Examples 1 in Composite Functions

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Find each of all the composite functions below using the definitions as follows.
$\boldsymbol{f}$**:** $\boldsymbol{x} \longrightarrow \boldsymbol{x}$, $\boldsymbol{g}$**:** $\boldsymbol{x} \longrightarrow \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{1}$, and $\boldsymbol{h}$**:** $\boldsymbol{x} \longrightarrow \mathbf{3}\boldsymbol{x}^{\mathbf{3}} + \mathbf{2}$.

0. $g \bullet f$

1. $f \bullet g$

2. $g \bullet h$

3. $h \bullet g$

4. $h \bullet (g \bullet f)$

5. $(h \bullet g) \bullet f$

6. $h \bullet (g \bullet g)$

7. $(h \bullet g) \bullet g$

Suggestions or Solutions

To the Problem in the Example 0

Find $g \bullet f$ using the definition as follows. f: $x \longrightarrow x$, and g: $x \longrightarrow 2x^2 + 1$.

$f(x) = x$, and $g(x) = 2x^2 + 1$, so $g \bullet f = g\{f(x)\} = 2\{f(x)\}^2 + 1 = 2x^2 + 1$ for x real.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, if not specified, the domain is assumed to be a set of all real numbers, or the set of all numbers the function can be defined for. More specifically, it is the set of all numbers the expression can hold for. For instance, $\sqrt{x+1}$ can hold for $x = 1$, but cannot hold for $x = -2$, because what's inside the square root cannot be negative.

Next, if R is a set of all real numbers, the expressions of f and g both can hold for R. So the domain of f is R, so is the range, and so is the domain of g.

And thus, the function f is defined in R, since R is the domain.

Where then, is a composite function defined?

A composite function is defined in the domain of the function blends into. Also, the range of the function blends into has to be a subset of the domain of the other function.

Now, if we set $p = g \bullet f$, p is a composite function, where f is the function blends into.

So p has to be defined in the domain of f, and the range of f has to be a subset of the domain of g. And for instance, a subset of a set A is equal to or a part of the set A.

We know the range of f is R, and so is the domain of g. That is, the range of f is a subset of the domain of g. Thus, we should be able to get the composite function p in this case.

So assuming $\boldsymbol{p = g \bullet f}$, we have $\boldsymbol{y = p(x) = (g \bullet f)(x)}$, and can put it this way, too.

$$\begin{tikzcd} X_f \arrow[r, "f"] \arrow[rr, bend right, "p"'] & Y_f \arrow[r, "g"] & Y_p \end{tikzcd} \quad \Rightarrow \quad X_f \xrightarrow{g \bullet f} Y_p.$$

And of course, $\boldsymbol{X_f = X_p}$, and $\boldsymbol{p}$'s range is $\boldsymbol{Y_p}$.

Now, in the diagram above, $\boldsymbol{X_f}$ is the domain of $\boldsymbol{f}$, the function blends into, and thus, is the domain of $\boldsymbol{p}$. So lets now get the expression of $\boldsymbol{p}$.

We have $\boldsymbol{y = p(x) = (g \bullet f)(x) = g(f)}$, $\boldsymbol{y = g(x) = 2x^2 + 1}$, and $\boldsymbol{y = f(x) = x}$.

So we get $\boldsymbol{p(x) = g(f) = 2f^2 + 1 = 2x^2 + 1}$, since $\boldsymbol{f(x) = x}$.

Thus, we get $\boldsymbol{y = (g \bullet f)(x) = 2x^2 + 1}$ for $\boldsymbol{x}$ real, which is the same as the function $\boldsymbol{g}$ given.

We know $\boldsymbol{f}$ is an identity function. So getting a composite function mixing two functions, we just get the other function if the function blends into is an identity function. It doesn't matter though, which one is the function blends into.

That is, if one of the two is an identity function, we just get the other function.

In short:

$\boldsymbol{f(x) = x}$, and $\boldsymbol{g(x) = 2x^2 + 1}$, so $\boldsymbol{g \bullet f = g\{f(x)\} = 2\{f(x)\}^2 + 1 = 2x^2 + 1}$ for $\boldsymbol{x}$ real.

