Examples Y in Arithmetic 2

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Examples Y

0. Show that $n^5 - n$ is a multiple of 30 if $|n| > 1$ is an integer.

1. Suppose a, b, and c are positive integers, and $p = a^2 + b^2 + c^2$. Then,

A. Show that if a is not a multiple of 3, the remainder in the division of a^2 by 3 is 1.
B. Show that if none of a, b, and c is a multiple of 3, p is a multiple of 3.
C. Show that if a, b, c, and p are primes, at least one of a, b, and c is 3.

2. Assuming a is an integer positive, and $n(a)$ is the number of all positive divisors of a, find the minimum of x for which $n(108)n(32)n(x) = 5400$.

3. Assuming a and b are positive integers, find the smallest a and b for which $a^2 = 2b^2$.

4. Assuming x, y, and z are integers, find the smallest y for which we get $x^2 + z^2 = y^3$ if $x > y > z > 0$.

5. Put 2003 in the sum of powers of 2. Putting for instance, 83 in the sum of powers of 2, we get $83 = 2^6 + 2^4 + 2^1 + 2^0$.

6. Put each of 135 and 598 in terms of the powers of digits in the number itself.
For instance, $153 = 1 + 5^3 + 3^3$, $24 = 2^3 + 4^2$, and $2427 = 2 + 4^2 + 2^3 + 7^4$.

7. Show that if x is a positive integer and has an odd number of divisors, there is an integer n for which we get $x = n^2$.

Suggestions or Solutions
To the Examples Y

0. Show that $n^5 - n$ is a multiple of 30 if $|n| > 1$ is an integer.

To begin with, 5 divides 30, and so does 6.
So 30 is a common multiple of 5 and 6. What then, about a multiple of 30?

It is a common multiple of 5 and 6.
And thus, if $(n^5 - n)$ is a common multiple of 5 and 6, it is a multiple of 30.
How do we know then, if $(n^5 - n)$ is a common multiple of 5 and 6?

We can have $n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = (n - 1)n(n + 1)(n^2 + 1)$. So what?

We know $(n - 1)n(n + 1)$ is a product of three contiguous integers, since n is an integer.
And a multiple of three contiguous integers is a multiple of 6.
That's because one the three integers is even, and one of the others is a multiple of 3.
For instance, $12\cdot13\cdot14 = 3\cdot4\cdot13\cdot2\cdot7 = 6\cdot4\cdot13\cdot7$. So $(n^5 - n)$ is a multiple of 6.

And next, we can put all integers this way: $5k$, $5k \pm 1$, and $5k \pm 2$, where k is an integer.
For instance, if $k = -1$, we get -7, -6, -5, -4, -3, if $k = 0$, we get -2, -1, 0, 1, 2, and if $k = 1$, we get 3, 4, 5, 6, 7.

Then, first, if $n = 5k$, we can say, of course, n is a multiple of 5.

Next, if $n = 5k \pm 1$, we get $n \pm 1 = 5k$. That is, $n + 1 = 5k$ or $n - 1 = 5k$.
So $(n - 1)$ or $(n + 1)$ is a multiple of 5.

And next, if $\boldsymbol{n} = \mathbf{5}\boldsymbol{k} \pm \mathbf{2}$, we get $\boldsymbol{n}^{\mathbf{2}} + \mathbf{1} = (\mathbf{5}\boldsymbol{k} \pm \mathbf{2})^{\mathbf{2}} + \mathbf{1} = \mathbf{25}\boldsymbol{k}^{\mathbf{2}} \pm \mathbf{20}\boldsymbol{k} + \mathbf{5}$.
So we can say that $(\boldsymbol{n}^{\mathbf{2}} + \mathbf{1})$ is a multiple of 5.

And thus, $(\boldsymbol{n} - \mathbf{1})\boldsymbol{n}(\boldsymbol{n} + \mathbf{1})(\boldsymbol{n}^{\mathbf{2}} + \mathbf{1})$, that is, $(\boldsymbol{n}^{\mathbf{5}} - \boldsymbol{n})$ is a multiple of 5, too.
So $(\boldsymbol{n}^{\mathbf{5}} - \boldsymbol{n})$ is a common multiple of 5 and 6, and thus, is a multiple of 30.

1. Suppose $a, b,$ and c are positive integers, and $p = a^2 + b^2 + c^2$. Then,

A. Show that if a is not a multiple of 3, the remainder in the division of a^2 by 3 is 1.
B. Show that if none of $a, b,$ and c is a multiple of 3, p is a multiple of 3.
C. Show that if $a, b, c,$ and p are primes, at least one of $a, b,$ and c is 3.

