radius from a particular point called the center. And showing a circle in math, we usually put it in a coordinate plane as the  $x$ - $y$  plane.

So for instance, collecting all points that are 3 cm away from a point (1, 2) in the  $x$ - $y$  plane, we get a circle as below, so its radius is 3 cm, and its center is (1, 2).



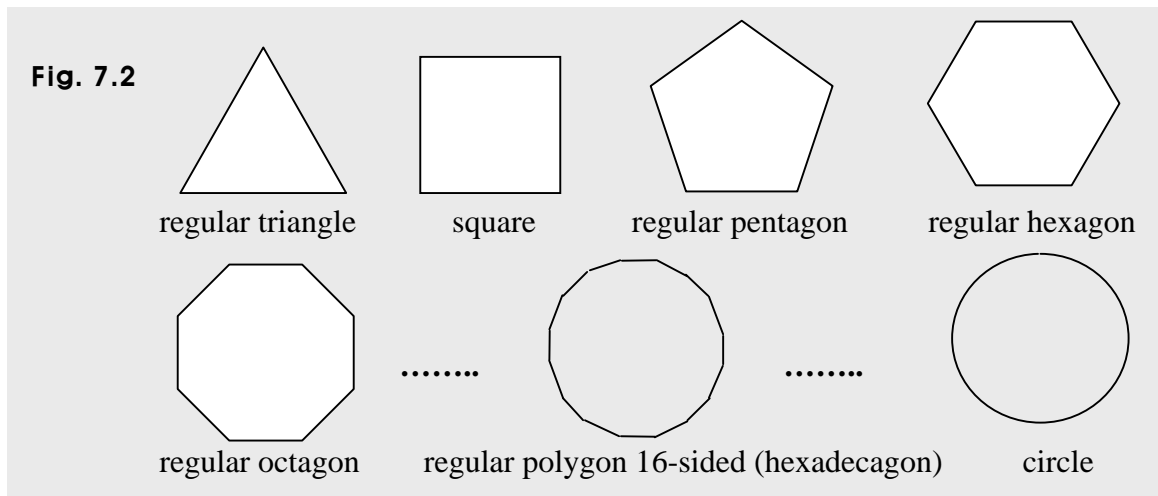
And as in the case of a triangle, we want to note that a circle is an idea and not a material object, and that saying just a circle, we mean a closed line segment and not a circular plate, which is full of points, and thus, is a plate. So *nothing* inside a circle is a part of the circle, which is therefore, a closed line segment only, and is empty inside.



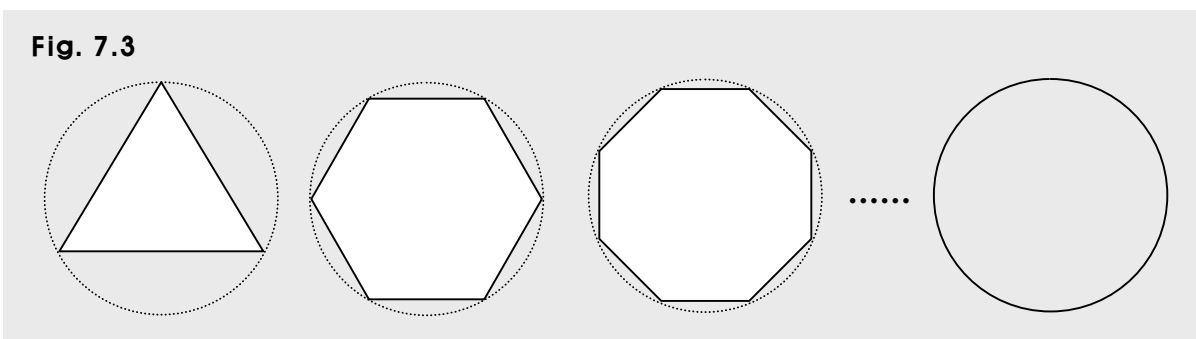
So a circle is a closed line segment, which is a collection of all the points, the distance from each of which to a particular point is the same. The distance is the radius, and the particular point is the center.

And technically, we can take a circle for a polygon that is a collection of all the same and smallest line segments that are the same distance away from a particular point called the center of the polygon.

Technically therefore, a circle can be taken as a regular polygon with infinitely many sides equal. So in short, a circle can be taken for an infinite regular polygon, technically, of course.



And we can put the diagram above the way below, too.



So a circle is a 2-D object, a collection of all the points the radius away from the center. Also, a circle has its diameter, a line segment passing through the center and connecting two points facing each other in the circle. So the diameter is twice the radius of the circle.

What then about the circumference of a circle?

It is no other than the circle. More specifically, the circumference of a circle is the length of the curved and closed line segment forming the circle. So for instance, assuming the radius is  $r$ , and  $C$  is the circumference, we get  $C = 2\pi r$ , where  $\pi$  is the circular ratio, which is an irrational number, and is 3.141592... And the diameter is twice the radius. So in short, the circumference of a circle is  $\pi$  times the diameter.

And saying just the radius of a circle, we mean the distance from the center to a point in the circle. So for instance, if the distance is 2, the radius is 2.

Saying however, a radius of a circle, we mean a line segment connecting the center and a point in the circle. How many points are there in a circle though?

Infinitely many, of course. So a circle can be said to have infinitely many radii, each of which has the same length. And the same is true for the diameter, too. So saying just the diameter of a circle, we mean the length of a line segment that is between two points facing each other in the circle, and is passing through the center.

- And we can put the diameter of a circle this way, too:

The diameter is *the largest distance* between two points in the circle.

Saying however, a diameter of a circle, we just mean such a line segment as stated above, that is, we mean a line segment connecting two points facing each other in a circle.

So a diameter passes through the center of the circle, and every diameter in a circle has the same length, which is twice the radius.

And a circle has infinitely many of such line segments with the same length. Thus, a circle has infinitely many diameters sharing the same length.

- So *what can* be said to *define* a particular circle?

The radius and the center can define a particular circle. So for instance, we can have only one circle of radius 3 centered at  $(1, 2)$  in the  $x$ - $y$  plane. Also, defining that circle, we can use an equation as follows.  $(x - 1)^2 + (y - 2)^2 = 3^2$ , which is therefore, called the equation of the circle of radius 3 centered at  $(1, 2)$  in the  $x$ - $y$  plane. And the equation is in fact, the distance formula, and is said to be in the standard form. How is it though, the distance formula?

Assuming for instance,  $a$  and  $b$  are two legs in a right triangle, and  $c$  is the hypotenuse, we can put the distance formula this way:  $a^2 + b^2 = c^2$ .

So in the equation of the circle,  $(x - 1)^2 + (y - 2)^2 = 3^2$ , taking  $(x - 1)$  as  $a$ ,  $(y - 2)$  as  $b$ , and 3 as  $c$ , we get  $a^2 + b^2 = c^2$ .

Thus,  $|x - 1|$  and  $|y - 2|$  are the two legs in a right triangle where 3 is the hypotenuse, and is the distance between an arbitrary point  $(x, y)$  to a point  $(1, 2)$ . So the equation above indicates a geometric object, which is collection of all the points, the distance from each of which to a point  $(1, 2)$  is 3.

That is to say that  $(x, y)$  is an arbitrary point in the circle where the radius is 3, and the center of the circle is  $(1, 2)$ .