Suggestions or Solutions
To the Problem in the Example 1

Find $f \bullet g$ using the definition as follows. f: $x \longrightarrow x$, and g: $x \longrightarrow 2x^2 + 1$.

$\boldsymbol{f(x) = x}$, and $\boldsymbol{g(x) = 2x^2 + 1}$, so $\boldsymbol{f \bullet g = f\{g(x)\} = g(x) = 2x^2 + 1}$ for $\boldsymbol{x}$ real.

If not quite sure of the idea behind the processes above, follow the steps below.

In this problem, too, we want to get a composite function mixing two functions, which are though, the same ones we used in the previous problem. In this problem though, the mixing sequence is the opposite.
So first, assuming $\boldsymbol{y = p(x) = (f \bullet g)(x)}$, we have

$$\begin{tikzcd} X_g \arrow[r, "g"] \arrow[rr, bend right, "p"'] & Y_g \arrow[r, "f"] & Y_p \end{tikzcd} \quad \Rightarrow \quad X_g \xrightarrow{f \bullet g} Y_p.$$

And in $\boldsymbol{p}$, we have $\boldsymbol{X_p = X_g}$, of course.

So this time, of the two functions $\boldsymbol{f}$ and $\boldsymbol{g}$, $\boldsymbol{g}$ is the one blends into. And thus, the range of $\boldsymbol{g}$ needs to fit the domain of $\boldsymbol{f}$.

In the previous problem, we've already found that the range of $\boldsymbol{g}$ is $\boldsymbol{y \geq 1}$, and that the domain of $\boldsymbol{f}$ is $\boldsymbol{R}$, which is a set of all real numbers. So the range of the functions blends into is a subset of the domain of the other function, and thus, we can get the composite function that can be defined in the domain of $\boldsymbol{g}$, which is $\boldsymbol{R}$, of course.

So next, we want to get the expression of $\boldsymbol{p}$.

We have $\boldsymbol{y = p(x) = (f \bullet g)(x) = f(g)}$, $\boldsymbol{y = g(x) = 2x^2 + 1}$, and $\boldsymbol{y = f(x) = x}$.

Thus, we get $\boldsymbol{p(x) = f(g) = g}$, since $\boldsymbol{f(x) = x}$. And we know $\boldsymbol{g = 2x^2 + 1}$.

So we get $\boldsymbol{p(x) = 2x^2 + 1}$, which has, of course, no problem with the domain $\boldsymbol{R}$, and therefore, we get $\boldsymbol{y = (f \bullet g)(x) = 2x^2 + 1}$ for $\boldsymbol{x}$ real, which is again, the same as the function $\boldsymbol{g}$ given. And we know $\boldsymbol{f}$ is an identity function.

So considering the previous problem, we can be sure that getting a composite function using two functions, one of which is an identity function, we just get the other function.
That is, if $\boldsymbol{f}$ is an identity function, we just get $\boldsymbol{f \bullet g = g \bullet f = g}$.

Suggestions or Solutions

To the Problem in the Example 2

Find $g \bullet h$ using the definition as follows. g: $x \longrightarrow 2x^2 + 1$, and h: $x \longrightarrow 3x^3 + 2$.

> $g(x) = 2x^2 + 1$, and $h(x) = 3x^3 + 2$, so $g \bullet h = g\{h(x)\} = g(3x^3 + 2) = 2(3x^3 + 2)^2 + 1$ for x real.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, assuming $y = p(x) = (g \bullet h)(x)$, we have

$$\begin{tikzcd} X_h \arrow[r, "h"] \arrow[rr, bend right, "p"'] & Y_h \arrow[r, "g"] & Y_p \end{tikzcd} \quad \Rightarrow \quad X_h \xrightarrow{g \bullet h} Y_p.$$

And in p, we have $X_p = X_h$, of course.

So of the two functions h and g we want to mix, h is the function blends into. And thus, the range of h has to fit the domain of g.

The range of h is $y \geq 2$, and the domain of g is R, which is a set of all real numbers.

So the range is a subset of the domain, and thus, we can get the composite function that can be defined in the domain of the function blends into.

And the domain is X_h, which is R, which is the domain of h, of course.

Thus next, we want to get the expression of p.