A.
To begin with, if $\boldsymbol{a}$ is not a multiple of 3, we can put it this way: $\boldsymbol{a} = \mathbf{3}\boldsymbol{k} + \mathbf{1}$ or $\mathbf{3}\boldsymbol{k} + \mathbf{2}$, where $\boldsymbol{k}$ is an integer ≥ 0. And more simply, we can put it this way, too: $\boldsymbol{a} = \mathbf{3}\boldsymbol{k} \pm \mathbf{1}$.

Then, we get $\boldsymbol{a}^{\mathbf{2}} = (\mathbf{3}\boldsymbol{k} \pm \mathbf{1})^{\mathbf{2}} = \mathbf{9}\boldsymbol{k}^{\mathbf{2}} \pm \mathbf{6}\boldsymbol{k} + \mathbf{1} = \mathbf{3}(\mathbf{3}\boldsymbol{k}^{\mathbf{2}} \pm \mathbf{2}\boldsymbol{k}) + \mathbf{1}$.
So the remainder in the division of $\boldsymbol{a}^{\mathbf{2}}$ by 3 is 1.

B.
To begin with, we know that if $\boldsymbol{a}$ is not a multiple of 3, the remainder in the division of $\boldsymbol{a}^{\mathbf{2}}$ by 3 is 1.
And the same is true, too, for $\boldsymbol{b}^{\mathbf{2}}$, and $\boldsymbol{c}^{\mathbf{2}}$, because neither of $\boldsymbol{b}$ and $\boldsymbol{c}$ is a multiple of 3.
So the remainder in the division of $\boldsymbol{b}^{\mathbf{2}}$ by 3 is 1.
And the remainder in the division of $\boldsymbol{c}^{\mathbf{2}}$ by 3 is 1, too.

Thus next, setting $\boldsymbol{a}^{\mathbf{2}} = \mathbf{3}\boldsymbol{A} + \mathbf{1}$, $\boldsymbol{b}^{\mathbf{2}} = \mathbf{3}\boldsymbol{B} + \mathbf{1}$, and $\boldsymbol{c}^{\mathbf{2}} = \mathbf{3}\boldsymbol{C} + \mathbf{1}$ where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are positive integers, we get $\boldsymbol{p} = \boldsymbol{a}^{2} + \boldsymbol{b}^{2} + \boldsymbol{c}^{2} = \mathbf{3}(\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}) + \mathbf{3}$.
So we can say that $\boldsymbol{p}$ is a multiple of 3.

C.

Suppose that none of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is 3.

Then, none of them is a multiple of 3, since they are primes. We know from the part $\boldsymbol{B}$, $\boldsymbol{p}$ is a multiple of 3. So we get $\boldsymbol{p = 3}$, because $\boldsymbol{p}$ is a prime.
However, we get $\boldsymbol{p = a^2 + b^2 + c^2 \geq 12}$, because $\boldsymbol{a \geq 2}$, $\boldsymbol{b \geq 2}$, and $\boldsymbol{c \geq 2}$, since $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are primes.

That is to say that it can be the case where $\boldsymbol{p = 12}$, which contradicts however, the fact that $\boldsymbol{p}$ is a multiple of 3. So at least one of a, b and c is 3.

2. Assuming *a* is an integer positive, and *n*(*a*) is the number of all positive divisors of *a*, find the minimum of *x* for which *n*(108)*n*(32)*n*(*x*) = 5400.

To begin with, $\boldsymbol{n(a)}$ is the number of all positive divisors of $\boldsymbol{a}$. So $\boldsymbol{n(108)}$ is the number of all positive divisors of 108. And the same is true for $\boldsymbol{n(32)}$, too.
So next, we may want to find $\boldsymbol{n(108)}$ and $\boldsymbol{n(32)}$. How to find them?

Factorizing first, 108 and 32, we get $108 = 2^2 3^3$, and $32 = 2^5$.
Next, we have a fact below.

Suppose $\boldsymbol{N}$ is an integer, and $\boldsymbol{N = A^{\alpha}B^{\beta}C^{\gamma} \ldots Z^{\delta}}$ where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C} \ldots \boldsymbol{Z}$ are primes, and α, β, γ, … δ are positive integers.
Then, the number of all positive divisors of $\boldsymbol{N}$ is $\boldsymbol{(\alpha + 1)(\beta + 1)(\gamma + 1)\ldots(\delta + 1)}$.

So we get $\boldsymbol{n(108) = (2 + 1)(3 + 1) = 12}$, and $\boldsymbol{n(32) = 5 + 1 = 6}$.
Thus we get $\boldsymbol{n(108)n(32)n(x) = 12 \cdot 6 \cdot n(x) = 72\ n(x) = 5400 \Rightarrow n(x) = 75}$.
So we can see that 75 is the number of all positive divisors of $\boldsymbol{x}$.
What then, can we do with the number of such divisors?

