Examples 9 in Irrationals in Arithmetic 3

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Examples 9 in Irrationals

0. Simplify $\sqrt{(x+\sqrt{x^2})^2}-\sqrt{(x-\sqrt{x^2})^2}$.

1. Assuming $x = 3a + b^3$, $y = 3b + a^3$, and $ab = 1$, find the value of the expression as follows.
$\sqrt[3]{(x+y)^2}-\sqrt[3]{(x-y)^2}$.

2. Assuming $u \geq 0$, and $\sqrt{u} = 2a+3$, simplify $\sqrt{u+4a+7}-\sqrt{u-36a+27}$.

Suggestions or Solutions
To the Examples 9 in Irrationals

0. Simplify $\sqrt{(x+\sqrt{x^2})^2}-\sqrt{(x-\sqrt{x^2})^2}$

Assuming first, $x \geq 0$, we get $\sqrt{x^2}=x$.

So we get $\sqrt{(x+\sqrt{x^2})^2}-\sqrt{(x-\sqrt{x^2})^2}=\sqrt{(x+x)^2}-0=\sqrt{(2x)^2}=2x$.

Assuming next, $x < 0$, we get $\sqrt{x^2}=-x$.

So we get $\sqrt{(x+\sqrt{x^2})^2}-\sqrt{(x-\sqrt{x^2})^2}=0-\sqrt{(x+x)^2}=-\sqrt{(2x)^2}=-(-2x)=2x$.

Thus, we get $\sqrt{(x+\sqrt{x^2})^2}-\sqrt{(x-\sqrt{x^2})^2}=2x$.

So putting threads together, we get $\sqrt{(x+\sqrt{x^2})^2}-\sqrt{(x-\sqrt{x^2})^2}=2x$.

1. Assuming $x = 3a + b^3$, $y = 3b + a^3$, and $ab = 1$, find the value of the expression as follows. $\sqrt[3]{(x+y)^2}-\sqrt[3]{(x-y)^2}$. This is just for a algebra practice.

We are going to use a factorization identity $(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a + b)$.

$x + y = a^3 + b^3 + 3(a + b) = (a + b)^3 - 3ab(a + b) + 3(a + b) = (a + b)^3$, since $ab = 1$.

$x - y = b^3 - a^3 + 3(a - b) = (b - a)^3 + 3ab(b - a) - 3(b - a) = (b - a)^3$, since $ab = 1$.

So we get

$\sqrt[3]{(x+y)^2}-\sqrt[3]{(x-y)^2}=\sqrt[3]{(a+b)^6}-\sqrt[3]{(b-a)^6}=(a + b)^2 - (b - a)^2 = 4ab = 4$.

2. Assuming $u \geq 0$, and $\sqrt{u}=2a+3$, simplify $\sqrt{u+4a+7}-\sqrt{u-36a+27}$.

This is just another practice for algebra.

We are given the expression $\sqrt{\boldsymbol{u}} = \mathbf{2}\boldsymbol{a} + \mathbf{3}$, so we want to use it, of course.

Looking at the expression to be simplified, $\boldsymbol{u}$ is a part of what's inside the radical sign.

So it's a good idea to get $\boldsymbol{u}$ out of $\sqrt{\boldsymbol{u}} = \mathbf{2}\boldsymbol{a} + \mathbf{3}$. How?

Squaring both sides, we can get it.

So we get $\sqrt{\boldsymbol{u}} = \mathbf{2}\boldsymbol{a} + \mathbf{3} \Rightarrow \boldsymbol{u} = (\mathbf{2}\boldsymbol{a} + \mathbf{3})^2 = \mathbf{4}\boldsymbol{a}^2 + \mathbf{12}\boldsymbol{a} + \mathbf{9}$.

And next, how are we going to put it into the expression to be simplified?

The expression is made of two radicals, so we may want to take care of each at a time.

And taking care of each, we want to take care of what's inside first.

So to begin with, we get

$\boldsymbol{u} + \mathbf{4}\boldsymbol{a} + \mathbf{7} = \mathbf{4}\boldsymbol{a}^2 + \mathbf{12}\boldsymbol{a} + \mathbf{9} + \mathbf{4}\boldsymbol{a} + \mathbf{7} = \mathbf{4}\boldsymbol{a}^2 + \mathbf{16}\boldsymbol{a} + \mathbf{16} = \mathbf{4}(\boldsymbol{a} + \mathbf{2})^2$.