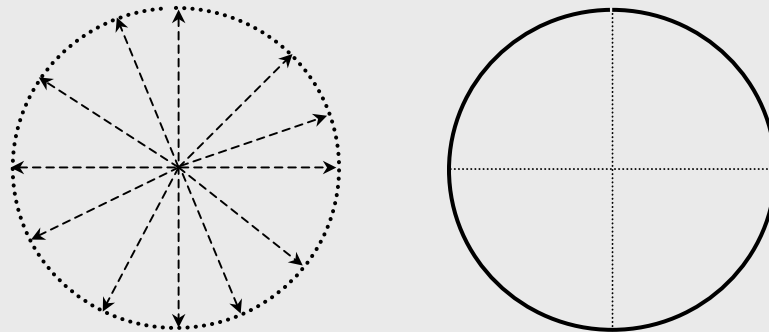
So using the distance formula, we can easily come up with the equation of any circle if given the radius and the center. What then about the general form?

Putting it in the general form, we just expand or simplify the standard equation, and then, set the right hand side to 0. Thus, expanding or simplifying the equation  $(x - 1)^2 + (y - 2)^2 = 3^2$ , and setting the right hand side to 0, we get  $x^2 + y^2 - 2x - 4y - 4 = 0$ .

And we know that the radius of a circle is the distance from its center to a point in the circle. Why do we call though, the distance the radius?

That's probably because vectors of the same length seem to radiate in all directions from a point called the center of a circle, and thus, probably for that reason, the length is called the radius.

Fig. 7.4

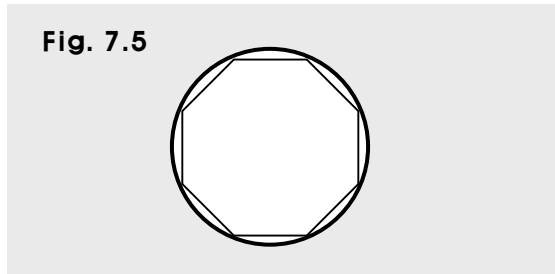


And each of the vectors is called a radius. In fact, it is called a radial vector. So the same radii can be said to radiate in all directions from the center of a circle.

- And circles are in two kinds in connection with polygons, one is a circumcircle, and the other is an incircle.

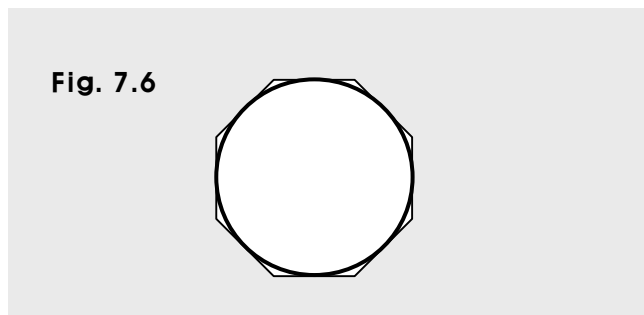
A circumcircle surrounds a polygon, passes through all the vertices of the polygon, and is said to circumscribe the polygon, which is then, said to be inscribed in its circumcircle.

Some polygons have a circumcircle. So not every polygon has a circumcircle. However, every regular (equilateral) polygon has a circumcircle.



And many polygons can have the same circumcircle. In other words, many different polygons can share one particular circumcircle.

On the other hand, if a circle is tangent to every side of a polygon, and is inside the polygon, the circle is said to be inscribed in the polygon, and is called the incircle of the polygon.



And if a polygon has an incircle, the polygon is said to circumscribe the incircle, and is called a circumscribed polygon.

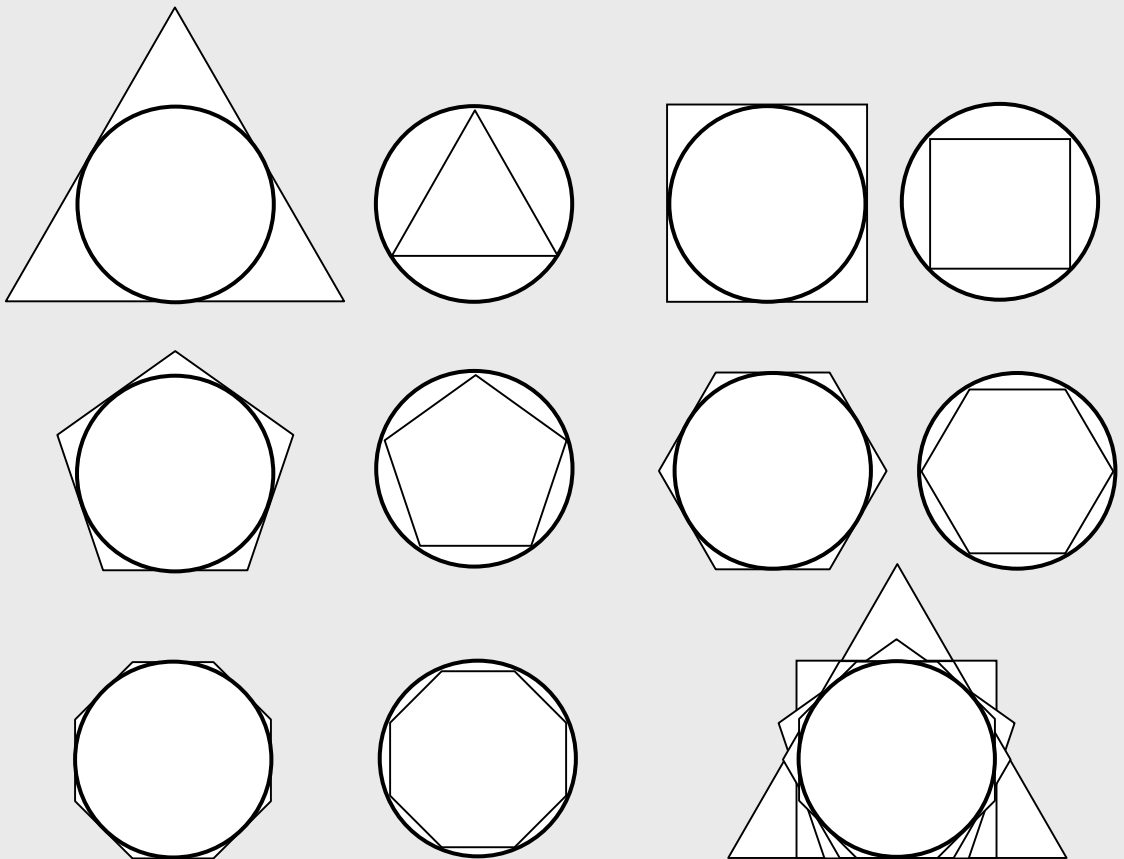
Some polygons can have an incircle. So not every polygon has an incircle.

However, every regular (or equilateral) polygon has an incircle.

And of course, many polygons can have the same incircle at the same time. In other words, many different polygons can share one particular incircle, which is inscribed in all the polygons.

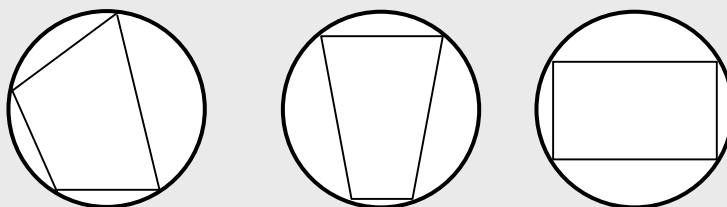
Examples of regular polygons with the incircles and circumcircles

Fig. 7.7



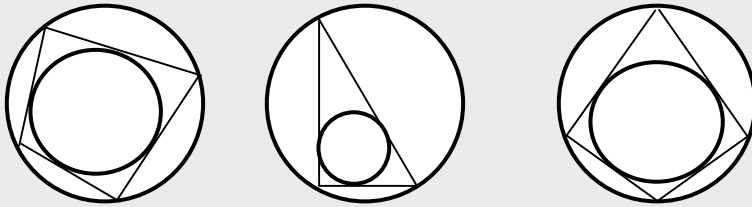
Some polygons can have a circumcircle only.

