Examples 8 in Irrationals in Arithmetic 3

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Examples 8 in Irrationals

0. Show that $\sqrt{\mathbf{2}}$ is an irrational number.

1. Show that $\sqrt{\mathbf{3}}$ is an irrational number.

2. Show that $\sqrt{\mathbf{6}}$ is an irrational number.

3. Show that $\sqrt{\mathbf{2}} + \sqrt{\mathbf{3}}$ is an irrational number.

Suggestions or Solutions
To the Examples 8 in Irrationals

0. Show that $\sqrt{2}$ is an irrational number.

A rational number can be put in a ratio between two integers prime to each other.
So assuming $\sqrt{2}$ is a rational number, we can put it this way: $\sqrt{2} = \frac{a}{b}$ where a and b are nonzero integers, and more importantly, are prime to each other.
Then, we can get $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2$.

In the problem 3 in the **Examples Y in Arithmetic 2** however, it was explained that if $a^2 = 2b^2$, a cannot be an integer if b is a nonzero integer. And vice versa.

So if $a^2 = 2b^2$, b cannot be an integer if a is a nonzero integer. And thus, we cannot get $a^2 = 2b^2$ if a and b are nonzero integers, which contradicts the assumption above. So it is not the case where $\sqrt{2}$ is a rational number. That is, it's an irrational number.

Anyway, if $2b^2 = a^2$, we can say that a^2 is an integer multiple of 2, because b^2 is an integer, since b is an integer. So in short, a^2 is an even integer.

And thus, assuming k is an integer, we can set $a^2 = 2k$, where a is an integer, of course.
What integer then, is a?

It is an even integer, too.
We cannot get an even integer squaring an odd integer.
And we get an even integer squaring an even integer.
So assuming n is an integer, we can set $a = 2n$.
Now, we have $2b^2 = a^2$, and $a = 2n$. So we get $2b^2 = 4n^2$, and thus, we get $b^2 = 2n^2$.
So b^2 is an even integer. And thus, b is an even integer, too.
Assuming thus, m is an integer, we can set $b = 2m$. What then, do we get?

We get a contradiction.
It was mentioned that $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.
However, we now have $\boldsymbol{a = 2n}$, and $\boldsymbol{b = 2m}$.
So it is not the case where $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.
And thus, the assumption that $\sqrt{\mathbf{2}}$ is a rational number is not true.
So $\sqrt{\mathbf{2}}$ is an irrational number.

1. Show that $\sqrt{\mathbf{3}}$ is an irrational number.

Suppose $\sqrt{\mathbf{3}}$ is a rational number.

Then, we can put it this way: $\sqrt{\mathbf{3}} = \boldsymbol{a/b}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.
And we can get $\mathbf{3} = \boldsymbol{a}^2/\boldsymbol{b}^2 \Rightarrow \mathbf{3}\boldsymbol{b}^2 = \boldsymbol{a}^2$.

So we can say that $\boldsymbol{a}^2$ is a multiple of 3.
And thus, assuming $\boldsymbol{k}$ is an integer, we can set $\boldsymbol{a}^2 = \mathbf{3}\boldsymbol{k}$, where $\boldsymbol{a}$ is an integer, of course.
What integer then is $\boldsymbol{a}$?

It is a multiple of 3, too.
We cannot get a multiple of 3 squaring an integer that is not a multiple of 3.
Note that a multiple of 6, 9, or such is a multiple of 3.
And we get a multiple of 3 squaring an integer that is a multiple of 3.
So assuming $\boldsymbol{n}$ is an integer, we can set $\boldsymbol{a} = \mathbf{3}\boldsymbol{n}$.

Now, we have $\mathbf{3}\boldsymbol{b}^2 = \boldsymbol{a}^2$, and $\boldsymbol{a} = \mathbf{3}\boldsymbol{n}$.
So we get $\mathbf{3}\boldsymbol{b}^2 = \mathbf{9}\boldsymbol{n}^2$, and thus, we get $\boldsymbol{b}^2 = \mathbf{3}\boldsymbol{n}^2$.
So $\boldsymbol{b}^2$ is a multiple of 3. And thus, $\boldsymbol{b}$ is a multiple of 3, too.
Assuming therefore, $\boldsymbol{m}$ is an integer, we can set $\boldsymbol{b} = \mathbf{3}\boldsymbol{m}$. What then, do we get?

