Examples D in Irrationals in Arithmetic 3

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Examples D in Irrationals

0. Assuming $\boldsymbol{a} = \frac{1}{\sqrt{2}}$, find the value of $\sqrt{\frac{1+a}{1-a}} - \sqrt{\frac{1-a}{1+a}}$.

1. Assuming $\boldsymbol{b} = \sqrt{3}$, find the value of $\frac{1}{\sqrt{b+1-2\sqrt{b}}} - \frac{1}{\sqrt{b+1+2\sqrt{b}}}$.

2. Assuming $\boldsymbol{c} = \frac{\sqrt{3}}{4}$, find the value of $\frac{1+2c}{1+\sqrt{1+2c}} + \frac{1-2c}{1-\sqrt{1-2c}}$.

3. Assuming $\boldsymbol{d} = \sqrt{2} - 1$, find the value of $\sqrt{\frac{d+1+\sqrt{d^2+2d}}{d+1-\sqrt{d^2+2d}}}$.

4. Simplify $\sqrt[3]{10+\sqrt{108}} + \sqrt[3]{10-\sqrt{108}}$.

5. Find $\boldsymbol{u}$ and $\boldsymbol{v}$ for which $\frac{u}{3-2\sqrt{2}} - \frac{v}{3+2\sqrt{2}} = 3 + 6\sqrt{2}$.

6. Assuming $(u\sqrt{2}+\sqrt{3})u + (v\sqrt{2}+\sqrt{3})v + (w\sqrt{2}+\sqrt{3})w = 5\sqrt{2} + 3\sqrt{3}$, find the value of $uv + vw + wu$.

Suggestions or Solutions
To the Examples D in Irrationals

0. Assuming $a = \frac{1}{\sqrt{2}}$, find the value of $\sqrt{\frac{1+a}{1-a}} - \sqrt{\frac{1-a}{1+a}}$.

$$\sqrt{\frac{1+a}{1-a}} - \sqrt{\frac{1-a}{1+a}} = \frac{\sqrt{1+a}}{\sqrt{1-a}} - \frac{\sqrt{1-a}}{\sqrt{1+a}} = \frac{(\sqrt{1+a})^2-(\sqrt{1-a})^2}{\sqrt{1-a}\sqrt{1+a}} = \frac{(1+a)-(1-a)}{\sqrt{(1-a)(1+a)}} = \frac{2a}{\sqrt{1-a^2}}.$$

And we have $a = \frac{1}{\sqrt{2}}$.

So we get $\frac{2a}{\sqrt{1-a^2}} = \frac{\frac{2}{\sqrt{2}}}{\sqrt{1-\frac{1}{2}}} = \frac{\frac{(\sqrt{2})^2}{\sqrt{2}}}{\sqrt{\frac{1}{2}}} = \frac{\sqrt{2}}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}\cdot\sqrt{2}}{\frac{1}{\sqrt{2}}\cdot\sqrt{2}} = \frac{2}{1} = 2.$

1. Assuming $b = \sqrt{3}$, find the value of $\frac{1}{\sqrt{b+1-2\sqrt{b}}} - \frac{1}{\sqrt{b+1+2\sqrt{b}}}$.

First, we can get

$b+1-2\sqrt{b} = b+1-2\sqrt{b\cdot 1} = (\sqrt{b}-1)^2$, and $b+1+2\sqrt{b} = b+1+2\sqrt{b\cdot 1} = (\sqrt{b}+1)^2$.

So we get $\frac{1}{\sqrt{b+1-2\sqrt{b}}} - \frac{1}{\sqrt{b+1+2\sqrt{b}}} = \frac{1}{\sqrt{(\sqrt{b}-1)^2}} - \frac{1}{\sqrt{(\sqrt{b}+1)^2}} = \frac{1}{\sqrt{b}-1} - \frac{1}{\sqrt{b}+1}.$

Next, we can get $\frac{1}{\sqrt{b}-1} - \frac{1}{\sqrt{b}+1} = \frac{(\sqrt{b}+1)-(\sqrt{b}-1)}{(\sqrt{b}-1)(\sqrt{b}+1)} = \frac{2}{b-1}$. And we have $b = \sqrt{3}$.