Fig. 7.8



Some polygons can have a circumcircle and an incircle both.

Fig. 7.9



In fact, if a circle is inscribed in a polygon, the polygon has its circumcircle. And every triangle has an incircle and a circumcircle both. Some polygons however, have *neither* an incircle *nor* a circumcircle.

Fig. 7.A

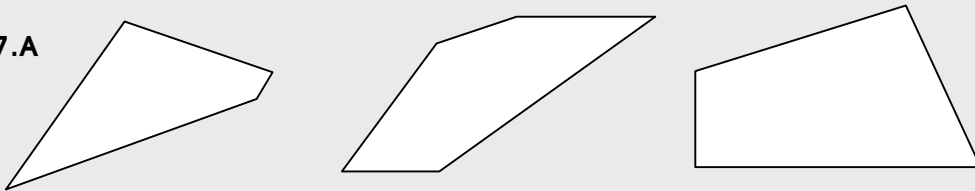
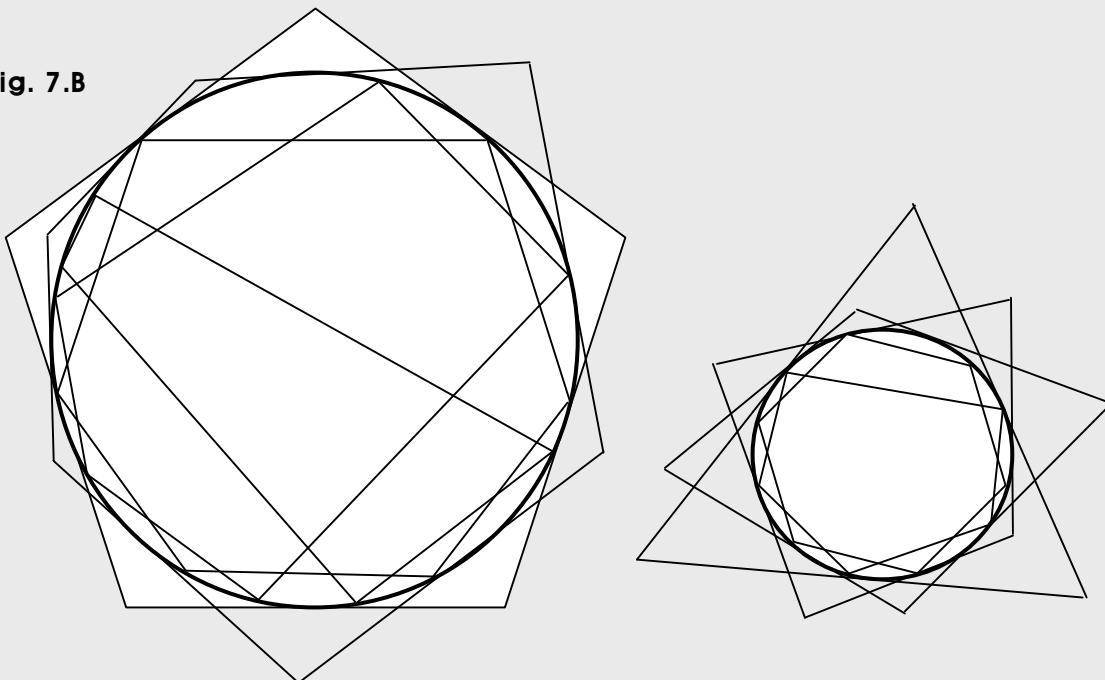


Fig. 7.B

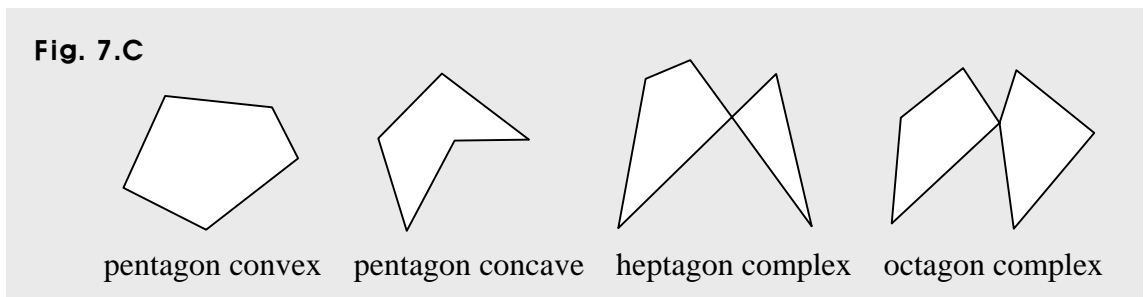


A circle can circumscribe or can be inscribed in infinitely many polygons.  
 What is a polygon though?

A polygon is a set of three or more line segments forming a closed plane object.  
 We have three kinds in polygons. One is convex, another is concave, and the other is complex.

Normally, just saying a polygon, we mean a polygon convex where every line segment meets two of the others at its both ends, and every internal angle is between 0 and 180°. So for instance, every internal angle in a rectangle is 90°, and every internal angle in a regular triangle is 60°.

Thus, a polygon convex is a closed plane object, and is a set of three or more line segments, each of which meets two of the others at its both ends only, and is a plane object where every internal angle is between 0 and 180°. And a line passing through such a polygon can cross up to two sides in the polygon, so it cannot cross three or more sides.



- Now, a circle is a line segment closed, and is curved evenly so that every point in the circle is the same distance away from a point called the center. And the same distance is called the radius. So anyway, a circle is a line segment. How then can we find the length of such a line segment?

We call such a length the circumference of a circle.

And the circumference is approximately 3 times the diameter of the circle.

So the ratio of the circumference to the diameter is **approximately** 3. Thus, assuming  $C$  is the circumference, and  $D$  is the diameter, we can set  $C \approx 3D$ . What then is the exact value of the ratio?

The ratio is not in fact, a rational number as 3.14, and is an irrational number. And more precisely, the ratio is 3.141592..., and is often taken as 3.14 as an approximation.

And the ratio is called the circular ratio, and we often use a Greek letter  $\pi$  to indicate the ratio. So we often set  $\pi = 3.14$ .

