Examples Z in Arithmetic 2

Click this to start.

Examples Z

0. Assuming $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ are integers, find the smallest $\boldsymbol{y}$ for which we get $\boldsymbol{x^2 + z^2 = y^3}$ if $\boldsymbol{x > y > z > 0}$.

1. Put 2003 in the sum of powers of 2. Putting for instance, 83 in the sum of powers of 2, we get $83 = 2^6 + 2^4 + 2^1 + 2^0$.

2. Put each of 135 and 598 in terms of the powers of digits in the number itself.
For instance, $153 = 1 + 5^3 + 3^3$, $24 = 2^3 + 4^2$, and $2427 = 2 + 4^2 + 2^3 + 7^4$.

3. Show that if $\boldsymbol{x}$ is a positive integer and has an odd number of divisors, there is an integer $\boldsymbol{n}$ for which we get $\boldsymbol{x = n^2}$.

Suggestions or Solutions
To the Examples Y

0. Assuming x, y, and z are integers, find the smallest y for which $x^2 + z^2 = y^3$ when $x > y > z > 0$.

Sometimes, trial-and-error works. It's not just random though, and is still mathematical. So it still has to make sense. So first, what do we mean by $\boldsymbol{x^2 + z^2 = y^3}$?

We know $\boldsymbol{x}$, $\boldsymbol{y}$, and z are integers. So we can try finding an integer cubed that can be the sum of two integers squared. And next, what do we mean by $\boldsymbol{x > y > z > 0}$?

The integer to be cubed is between the two integers to be squared.
So we can try some integers the way below.

$\boldsymbol{y = 1 \Rightarrow y^3 = 1}$, $\boldsymbol{y = 2 \Rightarrow y^3 = 8}$, $\boldsymbol{y = 3 \Rightarrow y^3 = 27 = 9 + 18}$.

$\boldsymbol{y = 4 \Rightarrow y^3 = 64 = 9 + 55 = 25 + 39 = 36 + 28}$.

$\boldsymbol{y = 5 \Rightarrow y^3 = 125 = 1 + 124 = 4 + 121 = 2^2 + 11^2}$. And we have $2 < 5 < 11$.

So we can say that $\boldsymbol{x = 11}$, $\boldsymbol{y = 5}$, and $\boldsymbol{z = 2}$.

1. Put 2003 in the sum of powers of 2. Putting for instance, 83 in the sum of powers of 2, we get $\boldsymbol{83 = 2^6 + 2^4 + 2^1 + 2^0}$.

Expressing a number in terms of powers of 10, we put the number in the decimal number system where the base is 10. So we can put 2003 this way: $2{\cdot}10^3 + 3{\cdot}10^0$.
And more specifically, we can put it this way, too: $2003 = 2{\cdot}10^3 + 0{\cdot}10^2 + 0{\cdot}10^1 + 3{\cdot}10^0$.

What then, about $83 = 2^6 + 2^4 + 2^1 + 2^0$?

We can put it this way, too: $83 = 1\cdot2^6 + 0\cdot2^5 + 1\cdot2^4 + 0\cdot2^3 + 0\cdot2^2 + 1\cdot2^1 + 1\cdot2^0$.
And we can put $1\cdot2^6 + 0\cdot2^5 + 1\cdot2^4 + 0\cdot2^3 + 0\cdot2^2 + 1\cdot2^1 + 1\cdot2^0$ this way, too: 1010011.
And changing 83 to 1010011, we put 38 in the binary number system. In other words, putting 83 in the binary system, we get 1010011, which is thus, called a binary number.

2003
$= 2002 + 1$
$= 2\cdot1001 + 1$
$= 2(1000 + 1) + 1$
$= 2\cdot1000 + 2 + 1$
$= 2\cdot(2\cdot500) + 2 + 1$
$= 2^2\cdot500 + 2 + 1$
$= 2^3\cdot250 + 2 + 1$
$= 2^4\cdot125 + 2 + 1$
$= 2^4(124 + 1) + 2 + 1$
$= 2^4\cdot124 + 2^4 + 2 + 1$
$= 2^5\cdot62 + 2^4 + 2 + 1$
$= 2^6\cdot31 + 2^4 + 2 + 1$
$= 2^6(30 + 1) + 2^4 + 2 + 1$
$= 2^6\cdot30 + 2^6 + 2^4 + 2 + 1$
$= 2^7\cdot15 + 2^6 + 2^4 + 2 + 1$
$= 2^7(14 + 1) + 2^6 + 2^4 + 2 + 1$
$= 2^7\cdot14 + 2^7 + 2^6 + 2^4 + 2 + 1$
$= 2^8\cdot7 + 2^7 + 2^6 + 2^4 + 2 + 1$
$= 2^8(6 + 1) + 2^7 + 2^6 + 2^4 + 2 + 1$
$= 2^8\cdot6 + 2^8 + 2^7 + 2^6 + 2^4 + 2 + 1$
$= 2^9\cdot3 + 2^8 + 2^7 + 2^6 + 2^4 + 2 + 1$
$= 2^9(2 + 1) + 2^8 + 2^7 + 2^6 + 2^4 + 2 + 1$
$= 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2 + 1$

So we can get $2003 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2 + 2^0$.
And putting 2003 in the binary system, we get 11111010011.
And we can say that the binary equivalent to 2003 is 11111010011.