So we get $\frac{1}{\sqrt{b+1-2\sqrt{b}}} - \frac{1}{\sqrt{b+1+2\sqrt{b}}} = \frac{2}{b-1} = \frac{2}{\sqrt{3}-1} = \frac{2(\sqrt{3}+1)}{3-1} = \sqrt{3}+1.$

2. Assuming $c = \frac{\sqrt{3}}{4}$, find the value of $\frac{1+2c}{1+\sqrt{1+2c}} + \frac{1-2c}{1-\sqrt{1-2c}}$.

First, we can get $c = \frac{\sqrt{3}}{4} \Rightarrow 1 + 2c = 1 + \frac{2\sqrt{3}}{4} = \frac{4+2\sqrt{3}}{4} = \frac{3+1+2\sqrt{3\cdot 1}}{4} = \frac{(\sqrt{3}+1)^2}{4}$, and also, we get

$c = \frac{\sqrt{3}}{4} \Rightarrow 1 - 2c = 1 - \frac{2\sqrt{3}}{4} = \frac{4-2\sqrt{3}}{4} = \frac{3+1-2\sqrt{3\cdot 1}}{4} = \frac{(\sqrt{3}-1)^2}{4}$.

So next, we get $\sqrt{1+2c} = \sqrt{\frac{(\sqrt{3}+1)^2}{4}} = \frac{\sqrt{3}+1}{2}$, and $\sqrt{1-2c} = \sqrt{\frac{(\sqrt{3}-1)^2}{4}} = \frac{\sqrt{3}-1}{2}$.

Thus next, we get $1 + \sqrt{1+2c} = 1 + \frac{\sqrt{3}+1}{2} = \frac{3+\sqrt{3}}{2}$, and $1 - \sqrt{1-2c} = 1 - \frac{\sqrt{3}-1}{2} = \frac{3-\sqrt{3}}{2}$.

And we have $1 + 2c = \frac{4+2\sqrt{3}}{4} = \frac{2+\sqrt{3}}{2}$, and $1 - 2c = \frac{4-2\sqrt{3}}{4} = \frac{2-\sqrt{3}}{2}$.

So we get $\frac{1+2c}{1+\sqrt{1+2c}} = \frac{\frac{2+\sqrt{3}}{2}}{\frac{3+\sqrt{3}}{2}} = \frac{2+\sqrt{3}}{3+\sqrt{3}}$, and $\frac{1-2c}{1-\sqrt{1-2c}} = \frac{\frac{2-\sqrt{3}}{2}}{\frac{3-\sqrt{3}}{2}} = \frac{2-\sqrt{3}}{3-\sqrt{3}}$.

Thus, we get $\frac{1+2c}{1+\sqrt{1+2c}} + \frac{1-2c}{1-\sqrt{1-2c}} = \frac{2+\sqrt{3}}{3+\sqrt{3}} + \frac{2-\sqrt{3}}{3-\sqrt{3}} = \frac{(2+\sqrt{3})(3-\sqrt{3})+(2-\sqrt{3})(3+\sqrt{3})}{9-3}$

$= \frac{(6+\sqrt{3}-3)+(6-\sqrt{3}-3)}{6} = \frac{6}{6} = 1$.

3. Assuming $d = \sqrt{2} - 1$, find the value of $\sqrt{\frac{d+1+\sqrt{d^2+2d}}{d+1-\sqrt{d^2+2d}}}$.

First, we get $d = \sqrt{2} - 1 \Rightarrow d + 1 = \sqrt{2}$.

So next, we get $(d + 1)^2 = 2 \Rightarrow d^2 + 2d + 1 = 2 \Rightarrow d^2 + 2d = 1$.

And thus, we get $\sqrt{\frac{d+1+\sqrt{d^2+2d}}{d+1-\sqrt{d^2+2d}}} = \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}} = \sqrt{\frac{(\sqrt{2}+1)^2}{2-1}} = \sqrt{2} + 1$.

4. Simplify $\sqrt[3]{10+\sqrt{108}} + \sqrt[3]{10-\sqrt{108}}$.

We have $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$.

So assuming first, $a = \sqrt[3]{10+\sqrt{108}}$, and $b = \sqrt[3]{10-\sqrt{108}}$, we want to find $a + b$.

Then first, we get $\boldsymbol{a}^3 = \mathbf{10} + \sqrt{\mathbf{108}}$, and $\boldsymbol{b}^3 = \mathbf{10} - \sqrt{\mathbf{108}}$, so we get $\boldsymbol{a}^3 + \boldsymbol{b}^3 = \mathbf{20}$.