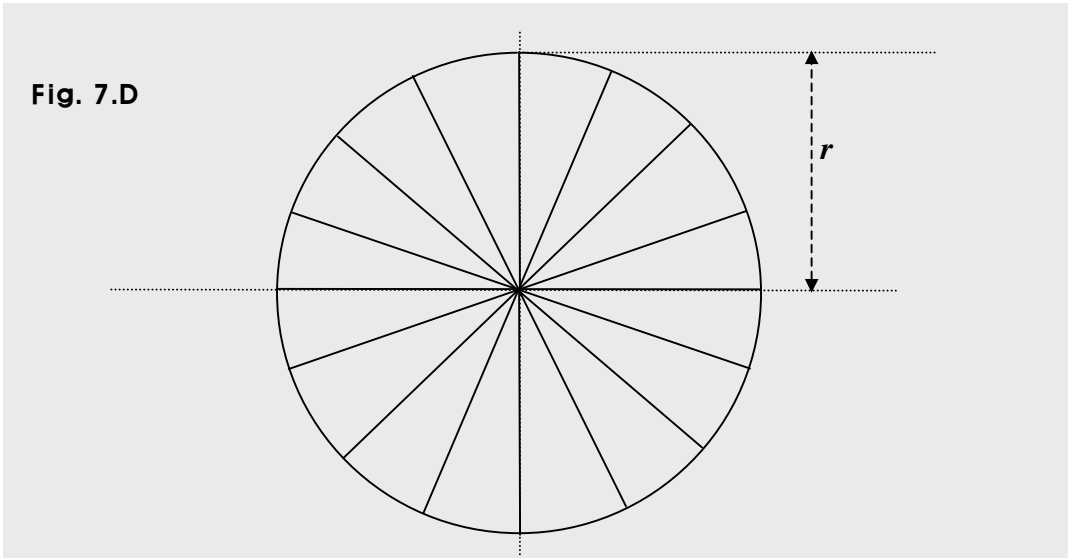
Thus, assuming  $C$  is the circumference, and  $D$  is the diameter, we get  $C = \pi D$ .

And we know the radius is half the diameter. So assuming  $r$  is the radius, we can put the circumference this way, too:  $C = 2\pi r$ .

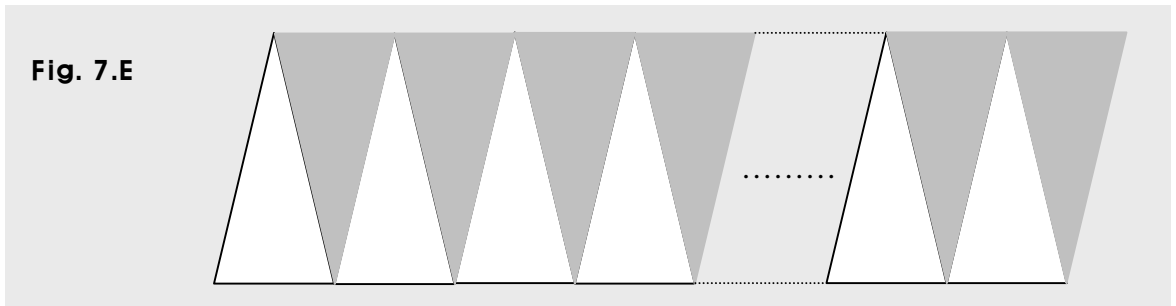
So if for instance, the radius of a circle is 1, the circumference is  $2\pi$ .

And next, we know a circle is a closed line segment. So we should be able to find the area taken by a circle, too. How then can we get the area?

We can partition a circular plate of radius  $r$  into circular wedges the way as follows.



Then, assuming we partition the disk above into infinitely many wedges, we can take the wedges as triangles isosceles, and thus, can put together all the triangles the way below.



Then, we can say that all the triangles in gray are from the upper half of the disk, and that all the triangles in white are from the lower half of the disk.

Also, we can say that putting together all the triangles in gray and white, we get a parallelogram plate, which is however, no other than a rectangular plate. What then is the length of the base of the rectangle above?

We know that the circumference of the disk is  $2\pi r$ , and all the triangles in white are from the lower half of the disk. So the base of the rectangle is the half the circumference, and thus, is  $\pi r$ . What then is the height of the rectangle?

We know that we partition the disk above into infinitely many wedges. So we can take the wedges as triangles isosceles, and also, the height of the triangle can be taken as the radius of the disk, that is,  $r$ . What then is the area of the rectangle?

The area of a rectangle is the product of the base and the height.

And the base is  $\pi r$ , and the height is  $r$ . So the area is  $\pi r^2$ .

And we know that circular disk is no other than the rectangular plate. So the area of the circle is no other than the area of the rectangle, and thus, is  $\pi r^2$ .

Thus, assuming  $A$  is the area of a circle of radius  $r$ , we get  $A = \pi r^2$ .

Also, assuming  $C$  is the circumference of a circle of radius  $r$ , we get  $C = 2\pi r$ .

So what matters in the area or the circumference of a circle is the radius.

Thus, if two circles share the same radius, that is, have the same radii, can we say that the two circles are the same?

In analytic geometry (called coordinate geometry, too), we put a circle in a coordinate plane as the  $x$ - $y$  plane. And working with a circle in analytic geometry, we can put it in an equation, too, which is called the equation of the circle, and is expressed in terms of  $x$  and  $y$  if we put the circle in the  $x$ - $y$  plane. And defining a particular circle in analytic geometry, we specify not only the radius but the center, too.

So what matters in a circle in analytic geometry is not only the radius but the center, too.

Thus, even if circles have the same radii, that is, share the same radius, we say that the circles are different if the centers are different.

In other words, though the circles themselves are the same, if they have different centers, they are different. That's because their equations are different, since their centers are specified in their equations.

So for instance, putting a circle in an equation  $(x - a)^2 + (y - b)^2 = r^2$ , we mean a circle of radius  $r$  centered at a point  $(a, b)$ .

So for instance, putting a circle in an equation  $(x - 1)^2 + (y + 1)^2 = 2^2$ , we mean a circle of radius 2 centered at a point  $(1, -1)$ .

And the same is true for many other curves, too. So even if the curves themselves are the same, if their equations are different, we say that the curves are different.

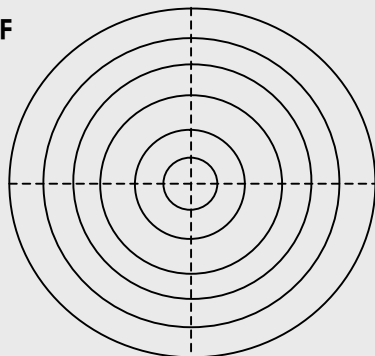
What if however, circles share the same center?

If circles have the same radii and the same centers, that is, if they share the same radius and the same center, we say that they are the same circles.

If however, the circles share the same center, but have different radii, the circles are said to be concentric. That is, they are called concentric circles. And of course, they are different circles.

So concentric circles share the same center, but their radii are different.

Fig. 7.F



concentric circles

And the space (area) between concentric circles is called an annulus.

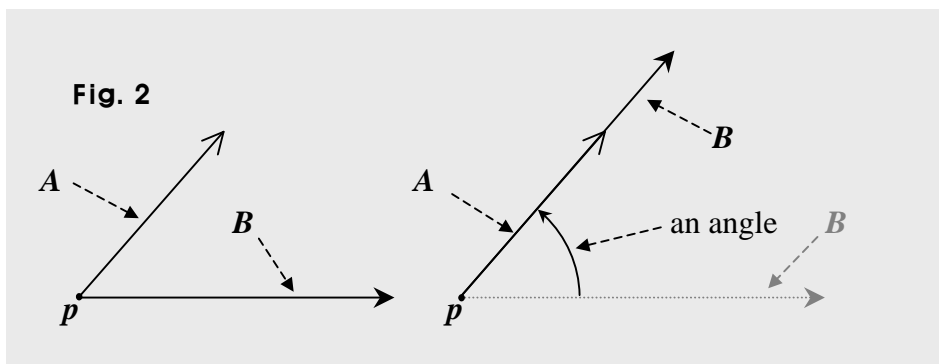
## 8. Circles and Angles

Angles are from circles. An angle looks in fact, like a part of a circle. How?

Let's see now first, what an angle is, and how it gets made.