We know that the number of all positive divisors of $\boldsymbol{N}$ is $\mathbf{(\alpha + 1)(\beta + 1)(\gamma + 1)\ldots(\delta + 1)}$, and that α, β, γ, … δ are positive integers, and more specifically, are the degrees of the prime factors that $\boldsymbol{N}$ has.

And we know $\mathbf{(\alpha + 1)(\beta + 1)(\gamma + 1)\ldots(\delta + 1)}$ is a product of integers.
So we may want to try putting 75 in a product of integers in all possible ways.

$75 = (74 + 1)\cdot 1 = (74 + 1)(0 + 1) = 25\cdot 3 = (24 + 1)(2 + 1) = 15\cdot 5 = (14 + 1)(4 + 1)$
$= 5\cdot 5\cdot 3 = (4 + 1)(4 + 1)(2 + 1)$.

Next, we want find the minimum of $\boldsymbol{x}$.
So we want to find the smallest of all the products below.
$2^{74}3^0 = 2^{74}1 = 2^{74}$, $2^{24}3^2$, $2^{14}3^4$, and $2^4 3^4 5^2$.

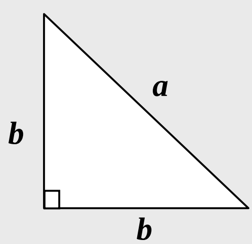
Of the three above, the smallest is $2^4 3^4 5^2 = 32400$, which is thus, the minimum of $\boldsymbol{x}$.

3. Assuming *a* and *b* are positive integers, find the smallest *a* and *b* for which we get $\boldsymbol{a^2 = 2b^2}$.

We can put $\boldsymbol{a^2 = 2b^2}$ this way, too: $\boldsymbol{a^2 = b^2 + b^2}$. What then, can we say about $\boldsymbol{a}$ and $\boldsymbol{b}$?

The equality looks like the distance formula.
So it seems that $\boldsymbol{a}$ is the hypotenuse in a right triangle isosceles, and $\boldsymbol{b}$ is each leg.

Fig. 3.0



Is there such a right triangle though? In other words, can we get a right triangle where two sides are equal and all the three sides are integers?

There is no right triangle isosceles where three sides are integers. How come?

No matter what right triangle it may be, its three sides have to satisfy the distance formula. So in the case above, too, we have to get $\boldsymbol{a^2 = b^2 + b^2}$, which is $\boldsymbol{a^2 = 2b^2}$, which is however, impossible if $\boldsymbol{a}$ and $\boldsymbol{b}$ are nonzero integers.
That is to say that if $\boldsymbol{a^2 = 2b^2}$, $\boldsymbol{a}$ cannot be an integer if $\boldsymbol{b}$ is a nonzero integer. And vice versa. So if $\boldsymbol{a^2 = 2b^2}$, $\boldsymbol{b}$ cannot be an integer if $\boldsymbol{a}$ is a nonzero integer.

And thus, we cannot get $\boldsymbol{a^2 = 2b^2}$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are nonzero integers.

And we can show the fact above the way below, too.

We know $\boldsymbol{a}$ and $\boldsymbol{b}$ are nonzero integers. So either prime or composite, we can put each in a form of a product of primes the way below.

Assuming first, $\boldsymbol{m}$, $\boldsymbol{n}$, $\boldsymbol{k \ldots}$ are nonnegative integers, we can set $\boldsymbol{a = 2^m 3^n 5^k \ldots}$.
So we get $\boldsymbol{a^2 = (2^m 3^n 5^k \ldots)^2 = 2^{2m} 3^{2n} 5^{2k} \ldots}$.

Assuming next, $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r \ldots}$ are nonnegative integers, we can set $\boldsymbol{b = 2^p 3^q 5^r \ldots}$.
So we get $\boldsymbol{b^2 = (2^p 3^q 5^r \ldots)^2 = 2^{2p} 3^{2q} 5^{2r} \ldots}$, and we get $\boldsymbol{2b^2 = 2 \cdot 2^{2p} 3^{2q} 5^{2r} \ldots = 2^{2p+1} 3^{2q} 5^{2r} \ldots}$

If two integers are the same, their factorizations have to be the same, too.

That is to say that if $\boldsymbol{a^2 = 2b^2}$, we have to get $\boldsymbol{2^{2m} 3^{2n} 5^{2k} \ldots = 2^{2p+1} 3^{2q} 5^{2r} \ldots}$, which is however, impossible even if $\boldsymbol{m = p}$, $\boldsymbol{n = q}$, $\boldsymbol{k = r}$, etc.