And next, we get

$\boldsymbol{u} - \mathbf{36}\boldsymbol{a} + \mathbf{27} = \mathbf{4}\boldsymbol{a}^2 + \mathbf{12}\boldsymbol{a} + \mathbf{9} - \mathbf{36}\boldsymbol{a} + \mathbf{27} = \mathbf{4}\boldsymbol{a}^2 - \mathbf{24}\boldsymbol{a} + \mathbf{36} = \mathbf{4}(\boldsymbol{a} - \mathbf{3})^2$.

And thus, we now have $\sqrt{\boldsymbol{u} + \mathbf{4}\boldsymbol{a} + \mathbf{7}} - \sqrt{\boldsymbol{u} - \mathbf{36}\boldsymbol{a} + \mathbf{27}} = \sqrt{\mathbf{4}(\boldsymbol{a} + \mathbf{2})^2} - \sqrt{\mathbf{4}(\boldsymbol{a} - \mathbf{3})^2}$.

So it seems we can rationalize the expression.

That is to say that we can now remove the radical signs.

We want to check the base though. What base?

What's inside the root sign is a power, which is made of a base and an exponent.

So in this case, $(\boldsymbol{a} + \mathbf{2})$ is the base in the power $(\boldsymbol{a} + \mathbf{2})^2$.

About what then, do we check the base?

We want to see if it can be negative or not.

That's because a square root of a real number is ≥ 0.

And in fact, both bases in both powers can be negative.

And we want to begin with the condition that $\boldsymbol{u} \geq \mathbf{0}$.
And we have this, too: $\sqrt{\boldsymbol{u}} = \mathbf{2}\boldsymbol{a} + \mathbf{3}$.
Note that conditions are not given for nothing.

So to begin with, we can get $\boldsymbol{u} \geq \mathbf{0} \Rightarrow \mathbf{2}\boldsymbol{a} + \mathbf{3} \geq \mathbf{0} \Rightarrow \boldsymbol{a} \geq \mathbf{-3/2}$.
And checking with $(\boldsymbol{a} + \mathbf{2})$, we get $\boldsymbol{a} + \mathbf{2} > \mathbf{0}$, since we have $\boldsymbol{a} \geq \mathbf{-3/2}$.
What then, about $(\boldsymbol{a} - \mathbf{3})$?

If $\boldsymbol{a} - \mathbf{3} \geq \mathbf{0}$, we need to have $\boldsymbol{a} \geq \mathbf{3}$. However, we are given $\boldsymbol{a} \geq \mathbf{-3/2}$.
That is to say that if $\mathbf{-3/2} \leq \boldsymbol{a} < \mathbf{3}$, we get $\boldsymbol{a} - \mathbf{3} < \mathbf{0}$.

So putting threads together, we have

$\mathbf{-3/2} \leq \boldsymbol{a} < \mathbf{3} \Rightarrow \boldsymbol{a} - \mathbf{3} < \mathbf{0}$, but $\boldsymbol{a} + \mathbf{2} > \mathbf{0}$.
$\boldsymbol{a} \geq \mathbf{3} \Rightarrow \boldsymbol{a} - \mathbf{3} \geq \mathbf{0}$, and $\boldsymbol{a} + \mathbf{2} > \mathbf{0}$.

Thus first, we get $\mathbf{-3/2} \leq \boldsymbol{a} < \mathbf{3} \Rightarrow \sqrt{\mathbf{4}(\boldsymbol{a}+\mathbf{2})^2} - \sqrt{\mathbf{4}(\boldsymbol{a}-\mathbf{3})^2}$

$= \mathbf{2}(\boldsymbol{a} + \mathbf{2}) - \{\mathbf{-2}(\boldsymbol{a} - \mathbf{3})\} = \mathbf{2}\boldsymbol{a} + \mathbf{4} + \mathbf{2}\boldsymbol{a} - \mathbf{6} = \mathbf{4}\boldsymbol{a} - \mathbf{2}$.

And next, we get $\boldsymbol{a} \geq \mathbf{3} \Rightarrow \sqrt{\mathbf{4}(\boldsymbol{a}+\mathbf{2})^2} - \sqrt{\mathbf{4}(\boldsymbol{a}-\mathbf{3})^2} = \mathbf{2}(\boldsymbol{a} + \mathbf{2}) - \mathbf{2}(\boldsymbol{a} - \mathbf{3}) = \mathbf{10}$.