Practice 4 on Domains and Ranges

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Practice 4 on Domains and Ranges

These are also, for familiarity with functions, particularly, domains and range, and also, for your algebra on inequalities, not quite straightforward sometimes.

Find the range of each function as follows.

0. $y = f(x) = -x^2 + 2x$ for $0 \le x \le 1$ or $-2 \le x \le -1$.

1. $y = f(x) = -x^2 + 2x$ for $0 \le x \le 3$.

2. $y = f(x) = -2x^2$ for $0 \le x < 0.1$ or $-2 \le x \le -1$.

3. $y = f(x) = \frac{1}{x^2}$ for x nonzero real.

4. $y = f(x) = -\frac{1}{x^2}$ for $x > 0$.

5. $y = f(x) = -\frac{1}{x^2}$ for $-1 \le x < 0$ or $x > 0$.

Suggestions or Solutions
To the Problems in Practice 4

0. Find the range of a function $y = f(x) = -x^2 + 2x$ for $0 \le x \le 1$ or $-2 \le x \le -1$.

As stated earlier in the practice set 3, when finding ranges of functions nonlinear, we may not want to apply inequalities only. And this is particularly the case.
So we don't want to apply inequalities only doing this example.

Usually, using the graph of a function, you can get the range readily and quickly.
And more importantly, you can get it safely. So graphing matters. And it does a lot.

If therefore, familiar with the nature of the curve of the function, we can get the range fast and easily, too, doing just simple algebra looking at the curve.

So let's now try getting the range considering the nature of the curve of f.

Examining the expression where $-x^2 + 2x$, we can see it's quadratic, and the coefficient of the quadratic term is -1, which is *negative*, so the curve is a parabola, *concave-down*.

So since the parabola is concave-down, as the value of x increases on the left of the *axis of symmetry*, the value of $f(x)$, that is, the value of y keeps increasing, and as the value of x increases on the right of the axis of symmetry, the value of $f(x)$ keeps decreasing.

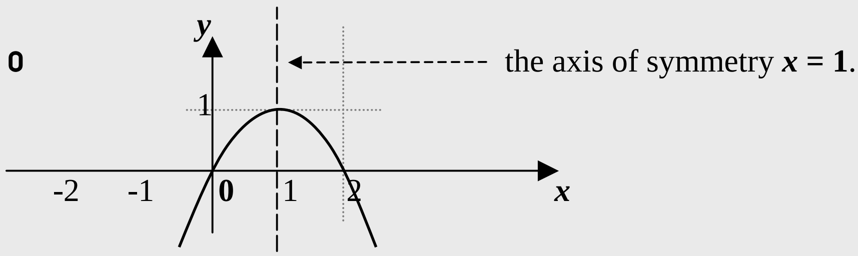
(If not sure of such a coefficient and such an axis of symmetry, refer to **CONICS 2**.)

And we have $y = f(x) = -x^2 + 2x$ for $0 \le x \le 1$ or $-2 \le x \le -1$.

Of this parabola, the axis of symmetry is $x = 1$.

The axis of symmetry of a parabola $y = ax^2 + bx + c$ is a line $x = -\frac{b}{2a}$.

Fig. 0



So the interval where $\mathbf{-2 \le x \le -1}$ is on the left of the axis of symmetry, which is $\mathbf{x = 1}$.
Thus, we can simply get $\mathbf{-2 \le x \le -1 \Rightarrow f(-2) \le f(x) \le f(-1)}$.

And the other interval where $\mathbf{0 \le x \le 1}$ is still on the left of the axis of symmetry.
Thus, we can simply get $\mathbf{0 \le x \le 1 \Rightarrow f(0) \le f(x) \le f(1)}$.

So next, we want to get $\mathbf{f(0)}$, $\mathbf{f(1)}$, $\mathbf{f(-2)}$, and $\mathbf{f(-1)}$.

Then, we get $\mathbf{f(0) = -x^2 + 2x = 0 + 0 = 0}$, and by the same token, we get

$\mathbf{f(1) = -1 + 2 = 1}$, $\mathbf{f(-2) = -4 - 4 = -8}$, and $\mathbf{f(-1) = -1 - 2 = -3}$.

So we get $\mathbf{f(-2) \le f(x) \le f(-1) \Rightarrow -8 \le f(x) \le -3}$, and $\mathbf{f(0) \le f(x) \le f(1) \Rightarrow 0 \le f(x) \le 1}$.