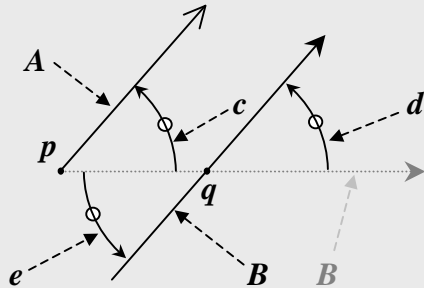
As shown in Fig. 2 below, suppose two rays diverge from a point  $p$ . Suppose also, one of the two ray turns about the point  $p$  towards the other until both rays cover each other.

Then, the amount of turning is an angle.



Suppose this time, as shown in Fig. 3 below, the ray  $B$  turns about a point  $q$  in the ray  $B$  itself until it gets parallel to the other ray  $A$ . Then, the amount of turning is an angle, too, which is the same as the angle mentioned above.

Fig. 3



The three angles  $c$ ,  $d$ , and  $e$  are the same.

Now, the ray  $B$  turned is parallel to the ray  $A$ , and has the same direction as that of the ray  $A$ . And we know the ray  $B$  before turning has a different direction. What then can we say about an angle?

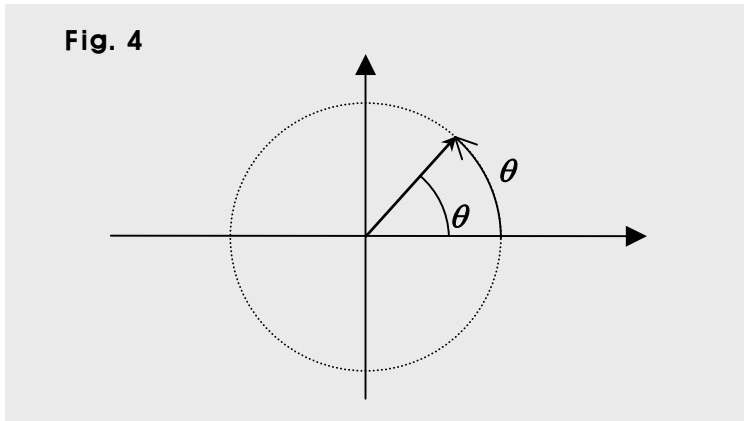
We can say that an angle is an amount of *difference in direction*, or an amount of *change in direction*. So in short, an angle can be called a change in direction, too.

Thus, we can use an angle specifying an amount of turning or a change in direction.

*An angle is an amount of turning  
or a change in direction.*

A dictionary says an angle is the figure formed by two lines diverging from the point where the two lines meet, or is the figure formed by two planes diverging from the line where the two planes meet.

What figure then do we normally use as the figure stated above?



We use as such a figure a part of a circle, which is called an arc.  
And we can calculate an angle by means of an arc and a circle. How?

We can get an angle using a ratio between the arc and the circle. And it's the angle the arc has. A circle has  $360^\circ$ . So we can get an angle the way as follows.

Take the ratio of an arc length to the circumference of the circle the arc belongs to.

Take the product of the ratio and  $360^\circ$ , the angle in a circle.

Then, the product is the angle the arc has.

In short, **an angle** can be put in terms of **a ratio of an arc to the circle** that has the arc.

So for instance, assuming  $\theta$  is an angle,  $k$  is such a ratio, and  $C$  is a constant, we can set  $\theta = Ck$ , where  $C$  is  $360^\circ$ . Thus, we get  $\theta = 360^\circ k$ .

So an arc is a part of a circle, and has an angle. What then is a circle?

A circle is in a plane, and is a collection of all the points that are the same distance away from a particular point called the center, and the same distance is the radius.

Thus, a circle is a 2-D object, and has its diameter, which is a line segment passing through the center and connecting two points facing each other in the circle, so is twice the radius of the circle.

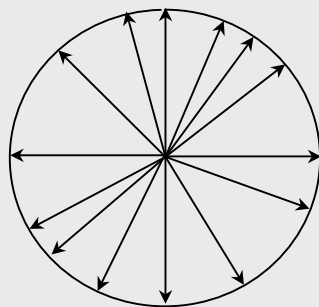
What then about the circumference of a circle?

It is the length of the curved and closed line segment forming the circle. For instance, assuming the radius is  $r$ , and  $C$  is the circumference, we get  $C = 2\pi r$ , where  $\pi$  is the circular ratio, which is an irrational number, and is 3.141592...

Suppose now, as in Fig. 0 above, all the radii in a circle are vectors, and all the vectors begin at the center of the circle. A vector has its length and direction, so is made of a line segment and an arrowhead, called the terminal point.

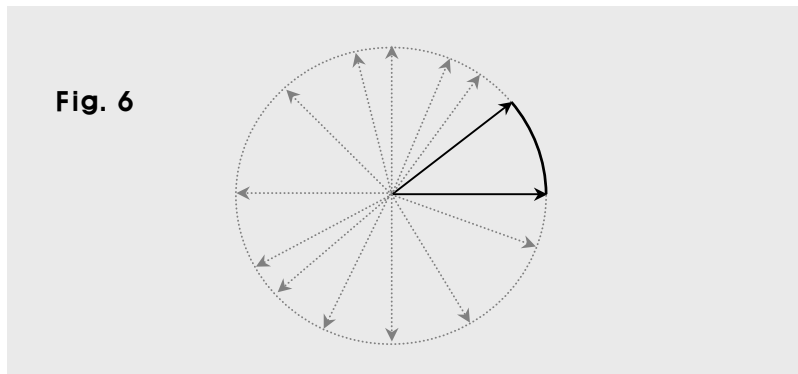
Then, all their lengths are equal, so all their terminal points are in the circle. That is to say that all the terminal points form the circle. And we can say that the center emits such vectors in all directions.

**Fig. 5**



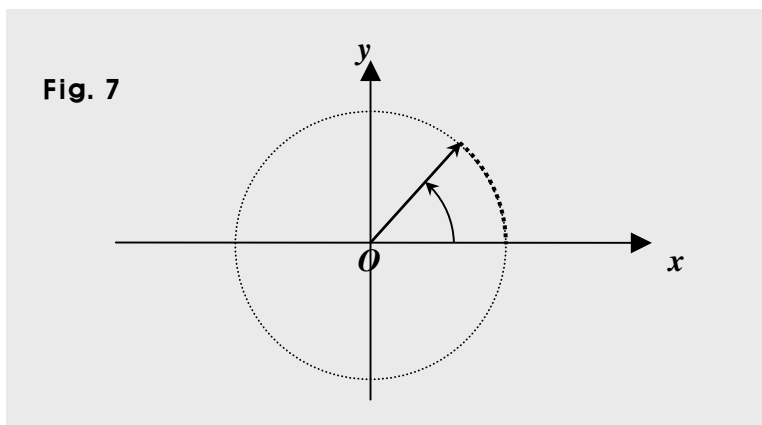
Also, we can say that each of all the vectors has a different direction. So every radius in a circle has a different direction. What then do we call the difference between the directions of two vectors?

We call it an angle, so it's the angle between two vectors. Thus, we can say that a difference in direction is an angle.



As shown above, an angle can be pictured by a circular wedge made of two radii and an arc, a part of a circle. Thus, a pair of radii in a circle can form a geometric object called an angle. Is there anything else though, we can call an angle?

Suppose as in Fig. 7 below, a vector is turning about the origin in the  $x$ - $y$  plane. What then can we call the amount of turning the vector makes?



We call it an angle, too. So we can say that an angle is an amount of turning, also. Thus, we can use an angle to specify an amount of turning or a change in direction.

In short, *an angle is an amount of turning or a change in direction.*

So an angle can tell us how much two radii in a circle are apart from each other. For instance, if in a circle, a radius is  $30^\circ$  away from another radius in the circle, the angle between the two radii is  $30^\circ$ .