Now, we have $y = p(x) = (g \bullet h)(x) = g(h)$, $y = g(x) = 2x^2 + 1$, and $y = h(x) = 3x^3 + 2$.

So we get $p(x) = g(h) = 2h^2 + 1$, since $g(x) = 2x^2 + 1$. And thus, $p(x) = 2(3x^3 + 2)^2 + 1$, which is perfectly defined for $x \in R$, which is the domain of the function h blends into.

And next, finding the ranges of p and g, we can see the range of g is $y \geq 1$, so is the range of p. So p is a composite function from the domain of h to the range of g.

How is $\boldsymbol{y \geq 1}$ though, the range of $\boldsymbol{p}$?

The expression in $\boldsymbol{p}$ is just an algebraic expression of degree 6, and we don't have to expand (or simplify) the expression if not necessary, of course.

Expanding it though, we get $\mathbf{2(3}\boldsymbol{x}^{\mathbf{3}} \mathbf{+ 2)^2 + 1 = 2(9}\boldsymbol{x}^{\mathbf{6}} \mathbf{+ 12}\boldsymbol{x}^{\mathbf{3}} \mathbf{+ 4) + 1 = 18}\boldsymbol{x}^{\mathbf{6}} \mathbf{+ 24}\boldsymbol{x}^{\mathbf{3}} \mathbf{+ 9}$.

Now, in the expression of $\boldsymbol{p}$, we have $\mathbf{2(3}\boldsymbol{x}^{\mathbf{3}} \mathbf{+ 2)^2}$, which is however, $\geq \mathbf{0}$.

That's because

First, $\boldsymbol{x}^{\mathbf{3}}$ can be any number negative as well as 0 or positive, since $\boldsymbol{x \in R}$, and thus, can be -2. So $\mathbf{3}\boldsymbol{x}^{\mathbf{3}} \mathbf{+ 2}$ can be 0.

Next, $\mathbf{(3}\boldsymbol{x}^{\mathbf{3}} \mathbf{+ 2)^2}$ cannot be $< \mathbf{0}$, because any number squared is $\geq \mathbf{0}$.

And thus, we get $\mathbf{2(3}\boldsymbol{x}^{\mathbf{3}} \mathbf{+ 2)^2 + 1 \geq 1} \Rightarrow \boldsymbol{y} \mathbf{\geq 1}$, which is the range of $\boldsymbol{p}$.

And of course, setting $\boldsymbol{t = x}^{\mathbf{3}}$, we just get $\mathbf{2(3}\boldsymbol{x}^{\mathbf{3}} \mathbf{+ 2)^2 + 1 = 2(3}\boldsymbol{t} \mathbf{+ 2)^2 + 1 \geq 1}$, too.

Therefore, we get $\boldsymbol{y} \mathbf{=} (\boldsymbol{g \bullet h})(\boldsymbol{x}) \mathbf{= 2(3}\boldsymbol{x}^{\mathbf{3}} \mathbf{+ 2)^2 + 1}$ for $\boldsymbol{x}$ real.

In short:

$\boldsymbol{g}(\boldsymbol{x}) \mathbf{= 2}\boldsymbol{x}^{\mathbf{2}} \mathbf{+ 1}$, and $\boldsymbol{h}(\boldsymbol{x}) \mathbf{= 3}\boldsymbol{x}^{\mathbf{3}} \mathbf{+ 2}$, so $\boldsymbol{g \bullet h} = \boldsymbol{g}\{\boldsymbol{h}(\boldsymbol{x})\} = \boldsymbol{g}(\mathbf{3}\boldsymbol{x}^{\mathbf{3}} \mathbf{+ 2}) \mathbf{= 2(3}\boldsymbol{x}^{\mathbf{3}} \mathbf{+ 2)^2 + 1}$ for $\boldsymbol{x}$ real.

Suggestions or Solutions

To the Problem in the Example 3

Find $h \bullet g$ using the definition as follows. g: $x \longrightarrow 2x^2 + 1$, and h: $x \longrightarrow 3x^3 + 2$.