And next, we get

$$\boldsymbol{ab} = \sqrt[3]{10+\sqrt{108}} \cdot \sqrt[3]{10-\sqrt{108}} = \sqrt[3]{(10+\sqrt{108})(10-\sqrt{108})} = \sqrt[3]{100-108} = \sqrt[3]{-8} = -2.$$

So we get $\boldsymbol{a^3 + b^3 = (a + b)^3 - 3ab(a + b) \Rightarrow 20 = (a + b)^3 + 6(a + b)}$.

How then, can we find $\boldsymbol{a + b}$?

Setting $\boldsymbol{a + b = c}$, we get $\boldsymbol{20 = c^3 + 6c \Rightarrow c^3 + 6c - 20 = 0}$.

So we want to solve the equation for $\boldsymbol{c}$. How?

Factorizing it, we can get $\boldsymbol{c}$.

Assuming $\boldsymbol{c - 2}$ is a factor, we can do a synthetic division the way below.

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 6 & -20 \\ & & 2 & 4 & 20 \\ \hline & 1 & 2 & 10 & 0 \end{array}$$

So we get $\boldsymbol{(c - 2)(c^2 + 2c + 10) = 0}$.

And we get $\boldsymbol{c^2 + 2c + 10 = c^2 + 2c + 1 + 9 = (c + 1)^2 + 9 > 0}$.

So we get $\boldsymbol{c = 2}$. And we know $\boldsymbol{a + b = c}$, where $\boldsymbol{a + b} = \sqrt[3]{10+\sqrt{108}} + \sqrt[3]{10-\sqrt{108}}$.

So we get $\sqrt[3]{10+\sqrt{108}} + \sqrt[3]{10-\sqrt{108}} = 2$.

5. Find $\boldsymbol{u}$ and $\boldsymbol{v}$ for which $\dfrac{u}{3-2\sqrt{2}} - \dfrac{v}{3+2\sqrt{2}} = 3 + 6\sqrt{2}$.

First, simplifying the left hand side, we get

$$\frac{u}{3-2\sqrt{2}} - \frac{v}{3+2\sqrt{2}} = \frac{u(3+2\sqrt{2}) - v(3-2\sqrt{2})}{9-8} = 3u - 3v + (u+v)2\sqrt{2}.$$

So we get $\frac{u}{3-2\sqrt{2}}-\frac{v}{3+2\sqrt{2}}=3+6\sqrt{2} \Rightarrow 3u-3v+(u+v)2\sqrt{2}=3+6\sqrt{2}$

$\Rightarrow 3u - 3v - 3 + (2u + 2v - 6)\sqrt{2} = 0$.

Thus, we get $3u - 3v - 3 = 0$, and $2u + 2v - 6 = 0$.

And next, solving the system above, we get first,

$3u - 3v - 3 = 0 \Rightarrow u - v = 1$, and $2u + 2v - 6 = 0 \Rightarrow u + v = 3$.

So next, we get $(u - v) + (u + v) = 1 + 3 = 4 \Rightarrow 2u = 4 \Rightarrow u = 2$.

And next, we have $u - v = 1$. So we get $2 - v = 1 \Rightarrow v = 1$.

6. Assuming $(u\sqrt{2}+\sqrt{3})u+(v\sqrt{2}+\sqrt{3})v+(w\sqrt{2}+\sqrt{3})w=5\sqrt{2}+3\sqrt{3}$, find the value of $uv + vw + wu$.

First, rewriting the left hand side, we get

$(u\sqrt{2}+\sqrt{3})u+(v\sqrt{2}+\sqrt{3})v+(w\sqrt{2}+\sqrt{3})w=(u^2+v^2+w^2)\sqrt{2}+(u+v+w)\sqrt{3}$.

So we get $(u\sqrt{2}+\sqrt{3})u+(v\sqrt{2}+\sqrt{3})v+(w\sqrt{2}+\sqrt{3})w=5\sqrt{2}+3\sqrt{3}$

$\Rightarrow (u^2+v^2+w^2-5)\sqrt{2}+(u+v+w-3)\sqrt{3}=0$. So we get

$u^2 + v^2 + w^2 - 5 = 0 \Rightarrow u^2 + v^2 + w^2 = 5$, and $u + v + w - 3 = 0 \Rightarrow u + v + w = 3$.

And we have $(u + v + w)^2 = u^2 + v^2 + w^2 + 2(uv + vw + wu)$.

So we get $3^2 = 5 + 2(uv + vw + wu) \Rightarrow uv + vw + wu = 2$.