Examples C in Irrationals in Arithmetic 3

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Examples C in Irrationals

Simplify each expression below.

0. $\sqrt{\frac{7}{6}-\sqrt{\frac{4}{3}}}$

1. $\dfrac{1}{\sqrt{9+4\sqrt{4+\sqrt{12}}}}$

2. $\sqrt{(a+b)^2-\sqrt{8(a^3b+ab^3)}}$

3. $\sqrt{a-\sqrt{a^2-b^2}}$ where $a > b > 0$.

Suggestions or Solutions
To the Examples C in Irrationals

3.0. Simplify $\sqrt{\frac{7}{6}-\sqrt{\frac{4}{3}}}$.

Multiplying 1 by a square root of a number twice, we get the number.
That is to say that squaring a square root, we get what's inside the root sign.

And doing algebra, we often use this tool: $\boldsymbol{a^2 \pm 2ab + b^2 = (a \pm b)^2}$, which is often called a *complete square*. So doing this problem, we are going to use the tool above.

Now, assuming $\boldsymbol{R} = \sqrt{\frac{7}{6}-\sqrt{\frac{4}{3}}}$, and setting $\frac{7}{6}-\sqrt{\frac{4}{3}} = \boldsymbol{(a-b)^2}$, we get $\boldsymbol{R} = \sqrt{\boldsymbol{(a-b)^2}}$.

Then, we get first, $\boldsymbol{R} > \mathbf{0}$, simply because a square root is $\geq \mathbf{0}$, and $\boldsymbol{a - b} \neq \mathbf{0}$.
So we get $\boldsymbol{R = a - b}$ if $\boldsymbol{a - b} > \mathbf{0}$, that is, if $\boldsymbol{a} > \boldsymbol{b}$.

And of course, if $\boldsymbol{a} < \boldsymbol{b}$, that is, if $\boldsymbol{a - b} < \mathbf{0}$, we get $\boldsymbol{R = -(a - b) = b - a}$.
So finding $\boldsymbol{a}$ and $\boldsymbol{b}$, we get $\boldsymbol{R}$. How?

Getting back to the complete square, we have $\boldsymbol{(a-b)^2 = a^2 - 2ab + b^2 = a^2 + b^2 - 2ab}$.

And we have $\frac{7}{6}-\sqrt{\frac{4}{3}} = \boldsymbol{(a-b)^2}$.

So converting $\frac{7}{6}-\sqrt{\frac{4}{3}}$ into a complete square, we can get $\boldsymbol{a}$ and $\boldsymbol{b}$. How?

Looking closely at $\boldsymbol{a^2 + b^2 - 2ab}$ and $\frac{7}{6}-\sqrt{\frac{4}{3}}$, we can notice that if 7/6 is <u>a sum of squares of two numbers</u>, and $\sqrt{\frac{4}{3}}$ is <u>twice the product of the two numbers</u>, we can put $\frac{7}{6}-\sqrt{\frac{4}{3}}$ into a complete square taking a form of $\boldsymbol{(\sqrt{u}-\sqrt{v})^2}$.

That is to say that it can be the case where we get $\frac{7}{6}-\sqrt{\frac{4}{3}} = \boldsymbol{(\sqrt{u}-\sqrt{v})^2}$.

And in fact, we can get $\frac{7}{6}-\sqrt{\frac{4}{3}} = \frac{7}{6} - \mathbf{2}\sqrt{\frac{1}{3}}$. How then can we get $\boldsymbol{u}$ and $\boldsymbol{v}$?

What we want to get is $\frac{7}{6}-\sqrt{\frac{4}{3}}=(\sqrt{u}-\sqrt{v})^2$.

And expanding the right hand side, we get $u + v - 2\sqrt{uv}$.

Then, we get $\frac{7}{6}-2\sqrt{\frac{1}{3}} = u + v - 2\sqrt{uv}$. What then, can we get?

We can get $u + v = \frac{7}{6}$, and $uv = \frac{1}{3}$. How then, can we get u and v?

We can set up an equation as follows. $x^2 - (u + v)x + uv = 0$.
Then, the solution is $x = u$ or v. Why?

We have $x^2 - (u + v)x + uv = 0 \Rightarrow (x - u)(x - v) = 0 \Rightarrow x = u$ or v.

We now have $u + v = \frac{7}{6}$, and $uv = \frac{1}{3}$.

So solving the equation $x^2 - \frac{7}{6}x + \frac{1}{3} = 0$, we can get u and v.

And thus, solving the equation, we get

$$x^2 - \frac{7}{6}x + \frac{1}{3} = 0 \Rightarrow 6x^2 - 7x + 2 = 0 \Rightarrow (3x - 2)(2x - 1) = 0 \Rightarrow x = \frac{2}{3} \text{ or } \frac{1}{2}.$$

That is to say that $(x - \frac{2}{3})(x - \frac{1}{2}) = 0$.
And the equation we began with is $(x - u)(x - v) = 0$.
So we can see that $(u, v) = (\frac{2}{3}, \frac{1}{2})$ or $(\frac{1}{2}, \frac{2}{3})$. Which one of the two then?

We have $\frac{7}{6}-\sqrt{\frac{4}{3}}=(\sqrt{u}-\sqrt{v})^2$. And we have $R=\sqrt{\frac{7}{6}-\sqrt{\frac{4}{3}}}$.

So we get $R=\sqrt{(\sqrt{u}-\sqrt{v})^2}$. And we know $R > 0$.