$\boldsymbol{g(x) = 2x^2 + 1}$, and $\boldsymbol{h(x) = 3x^3 + 2}$, so $\boldsymbol{h \bullet g = h\{g(x)\} = h(2x^2 + 1) = 3(2x^2 + 1)^3 + 2}$ for $\boldsymbol{x}$ real.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, assuming $\boldsymbol{y = p(x) = (h \bullet g)(x)}$, we have

$$\begin{tikzcd} X_g \arrow[r, "g"] \arrow[rr, bend right, "p"'] & Y_g \arrow[r, "h"] & Y_p \end{tikzcd} \quad \Rightarrow \quad X_g \xrightarrow{h \bullet g} Y_p.$$

And in $\boldsymbol{p}$, we have $\boldsymbol{X_p = X_g}$, of course.

So on the contrary to the previous problem, of the two functions $\boldsymbol{h}$ and $\boldsymbol{g}$ we want to mix, which one is the function blends into?

It is the function $\boldsymbol{g}$.

And thus, the range of $\boldsymbol{g}$ has to fit the domain of $\boldsymbol{h}$.

What then, is the range of $\boldsymbol{g}$?

The range of $\boldsymbol{g}$ is $\boldsymbol{y \geq 1}$, and the domain of $\boldsymbol{h}$ is $\boldsymbol{R}$, which is a set of all real numbers.

So the range of $\boldsymbol{g}$ is a subset of the domain of $\boldsymbol{h}$, and thus, we can get the composite function $\boldsymbol{p}$ that can be defined in the domain of the function blends into.

And the domain of $\boldsymbol{p}$ is $\boldsymbol{X_g}$, which is $\boldsymbol{R}$, which is of course, the domain of $\boldsymbol{g}$.

So next, we want to get the expression of $\boldsymbol{p}$.

We have $\boldsymbol{y = p(x) = (h \bullet g)(x) = h(g)}$, $\boldsymbol{y = g(x) = 2x^2 + 1}$, and $\boldsymbol{y = h(x) = 3x^3 + 2}$.

So we get $\boldsymbol{p(x) = h(g) = 3g^3 + 2}$, since $\boldsymbol{h(x) = 3x^3 + 2}$. And we have $\boldsymbol{g = 2x^2 + 1}$.

Thus, we get $\boldsymbol{p(x) = 3(2x^2 + 1)^3 + 2}$, which can be defined for $\boldsymbol{x \in R}$, which is $\boldsymbol{X_g}$, which is the function blends into. And let's next, check the ranges of $\boldsymbol{p}$ and $\boldsymbol{g}$.

Beginning with the range of $\boldsymbol{g}$, we get $\boldsymbol{y \geq 1}$. And next, finding the range of $\boldsymbol{p}$, we get $\boldsymbol{2x^2 + 1 \geq 1 \Rightarrow (2x^2 + 1)^3 \geq 1 \Rightarrow 3(2x^2 + 1)^3 \geq 3 \Rightarrow 3(2x^2 + 1)^3 + 2 \geq 5 \Rightarrow y \geq 5}$.

So the range of $\boldsymbol{p}$ is just a part of the range of $\boldsymbol{g}$. That's simply because the range in the function blends into is a part of the domain of the other function.

And thus, $\boldsymbol{p}$ is not a function from the domain of the function blends into to the range of the other function.

The function $\boldsymbol{p}$ is however, a composite function perfectly working in the domain of the function blends into.

Therefore, the solution to this problem is $\boldsymbol{y = (h \bullet g)(x) = 3(2x^2 + 1)^3 + 2}$ for $\boldsymbol{x}$ real.

And comparing the expression above to the one in the previous problem, we can see again, that it's *not* the case where we get $\boldsymbol{h \bullet g = g \bullet h}$ if $\boldsymbol{g}$ and $\boldsymbol{h}$ are the two functions given.

So we can say that in general, we get $\boldsymbol{h \bullet g \neq g \bullet h}$.

In short:

$\boldsymbol{g(x) = 2x^2 + 1}$, and $\boldsymbol{h(x) = 3x^3 + 2}$, so $\boldsymbol{h \bullet g = h\{g(x)\} = h(2x^2 + 1) = 3(2x^2 + 1)^3 + 2}$ for $\boldsymbol{x}$ real.