And of course, expressing an amount of angle, we use a unit of measure as in the cases of other objects as lengths, areas, volumes, weights, etc. For instance, we use kilograms or pounds for weights. What then is the unit of measure for angles?

We have two kinds in such a unit. One is *degree*, and the other is *radian*, called *rad* for short. Thus, we can use degrees or radians for angles. So for instance, an angle can be 25 degrees, 1 degree, 3 radians or 3 rad, or 1 radian or 1 rad.

And putting an angle in writing using degrees, we use a small circle, and put it at the upper right-hand corner of the number indicating the amount of angle. So we use it as a superscript as in  $25^\circ$ . What then about radians?

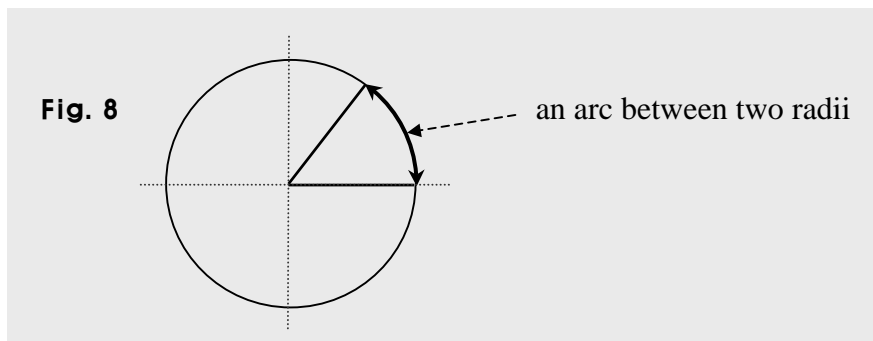
Usually, we use no symbol for radians. So assuming for instance,  $A$  is an angle in radian, we just put it in writing this way:  $A = 2$ , which means  $A$  is 2 radians.

Having to clarify though, an angle is in radians, we put 'rad' after the number. So for instance, if an angle  $B$  is 3 radians, we can put it this way:  $B = 3 \text{ rad}$ .

Thus, we have two systems of measurement for angles. And one system can get converted to the other. So two different amounts in angle can mean the same angle.

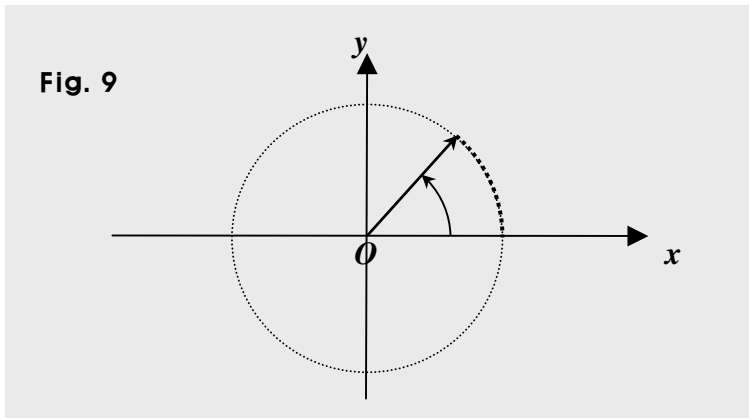
For instance, we have  $180^\circ = \pi \text{ rad}$  where  $\pi$  is the circular ratio, 3.141592... And briefly, we put it this way:  $180^\circ = \pi$ . How can we calculate though, an amount of angle?

As explained above, an angle is a geometric object, which is a wedge formed by two radii in a circle and an arc, a figure that can show not only a difference (or a change) in direction but an amount of turning, too.



And we can get an amount of angle using a ratio of an arc length to the circumference of the circle the arc belongs to. In short, an angle can be put in terms of a ratio of an arc to the circle that has the arc. How though?

Let's get back to the vector turning about the origin in the **x-y** plane.



Then, first, an angle is an amount of turning. And an arc has an angle.

Next, as the vector turns, the arrowhead traces an arc, and if the vector makes a complete turn, the arrowhead traces a circle. So if the vector makes a complete turn, the arc made is the circumference of the circle, that is, the arc is the circle itself.

And if half a complete turn is made, the arc is half the circle. If a third of a complete turn is made, the arc is a third of the circle.

Thus, an amount of turning, that is, the angle an arc has is proportional to the ratio of the length of the arc to the circumference of the circle the arc belongs to. So in short,

an angle is proportional to the ratio of an arc to its circle,

which is the case if the angle is in degrees.

In radians, an angle is simply, the ratio of an arc to its circle.

Thus, in degrees, assuming  $\theta$  is an angle,  $k$  is such a ratio, and  $C$  is a constant, we can set the equation as follows.  $\theta = Ck$ . What then is the constant  $C$ ?

It is a constant of proportionality, and is  $360^\circ$ . Why though?

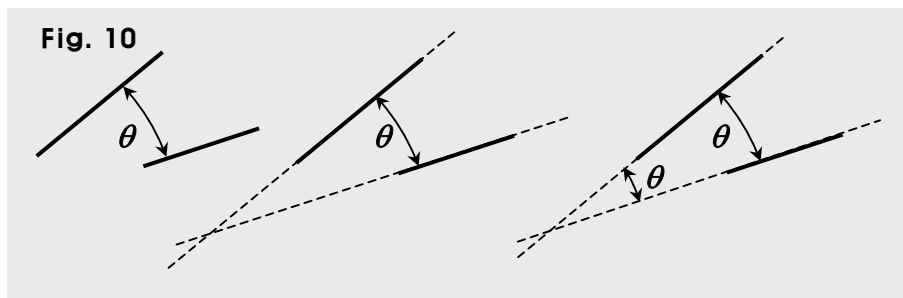
If the vector makes a complete turn, that is, if the arc made is a circle, the angle made is  $360^\circ$  by definition. In short, a complete turn is  $360^\circ$ . So the constant is  $360^\circ$ , and the angle a part of a circle has is a part of  $360^\circ$ .

So for instance, if the length of an arc is 2, and 12 is the circumference of the circle the arc belongs to, the ratio of the arc to its circle is  $\frac{2}{12} = \frac{1}{6}$ . Then, the angle is  $\frac{1}{6}$  of  $360^\circ$ , which is  $60^\circ$ .

So assuming  $\theta$  is an angle, and  $k$  is such a ratio, we can set  $\theta = 360^\circ k$ .

And if two lines are parallel to each other, the angle between the two is 0. What if two line segments not parallel are away from each other, and do not meet at a point?

A line segment is in a line, so is a part of a line, and two lines not parallel meet at a point. So the angle between the two lines that the two line segments belong to is the angle between the two line segments as shown in Fig. 10.



How small is  $1^\circ$ ?

Dividing a circle into 360 equal parts around the center, we get 360 equal wedges. So taking one of the wedges, we get  $\frac{360^\circ}{360} = 1^\circ$ . So if the vector turning counterclockwise makes a three hundred and sixtieth of a complete turn, we say that the angle made is  $1^\circ$ .

And again, dividing one wedge worth  $1^\circ$  into 60 equal parts, we get 60 equal smaller wedges, each of which has an angle called 1 *minute* denoted by  $1'$ .

So we get  $\frac{1^\circ}{60} = 1'$ , which is pretty small.

Thus, if the vector tuning counterclockwise makes a twenty one thousand and six hundredth of a complete turn, the angle made is  $1'$ .