And we have $\mathbf{y = f(x)}$. So we get $\mathbf{-8 \le y \le -3}$ or $\mathbf{0 \le y \le 1}$.

Therefore, the range is $\mathbf{-8 \le y \le -3}$ or $\mathbf{0 \le y \le 1}$.

1. Find the range of a function $y = f(x) = -x^2 + 2x$ for $0 \le x \le 3$.

As stated earlier, when finding ranges of functions nonlinear, we may not want to apply inequalities only. And this is particularly the case, and is in fact, worse than the one in the example 0 above. So we don't want to apply inequalities only, doing this example.

And as mentioned earlier, too, if familiar with the nature of the curve and the expression of the function, we can get the range fast and easily, too, doing just simple algebra, actually looking at the curve.

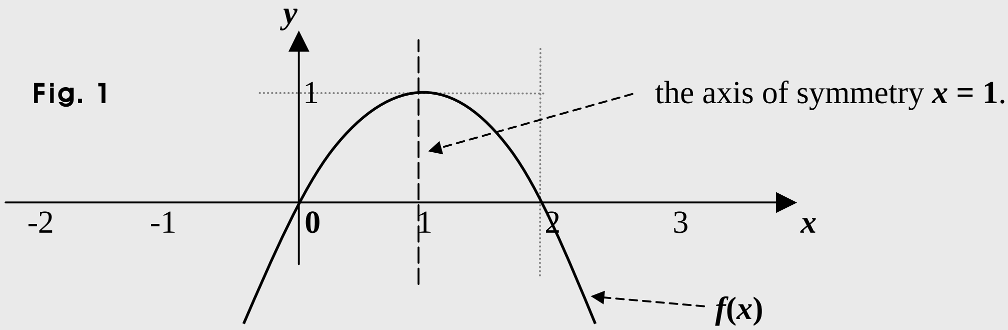
So let's now try getting the range considering the nature of the curve of $\mathbf{f}$.

Examining the expression where $\mathbf{-x^2 + 2x}$, we can see it's quadratic, and the coefficient of the quadratic term is -1, which is *negative*, so the curve is a parabola, *concave-down*.

So since the parabola is concave-down, as the value of $\mathbf{x}$ increases on the left of the *axis of symmetry*, the value of $\mathbf{f(x)}$ keeps increasing, and as the value of $\mathbf{x}$ increases on the right of the axis of symmetry, the value of $\mathbf{f(x)}$ keeps decreasing.

And we have $\boldsymbol{y = f(x) = -x^2 + 2x}$ for $\mathbf{0 \le x \le 3}$, where the axis of symmetry is $\boldsymbol{x = 1}$, because the axis of symmetry of a parabola $\boldsymbol{y = ax^2 + bx + c}$ is a line $\boldsymbol{x = -\frac{b}{2a}}$.

Fig. 1



So in this case, the interval $\mathbf{0 \le x \le 3}$ is not on either side, that is, the axis of symmetry, that is the line $\boldsymbol{x = 1}$ is passing through between both ends of the interval.   So?

We know that the parabola is concave-down. So as $\boldsymbol{x}$ changes from 0 to 3, $\boldsymbol{f}$**(x)** increases up to the point called the vertex, and then, decreases after the vertex. The vertex is thus, the highest point in the parabola.

So we want to get three values, one is $\boldsymbol{f(0)}$, and another is $\boldsymbol{f(3)}$.   What then, is the other?

It is the $\boldsymbol{y}$-coordinate at the vertex. So it is $\boldsymbol{f(1)}$, because the axis of symmetry passes through the vertex, and the axis is a line $\boldsymbol{x = 1}$, so the vertex is at $\boldsymbol{(1, f(1))}$.

Since the parabola is concave-down, the value of $\boldsymbol{f(1)}$ is the greatest value of the function $\boldsymbol{f(x)}$. So it can let us know one end of the interval that is the range of the function $\boldsymbol{f}$. Finding thus, the three values, we can get first

$\boldsymbol{f(0) = -x^2 + 2x = 0 + 0 = 0}$, and by the same token, we get $\boldsymbol{f(1) = -1 + 2 = 1}$, and $\boldsymbol{f(3) = -9 + 6 = -3}$.

So for $\mathbf{0 \le x \le 3}$, we get $\boldsymbol{f(3) \le f(x) \le f(1) \Rightarrow -3 \le f(x) \le 1}$

And we have $\boldsymbol{y = f(x)}$.   So we get $\boldsymbol{-3 \le y \le 1}$, which is the range.