So if $\sqrt{u} > \sqrt{v}$, we get $R = \sqrt{u} - \sqrt{v}$, and if $\sqrt{u} < \sqrt{v}$, we get $R = \sqrt{v} - \sqrt{u}$.

And thus, either way, we want to get $\boldsymbol{R} > \mathbf{0}$.

So if we set $\boldsymbol{R} = \sqrt{(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}})^2}$, we get $\boldsymbol{R} = \sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}$, since 2/3 > 1/2.

And even if we set $\boldsymbol{R} = \sqrt{(\sqrt{\frac{1}{2}} - \sqrt{\frac{2}{3}})^2}$, we get $\boldsymbol{R} = \sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}$, too.

And thus, we get $\sqrt{\frac{7}{6} - \sqrt{\frac{4}{3}}} = \sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}$.

3.1. Simplify $\dfrac{1}{\sqrt{9 + 4\sqrt{4 + \sqrt{12}}}}$

Adding together the sum of squares of two numbers and twice the product of the two numbers, we get the square of the sum of the two numbers, and the square is called a complete square. That is to say that

we have $\boldsymbol{a}^2 \pm 2\boldsymbol{ab} + \boldsymbol{b}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 \pm 2\boldsymbol{ab} = (\boldsymbol{a} \pm \boldsymbol{b})^2$, often called a complete square.

So setting $\boldsymbol{Q} = \dfrac{1}{\sqrt{9 + 4\sqrt{4 + \sqrt{12}}}}$, we can simplify $\boldsymbol{Q}$ using the idea above.

First, we can convert $4 + \sqrt{12}$ into a form of $(\sqrt{\boldsymbol{u}} - \sqrt{\boldsymbol{v}})^2$ the way below.

$$4 + \sqrt{12} = 4 + 2\sqrt{3} = 3 + 1 + 2\sqrt{3 \cdot 1} = (\sqrt{3} + \sqrt{1})^2 = (\sqrt{3} + 1)^2.$$

Then, we get $\boldsymbol{Q} = \dfrac{1}{\sqrt{9 + 4\sqrt{4 + \sqrt{12}}}} = \dfrac{1}{\sqrt{9 + 4\sqrt{(\sqrt{3} + 1)^2}}} = \dfrac{1}{\sqrt{9 + 4(\sqrt{3} + 1)}} = \dfrac{1}{\sqrt{13 + 4\sqrt{3}}}$.

So next, looking at $\sqrt{13 + 4\sqrt{3}}$, we can notice that we can get

$$13 + 4\sqrt{3} = 13 + 2\sqrt{12} = 12 + 1 + 2\sqrt{12 \cdot 1} = (\sqrt{12} + 1)^2.$$

Thus, we get $\boldsymbol{Q} = \frac{1}{\sqrt{(\sqrt{12}+1)^2}} = \frac{1}{\sqrt{12}+1}$.

And making the denominator rational, that is, rationalizing the denominator, we get

$$\boldsymbol{Q} = = \frac{1}{\sqrt{12}+1} = \frac{\sqrt{12}-1}{(\sqrt{12}+1)(\sqrt{12}-1)} = \frac{\sqrt{12}-1}{12-1} = \frac{\sqrt{12}-1}{11}.$$

3.2. Simplify $\sqrt{(a+b)^2 - \sqrt{8(a^3b+ab^3)}}$

We have $a^2 \pm 2ab + b^2 = a^2 + b^2 \pm 2ab = (a \pm b)^2$.
So we have $(a + b)^2 = a^2 + b^2 + 2ab$.
And we can get notice that $a^3b + ab^3 = ab(a^2 + b^2)$.
So we can get

$$(a+b)^2 - \sqrt{8(a^3b+ab^3)} = a^2 + b^2 + 2ab - 2\sqrt{2ab(a^2+b^2)} = (\sqrt{a^2+b^2} - \sqrt{2ab})^2.$$

Thus, we get $\sqrt{(a+b)^2 - \sqrt{8(a^3b+ab^3)}} = \sqrt{(\sqrt{a^2+b^2} - \sqrt{2ab})^2}$.

And we know $a^2 + b^2 + 2ab = (a + b)^2 \geq 0 \Rightarrow a^2 + b^2 \geq 2ab$.

So we get $\sqrt{(a+b)^2 - \sqrt{8(a^3b+ab^3)}} = \sqrt{a^2+b^2} - \sqrt{2ab}$.

3.3. Simplify $\sqrt{a - \sqrt{a^2 - b^2}}$ where $a > b > 0$.

We have $a - \sqrt{a^2 - b^2} = \frac{2a - 2\sqrt{a^2-b^2}}{2}$.

And we can get $a^2 - b^2 = (a + b)(a - b)$. So what?

We can get this, too: $2a = a + b + a - b = (a + b) + (a - b)$.

So we can get $2a - 2\sqrt{a^2 - b^2} = (a+b) + (a-b) - 2\sqrt{(a+b)(a-b)} = (\sqrt{a+b} - \sqrt{a-b})^2$.

And we have $a > b > 0$. So we get $a + b > a - b$.

Thus, we get $\sqrt{a - \sqrt{a^2 - b^2}} = \sqrt{\frac{2a - 2\sqrt{a^2 - b^2}}{2}} = \sqrt{\frac{(\sqrt{a+b} - \sqrt{a-b})^2}{2}} = \frac{\sqrt{a+b} - \sqrt{a-b}}{\sqrt{2}}$.