We can simplify the conversion process the way below.
First, divide 2003 by 2, and put the quotient and the remainder the way below.

2 | 2003
    1001 … 1

Next, divide 1001 by 2, and put the quotient and the remainder the way below.

2 | 2003
2 | 1001 … 1
      500 … 1

Next, divide 1001 by 2, and put the quotient and the remainder the way below.

2 | 2003
2 | 1001 … 1
2 | 500 … 1
      250 … 0

And keep doing the same until the quotient is less than 2.

2 | 2003
2 | 1001 … 1
2 | 500 … 1
2 | 250 … 0
2 | 125 … 0
2 | 62 … 1
2 | 31 … 0
2 | 15 … 1
2 | 7 … 1
2 | 3 … 1
          1 … 1

Then, collecting the numbers in the direction of the arrow above, we get 11111010011, which is the binary equivalent to 2003, and can be put more specifically this way: $11111010011_{[2]}$.

Let’s now, put 2003 in the octal system where we use octal numbers.

$$\begin{array}{l|l} 8 & 2003 \\ \hline 8 & \;\;250 \dots 3 \\ \hline 8 & \;\;\;\;31 \dots 2 \\ \hline & \;\;\;\;\;\;3 \dots 7 \end{array}$$

That is to say that we get $2003 = 3{\cdot}8^3 + 7{\cdot}8^2 + 2{\cdot}8 + 3{\cdot}8^0$. More specifically,

$$\begin{aligned}
&2003 \\
&= 2000 + 3 \\
&= 8{\cdot}250 + 3 \\
&= 8(248 + 2) + 3 \\
&= 8{\cdot}248 + 2{\cdot}8 + 3 \\
&= 8{\cdot}8{\cdot}31 + 2{\cdot}8 + 3 \\
&= 8^2 31 + 2{\cdot}8 + 3 \\
&= 8^2(24 + 7) + 2{\cdot}8 + 3 \\
&= 8^2 24 + 7{\cdot}8^2 + 2{\cdot}8 + 3 \\
&= 8^3 3 + 7{\cdot}8^2 + 2{\cdot}8 + 3 \\
&= 3{\cdot}8^3 + 7{\cdot}8^2 + 2{\cdot}8 + 3 \\
&= 3723_{[8]}
\end{aligned}$$

So the octal equivalent to 2003 is 3723, and more specifically, it is $3723_{[8]}$.

2. Put each of 135 and 598 in terms of the powers of digits in the number itself. For instance, $153 = 1 + 5^3 + 3^3$, $24 = 2^3 + 4^2$, and $2427 = 2 + 4^2 + 2^3 + 7^4$.

This problem can look difficult, but is not.
This is in particular, for those students who enjoy mental math a lot.
Sometimes, trial-and-error works. It's not random though.

So assuming first, m and n are integers, we can set $135 = 1 + 3^m + 5^n = 1 + 9 + 125$.
So we can take 2 as m, and take 3 as n.
And thus, we get $135 = 1+ 3^2 + 5^3$.

And assuming next, m, n, and k are integers, we can set $598 = 5^m + 9^n + 8^k$.

$5^m = 5, 25$, and 125 if $m = 1, 2$, and 3. And m cannot be 4 or larger, because $5^4 = 625$.

$9^n = 9$ and 81 if $n = 1$ and 2. And n cannot be 3 or larger, because $9^3 = 729$.

$8^k = 8, 64$, and 512 if $k = 1, 2$, and 3. And k cannot be 4 or larger, because $8^4 = 4096$.

And finding the numbers that can make 598, we can get

$598 = 512 + 81 + 5 = 5^1 + 9^2 + 8^3$.

3. Show that if x is a positive integer and has an odd number of divisors, there is an integer n for which we get $x = n^2$.

To begin with, we have a fact below.

Suppose N is an integer, and $N = A^{\alpha}B^{\beta}C^{\gamma} \dots Z^{\delta}$ where A, B, C … Z are primes, and α, β, γ, … δ are positive integers.
Then, the number of all positive divisors of N is $(\alpha + 1)(\beta + 1)(\gamma + 1)\dots(\delta + 1)$.

So for instance,

$12 = 2^2 3$ has 1, 2, 3, 4, 6, and 12 as divisors, and thus, has 6 divisors.
$16 = 2^4$ has as divisors 1, 2, 4, 8, and 16, and thus, has 5 divisors.
$180 = 2^2 3^2 5$ has as divisors the ones below.
1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, and 180
So it has 18 divisors.
Assuming now, m, n, k … are integers, we can set $x = 2^m 3^n 5^k \dots$.

Then, the number of the divisors of x is $(m + 1)(n + 1)(k + 1)\dots$, and is odd.
So we can say that every one of $m + 1$, $n + 1$, $k + 1$, etc. is odd.
That is to say that all the exponents, m, n, k, etc. are all even.

So assuming p, q, r, … are integers, we can set $m = 2p$, $n = 2q$, $k = 2r$, etc.

And we can get $x = 2^m 3^n 5^k \dots = 2^{2p} 3^{2q} 5^{2r} \dots = (2^p 3^q 5^r \dots)^2$, which is an integer squared.