2. Find the range of a function $y = f(x) = -2x^2$ for $0 \le x < 0.1$ or $-2 \le x \le -1$.

Though the expression, that is, the function is not linear in this case, it is quite simple. Anyway, the axis of symmetry is the y-axis, that is, a line $x = 0$. So the axis of symmetry is just at one end of the domain, not within the domain, and thus, should not cause much trouble in finding the range. So let's try this time, getting the range applying inequalities.

The domain is $0 \le x < 0.1$ or $-2 \le x \le -1$. So f holds for $0 \le x < 0.1$ or $-2 \le x \le -1$.
And we have $y = f(x) = -2x^2$. So we get

$$0 \le x < 0.1 \Rightarrow 0 \le x^2 < 0.01 \Rightarrow -0.01 < -x^2 \le 0 \Rightarrow -0.02 < -2x^2 \le 0 \Rightarrow -0.02 < y \le 0.$$

$$-2 \le x \le -1 \Rightarrow 1 < x^2 \le 4 \Rightarrow -4 \le -x^2 < -1 \Rightarrow -8 \le -2x^2 < -2 \Rightarrow -8 < y \le -2.$$

Thus, we get $-0.02 < y \le 0$ for $0 \le x < 0.1$, and also, $-8 \le y < -2$ for $-2 \le x \le -1$.
Therefore, the range is $-0.02 < y \le 0$ or $-8 \le y < -2$.

3. Find the range of a function $y = f(x) = \frac{1}{x^2}$ for x nonzero real.

That x is nonzero real means that the input can be any real number other than 0.
In fact, the function cannot hold for 0 since the denominator is 0 if $x = 0$.

So the domain can be put this way: $x < 0$ or $x > 0$.
Therefore, the function f holds for $x < 0$ or $x > 0$.

Considering the nature of the expression however, we may want to put the domain this way: $0 < |x| < 1$, and $|x| \ge 1$.

Then first, if the magnitude of x is between 0 and 1, x^2 is between 0 and 1, too.
That is, we have $0 < |x| < 1 \Rightarrow 0 < x^2 < 1$. Thus, we get $\frac{1}{x^2} > 1$. How?

If for instance, $x^2 = \frac{1}{4}$, we get $\frac{1}{x^2} = 4$. Thus, we get $0 < |x| < 1 \Rightarrow \frac{1}{x^2} > 1$.
And we have $y = f(x) = \frac{1}{x^2}$. So the range is $y > 1$ if the domain is $0 < |x| < 1$.

And next, if $|x| \geq 1$, we get $x^2 \geq 1$.

So if $|x| \geq 1$, we get $0 < \frac{1}{x^2} \leq 1$. And we have $y = f(x) = \frac{1}{x^2}$, too.

So we get $0 < y \leq 1$ for $|x| \geq 1$. And we have $y > 1$ for $0 < |x| < 1$, too.

Putting thus, threads together, we have $y > 1$ or $0 < y \leq 1$. So the range is $y > 0$.

4. Find the range of a function $y = f(x) = -\frac{1}{x^2}$ for $x > 0$.

The domain is $x > 0$. So the function f is defined for $x > 0$.

Though the domain is just one interval, we want to consider two cases. Why?

It's because of the nature of the expression. And the two cases are $0 < x < 1$, and $x \geq 1$.

Now first, if x is between 0 and 1, x^2 is between 0 and 1, too.

That is, we have $0 < x < 1 \Rightarrow 0 < x^2 < 1$. So we get $\frac{1}{x^2} > 1$ when $0 < x < 1$.

We have $y = f(x) = -\frac{1}{x^2}$, though. Thus, we get $y = -\frac{1}{x^2} < -1$ for $0 < x < 1$.

So the range is $y < -1$ if the domain is $0 < x < 1$.

And next, if $x \geq 1$, we get $x^2 \geq 1$. So we get $0 < \frac{1}{x^2} \leq 1$, since $\frac{1}{x^2}$ cannot be ≤ 0.

So if $x \geq 1$, we get $0 < \frac{1}{x^2} \leq 1$. We have $y = f(x) = -\frac{1}{x^2}$, though.