Then again, dividing  $1'$  by 60, we get an angle called 1 *second* denoted by  $1''$ .

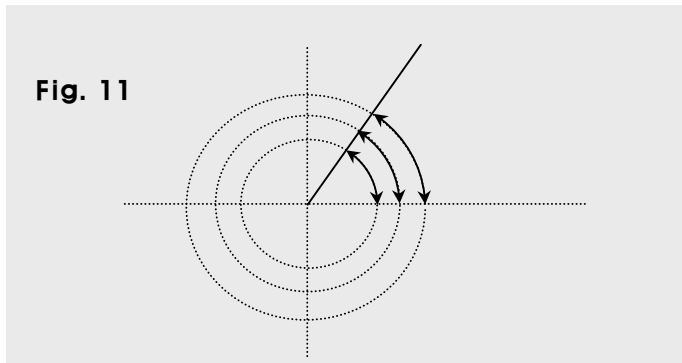
So we get  $\frac{1^\circ}{360} = \frac{1'}{60} = 1''$ , which is very small.

And we can keep going taking the smaller wedges. We do not normally though, get a wedge worth less than 1 second,  $1''$ . What then about  $0.78^\circ$ ?

It is an angle that is 78% of  $1^\circ$ , so  $0.78^\circ = \frac{78 \cdot 1^\circ}{100}$ . Thus, we get  $0.1^\circ = \frac{1^\circ}{10}$ .

So the bigger the wedge is, is the bigger the angle, in other words, the bigger the arc is, is the bigger the angle?

Not necessarily. Many different arcs can have the same angle.

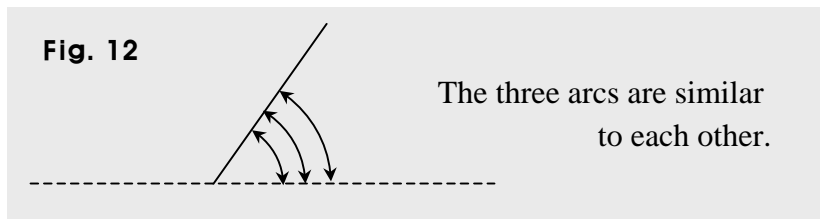


So what matters in an angle is not an arc length itself but the ratio of an arc to the circle the arc belongs to.

That's because every circle has  $360^\circ$ , but the radius of each can be different. So what?

First, many arcs with different lengths can have the same angle. All those arcs belong to circles concentric. Concentric circles have the same center as shown above.

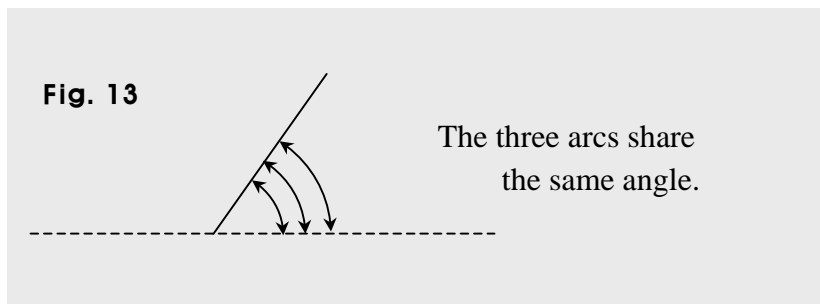
Next, holding an angle constant, we can see the larger the radius, the larger the arc. That is, for a particular angle, as the radius grows, the arc grows, too.



So the ratio between the radii is the same as the ratio between the arcs.

For instance, assuming in Fig. 12 above,  $R_1$ ,  $R_2$ , and  $R_3$  are the radii, and  $A_1$ ,  $A_2$ , and  $A_3$  are the arcs, and also, assuming  $R_1 < R_2 < R_3$ , and  $A_1 < A_2 < A_3$ , since the arcs are similar, we get  $R_1 : R_2 : R_3 = A_1 : A_2 : A_3$ .

Thus, we can get  $\frac{A_1}{R_1} = \frac{A_2}{R_2} = \frac{A_3}{R_3}$ . What then is the value of  $\frac{A_1}{R_1}$ ?



In other words, if  $\frac{A_1}{R_1} = \frac{A_2}{R_2} = \frac{A_3}{R_3} = D$ , what is  $D$ ?

We know  $D$  is the ratio of each arc to its corresponding radius, and is constant.

Also, the angle in each of the three wedges, that is, the angle of each arc is the same, and thus, is constant. So we can reasonably expect that the ratio is the angle.

And the ratio is in fact, the angle. So  $D$  is the angle.

We know however, a ratio is a number. How then can an angle be a number?

As mentioned earlier, we have two systems of measurement for angles. One is radian system, and the other is degree system.

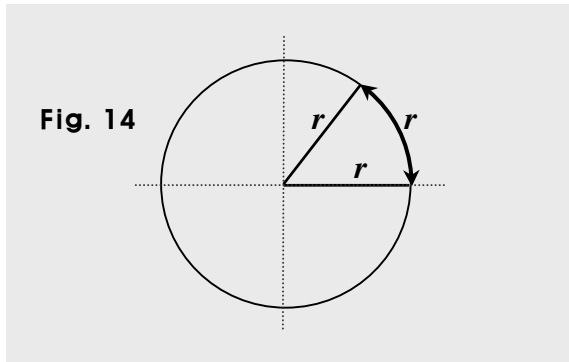
And using the radian system, we can use as angles all real numbers, which are therefore, angles in radians. And we will see now how we get angles in radians.

To begin with, stating angles, we can put angles in degree system.

We have two systems where we can put angles. What then is the other system?

It is the radian system, where we put angles in radians. So 1 radian is a unit of measure for angles. What is 1 radian though?

If the arc length is the radius of the circle that has the arc, the angle the arc has is 1 radian. That is, if an arc length is its radius, the angle of the arc is 1 radian.



So in short, if an arc is its radius, the angle is 1 rad.

Thus, if a circular wedge is like a regular (equilateral) triangle where three sides are equal, the angle of the arc is 1 rad. So if an arc is its radius, we get 1 rad.

Thus, assuming  $A$  is the arc,  $r$  is its radius, and  $A = r$ , we can get  $\frac{A}{r} = \frac{r}{r} = 1$ , which is called 1 radian.

So 1 radian is a **radius angle**, the angle of the arc the length of which is the radius.

Now, we know in a circle, the radius  $r$  is constant, and an arc  $A$  can change. So we can see that the larger the arc, the larger its angle.

Thus, we can see that  $A$  is proportional to its angle  $\theta$ , and that  $r$  is the proportionality constant.

So we can have  $\frac{A}{r} = \theta$  where  $A$  is the arc,  $r$  is its radius, and  $\theta$  is the angle  $A$  has.

And getting angles the way above, we call the angles radians.

Also, getting an arc length, we can use  $A = r\theta$ , because we have  $\frac{A}{r} = \theta$ .

For instance, the circumference  $C$  of a circle of radius  $r$  is  $2\pi r$ . And the angle in a circle is  $2\pi$ . So in the equation  $\frac{A}{r} = \theta$ , we have  $A = C$ , and  $\theta = 2\pi$ .

Then, we get  $\frac{A}{r} = \theta \Rightarrow \frac{C}{r} = 2\pi \Rightarrow C = 2\pi r$ .

Assuming for instance, the arc is a sixth of a circle of radius  $r$ ,

we get  $A = \frac{2\pi r}{6} = \frac{\pi r}{3}$ , so we get  $\theta = \frac{A}{r} = \frac{\frac{\pi r}{3}}{r} = \frac{\pi}{3}$ , called  $\frac{\pi}{3}$  radian. And we know the angle the arc has is  $60^\circ$ .