We get a contradiction.
It was mentioned that $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.
However, we now have $\boldsymbol{a} = \mathbf{3}\boldsymbol{n}$, and $\boldsymbol{b} = \mathbf{3}\boldsymbol{m}$.
So it is not the case where $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.
And thus, the assumption that $\sqrt{\mathbf{3}}$ is a rational number is not true.

So it is not the case where $\sqrt{\mathbf{3}}$ is a rational number.
And thus, $\sqrt{\mathbf{3}}$ is an irrational number.

2. Show that $\sqrt{\mathbf{6}}$ is an irrational number.

Suppose $\sqrt{\mathbf{6}}$ is a rational number.
Then, we can put it this way: $\sqrt{\mathbf{6}} = \boldsymbol{a}/\boldsymbol{b}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.
And we can get $\mathbf{6} = \boldsymbol{a}^2/\boldsymbol{b}^2 \Rightarrow \mathbf{6}\boldsymbol{b}^2 = \boldsymbol{a}^2$.

So we can say that $\boldsymbol{a}^2$ is a common multiple of 3 and 2, because it is a multiple of 6.
And thus, assuming $\boldsymbol{k}$ is an integer, we can set $\boldsymbol{a}^2 = \mathbf{6}\boldsymbol{k}$, where $\boldsymbol{a}$ is an integer, of course.
What integer then, is $\boldsymbol{a}$?

It is a multiple of 6.
We cannot get a multiple of 6 squaring an integer that is not a multiple of 6.
Note that a multiple of 12, 18, or such is a multiple of 6.

And we get a multiple of 6 squaring an integer that is a multiple of 6.
So assuming $\boldsymbol{n}$ is an integer, we can set $\boldsymbol{a} = \mathbf{6}\boldsymbol{n}$.

Now, we have $\mathbf{6}\boldsymbol{b}^2 = \boldsymbol{a}^2$, and $\boldsymbol{a} = \mathbf{6}\boldsymbol{n}$.
So we get $\mathbf{6}\boldsymbol{b}^2 = \mathbf{36}\boldsymbol{n}^2$, and thus, we get $\boldsymbol{b}^2 = \mathbf{6}\boldsymbol{n}^2$.
So $\boldsymbol{b}^2$ is a multiple of 6. And thus, $\boldsymbol{b}$ is a multiple of 6, too.
Thus, assuming $\boldsymbol{m}$ is an integer, we can set $\boldsymbol{b} = \mathbf{6}\boldsymbol{m}$. What then, do we get?

We get a contradiction.
It was mentioned that $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.
However, we now have $\boldsymbol{a} = \mathbf{6}\boldsymbol{n}$, and $\boldsymbol{b} = \mathbf{6}\boldsymbol{m}$.
So it is not the case where $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime to each other.
And thus, the assumption that $\sqrt{\mathbf{6}}$ is a rational number is not true.
So $\sqrt{\mathbf{6}}$ is an irrational number.

3. Show that $\sqrt{2}+\sqrt{3}$ is an irrational number.

Suppose $\sqrt{\mathbf{2}}+\sqrt{\mathbf{3}} = \boldsymbol{r}$, where $\boldsymbol{r}$ is a rational number.
Then first, we can put it this way: $\sqrt{\mathbf{2}}=\boldsymbol{r}-\sqrt{\mathbf{3}}$.

So next, squaring both sides, we get

$$\mathbf{2}=(\boldsymbol{r}-\sqrt{\mathbf{3}})^2=\boldsymbol{r}^2-2\boldsymbol{r}\sqrt{\mathbf{3}}+\mathbf{3}\Rightarrow 2\boldsymbol{r}\sqrt{\mathbf{3}}=\boldsymbol{r}^2-\mathbf{1}\Rightarrow\sqrt{\mathbf{3}}=\frac{\boldsymbol{r}^2-\mathbf{1}}{2\boldsymbol{r}}.$$

What then, do we get?

We get a contradiction.

We know $\sqrt{\mathbf{3}}$ is proven irrational in the problem 1.2 above.

However, $\frac{\boldsymbol{r}^2-\mathbf{1}}{2\boldsymbol{r}}$ is a rational number, because $\boldsymbol{r}$ is a rational number.

Doing arithmetic with rational numbers, we get a rational number.

So the assumption that $\sqrt{\mathbf{2}}+\sqrt{\mathbf{3}}$ is a rational number is not true.

And thus, $\sqrt{\mathbf{2}}+\sqrt{\mathbf{3}}$ is an irrational number.