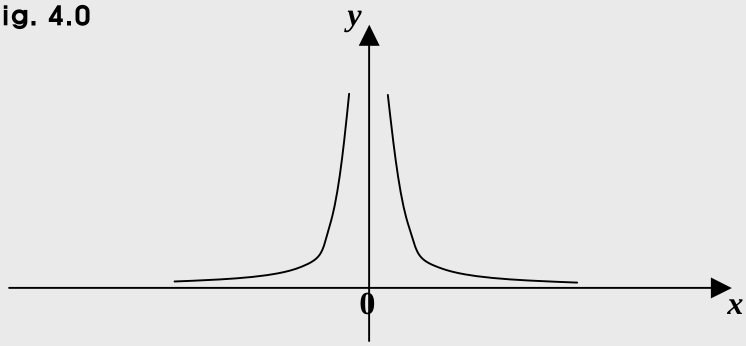
Thus, we get $-1 \leq -\frac{1}{x^2} < 0$.

So we get $-1 \leq y < 0$ for $x \geq 1$. And yet, we have $y < -1$ for $0 < x < 1$, too.

Thus, we have $-1 \leq y < 0$ or $y < -1$. So we get $y < 0$, and therefore, the range is $y < 0$.

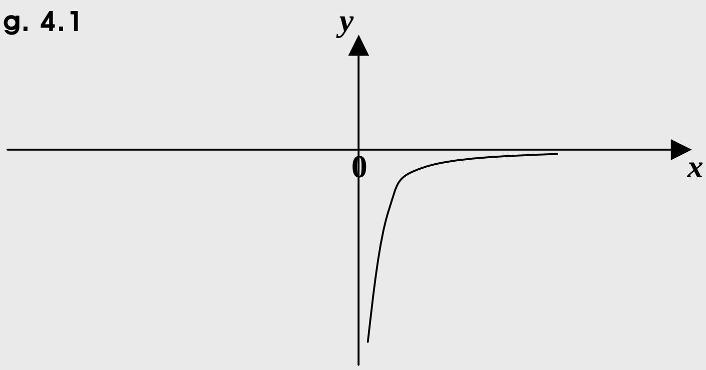
And in fact, putting in a graph the curve of $y = f(x) = \frac{1}{x^2}$ for x nonzero real, we get

Fig. 4.0



And putting in a graph the curve of $y = f(x) = -\frac{1}{x^2}$ for $x > 0$, we get

Fig. 4.1



5. Find the range of a function where $y = f(x) = -\frac{1}{x^2}$ for $-1 \le x < 0$ or $x > 0$.

The domain is $-1 \le x < 0$ or $x > 0$. So f is defined for $-1 \le x < 0$ or $x > 0$.

Though the domain is made of two intervals, yet we may want to consider three cases.

That is because of the nature of the expression. And the three cases are as follows.

- $-1 < x < 0$ or $0 < x < 1$, that is, the magnitude of x is between 0 and 1, so $0 < |x| < 1$.
- $x \ge 1$.
- $x = -1$.

Then first, if $\mathbf{0 < |x| < 1}$, we get $\mathbf{0 < x^2 < 1}$. So we get $\frac{1}{x^2} > \mathbf{1}$.

We have $y = f(x) = -\frac{1}{x^2}$, though. Then, we get $-\frac{1}{x^2} < \mathbf{-1}$. So we get $\boldsymbol{y < -1}$ for $\mathbf{0 < |x| < 1}$.

Next, if $\boldsymbol{x \geq 1}$, we get $\boldsymbol{x^2 \geq 1}$. So we get $\mathbf{0} < \frac{1}{x^2} \leq \mathbf{1}$, since $\frac{1}{x^2}$ cannot be $\leq \mathbf{0}$.

We have $y = f(x) = -\frac{1}{x^2}$, though. Thus, we get $\mathbf{-1} \leq -\frac{1}{x^2} < \mathbf{0}$.

So we get $\boldsymbol{-1 \leq y < 0}$ for $\boldsymbol{x \geq 1}$.

And next, if $\boldsymbol{x = -1}$, we simply get $-\frac{1}{x^2} = \mathbf{-1}$.

Now, putting threads together, we get $\boldsymbol{y = -1}$, $\boldsymbol{y < -1}$, or $\boldsymbol{-1 \leq y < 0}$.

Thus, the range is $\boldsymbol{y < 0}$.

And, putting in a graph the curve of $y = f(x) = -\frac{1}{x^2}$ for $\boldsymbol{-1 \leq x < 0}$ or $\boldsymbol{x > 0}$, we get

Fig. 5.0