So we get  $60^\circ = \frac{\pi}{3}$  rad. What then about a quarter circle?

The length of a quarter circle of radius  $r$  is  $\frac{2\pi r}{4}$ , and the angle a quarter circle has is  $90^\circ$ .

So we get  $\theta = \frac{A}{r} = \frac{\frac{2\pi r}{4}}{r} = \frac{\frac{\pi r}{2}}{r} = \frac{\pi}{2}$ , and thus, we get  $90^\circ = \frac{\pi}{2}$ .

And by the same token, since the angle of an eighth of a circle is  $45^\circ$ , if the arc

is an eighth of a circle, its angle  $\theta$  is  $\frac{A}{r} = \frac{\frac{2\pi r}{8}}{r} = \frac{\frac{\pi r}{4}}{r} = \frac{\pi}{4}$ .

Thus, we get  $45^\circ = \frac{\pi}{4}$ .

Also, the arc can be a half circle. Then, the circular wedge is a half of a circular

disk. So assuming  $A = \frac{2\pi r}{2} = \pi r$ , we get  $\theta = \frac{A}{r} = \frac{\pi r}{r} = \pi$ , called  $\pi$  radian. And we

know the length of a half circle of radius  $r$  is  $\pi r$ , and the angle a half circle has is  $180^\circ$ . So we get  $180^\circ = \pi$ .

And of course, the arc itself can be a circle, too, that is, the wedge can be a circular disk. Then, we get  $A = 2\pi r$ , so its angle  $\theta$  is  $\frac{A}{r} = \frac{2\pi r}{r} = 2\pi$ , and thus, we get  $360^\circ = 2\pi$  rad, since the angle a circle has is  $360^\circ$ .

What angle in the degree system then is 1 radian? That is, what degree is 1 rad?

We have  $60^\circ = \frac{\pi}{3}$  rad,  $90^\circ = \frac{\pi}{2}$  rad,  $45^\circ = \frac{\pi}{4}$  rad,  $180^\circ = \pi$  rad,  $360^\circ = 2\pi$ , etc.

And using any of the ones above, we can get the angle in degree equivalent to 1 rad.

Using the fourth one above, we get  $180^\circ = \pi$  rad  $\Rightarrow 1$  rad  $= \frac{180^\circ}{\pi} = 57.29657\dots^\circ$ .

So we get  $1$  rad  $\approx 57^\circ$ . What angle then in radians is  $1^\circ$ ?

We have  $45^\circ = \frac{\pi}{4}$  rad. So we get  $1^\circ = \frac{\frac{\pi}{4}}{45} = \frac{\pi}{4 \cdot 45} = \frac{\pi}{180}$  rad  $= 0.01745\dots$  rad.

So we get:  $1^\circ \approx 0.0175$  rad. Why angles in degrees or radians though?

We can use either system working with angles. Working with angles though, we can say in large that we use angles in two different cases. In one, we use angles that do not change, and in the other, we use angles that can change.

And angles that do not change can be said to be static, and angles that can change can be said to be dynamic. And using angles static, we are likely to use the degree system, and using angles dynamic, we are likely to use the radian system. In either case though, we can still use either system.

Using angles in degrees, we have to use a symbol, which is a small circle as in  $90^\circ$ .

Using angles in radians however, we just use real numbers as the angles. Usually, using angles in radians, we just use numbers only, and do not put rad or radian to the numbers.

So for instance, assuming  $A$  is  $\pi$  radian, we just put it this way:  $A = \pi$ , which is a number 3.141592..., which is an irrational number, which is a real number.

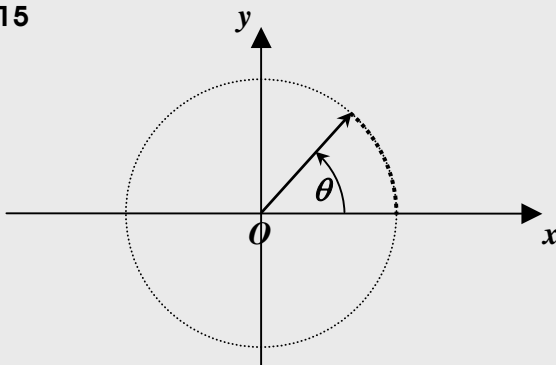
And for another instance, assuming  $B$  is 7 rad, we simply put it this way:  $B = 7$ . What then, can be the advantage of using angles in radians?

Suppose we work with a function, where the inputs are angles. Then, the input variable is an angle that can change, and thus, is dynamic (characterized by continuous change).

And it's convenient to use real numbers as angles. So in such cases, we normally use angles in radians rather than degrees.

Let's now get back again to the vector turning about the origin in the  $x$ - $y$  plane, and look at angles from a bit different perspective.

Fig. 15



Then, we can see that the vector keeps changing its direction, and that the angle between the vector and the  $x$ -axis changes as the vector turns. We know that the  $x$ -axis is fixed.

So what makes an angle is the vector turning; thus, we can say for simplicity that the angle between the vector and the  $x$ -axis is the angle the vector makes. When the vector is at rest sitting on the  $x$ -axis on the right of the origin, its angle is assumed to be  $0^\circ$ .

Suppose now, the vector starts turning counterclockwise.

Then, as the vector keeps turning, it keeps changing its direction, and its angle keeps changing. When turning counterclockwise, the angle is positive.

Technically, a circle is an arc, too. And by definition, a circle has  $360^\circ$ , which is  $2\pi$ , and a half circle has  $180^\circ$ , which is  $\pi$ . So if the vector turning counterclockwise makes a complete turn about the origin, its arrowhead traces a circle; thus, the angle made is  $360^\circ$ , that is,  $2\pi$ . And if two complete turns are made, the arrowhead traces a circle twice, so the angle made is twice  $360^\circ$ , and is  $720^\circ$ , that is,  $4\pi$ . Why not though,  $360^\circ$  but  $720^\circ$ ?

The angle made in this case is the amount of turning, which is in this case, twice a complete turn. So the angle can be more than  $2\pi$ , and can be  $3\pi$ ,  $10.7$ ,  $19$ ,  $27\pi$  etc.

And if a quarter of a turn is made, the arrowhead traces a quarter of a circle, so the angle made is a quarter of  $360^\circ$ , and is  $90^\circ$ , that is,  $\frac{\pi}{2}$ . What if the vector is turning clockwise?

Then, its angle is negative. So for instance, if the vector makes a complete turn clockwise, its angle is  $-360^\circ$ , that is,  $-2\pi$ , and if a quarter of a complete turn is made clockwise, the angle made is  $-90^\circ$  or  $-\frac{\pi}{2}$ .

Thus, tuning the vector counterclockwise or clockwise, we can get all angles. And indicating all such angles, we can use all real numbers. So what?

We can come up with functions where inputs are angles. And it's convenient to use real numbers as inputs. So in such a function, using as inputs angles in radians, we can use all real numbers as such inputs. And such a function is called a trigonometric function, often just called a trig function. What are angles for though?

Working with objects that have directions and can change their amounts if changing their directions, we need to work with angles. We don't just work with angles though. The angle we get to work with in such a case is one of the angles in a triangle. It's not just any triangle though. What triangle then?

Of all kinds in triangles, the simplest and the most basic, and thus, the most fundamental is a triangle where one angle is  $90^\circ$ , called a right angle. So such a triangle is called a right triangle. And a right triangle is the place where a special geometry called trigonometry begins.