Suggestions or Solutions

To the Problem in the Example 4

Find $h\bullet(g\bullet f)$ using the definition below.
$f: x \longrightarrow x$, $g: x \longrightarrow 2x^2 + 1$, and $h: x \longrightarrow 3x^3 + 2$.

$f(x) = x$, $g(x) = 2x^2 + 1$, and $h(x) = 3x^3 + 2$.

So first, we get $g\bullet f = g\{f(x)\} = 2f^2 + 1 = 2x^2 + 1$ for x real.

Thus, $h\bullet(g\bullet f) = h\bullet g = h(g) = h(2x^2 + 1) = 3(2x^2 + 1)^3 + 2 = 24x^6 + 36x^4 + 18x^2 + 5$ for x real.

If not quite sure of the idea behind the processes above, follow the steps below.

We can get a composite function mixing more than two functions. We don't however, just mix together all at once. We mix two at a time sequential manner.

Assuming for instance, mixing three functions, we mix first, two of the three, and get the blend of the two.

Then, mixing together the blend and the other function, we get a composite function, which is therefore, the blend of all the three functions.

We cannot do so however, if all the domains and ranges facing each other do not fit each other. That is, of the two functions to mix, if the range of the function blends into is not a subset of the domain of the other function, the two functions don't mix, so we get no composite function.

To begin with, $g\bullet f$ is a composite function, which is thus, a function, too, so calling it p, we get $h\bullet(g\bullet f) = h\bullet k$, which is a composite function, too, and is a function, also, so calling it q, we get $q = h\bullet k$. And thus, we have $q = h\bullet(g\bullet f)$.

And the same is true, too, for $h\bullet g\bullet f$, which is therefore, equal to $h\bullet(g\bullet f)$.

So making a composite function using more than two functions, we combine two at a time in sequential manner as stated above. Then, the last composite function is the one we want, and is a blend of all the functions used. And thus, we want to begin with $\boldsymbol{g \bullet f}$.

So first, assuming $\boldsymbol{y = p(x) = (g \bullet f)(x)}$, we have

$$\begin{tikzcd} X_f \arrow[r, "f"] \arrow[rr, bend right, "p"'] & Y_f \arrow[r, "g"] & Y_p \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} X_f \arrow[r, "g \bullet f"] & Y_p \end{tikzcd}$$

In $\boldsymbol{p}$, we have $\boldsymbol{X_f = X_p}$, of course.

And assuming next, $\boldsymbol{R}$ is a set of all real numbers, we need to assume that if the domain is unspecified in a function, it is a set of all real numbers.

And if in a function, the expression is a monomial or polynomial as the one in $\boldsymbol{f}$, $\boldsymbol{g}$ or $\boldsymbol{h}$, the domain is $\boldsymbol{R}$. It is *not* always the case however, the range is $\boldsymbol{R}$, too. The range of $\boldsymbol{h}$ is $\boldsymbol{R}$, but that of $\boldsymbol{g}$ is $\boldsymbol{y \geq 1}$.

And a composite function is defined in the domain of the function blends into. Also, the range of the function blends into has to be a subset of the domain of the other function.

Now, the range of $\boldsymbol{f}$ is $\boldsymbol{R}$, so is the domain of $\boldsymbol{g}$, so we can get the composite function $\boldsymbol{p}$. And $\boldsymbol{f}$ is the function blends into, and its domain is $\boldsymbol{R}$, so the domain of $\boldsymbol{p}$ is $\boldsymbol{R}$.

Thus next, we want to get the expression of $\boldsymbol{p}$.

We have $\boldsymbol{y = p(x) = (g \bullet f)(x) = g(f)}$, $\boldsymbol{y = g(x) = 2x^2 + 1}$, and $\boldsymbol{y = f(x) = x}$, which is an identity function. So we just get $\boldsymbol{p(x) = g(x) = 2x^2 + 1}$ for $\boldsymbol{x}$ real.

So next, we want to move on to $\boldsymbol{h \bullet (g \bullet f)}$, which is equal to $\boldsymbol{h \bullet p}$.

Thus first, assuming $\boldsymbol{y = q(x) = (h \bullet p)(x)}$, we have

$$\begin{tikzcd} X_p \arrow[r, "p"] \arrow[rr, bend right, "q"'] & Y_p \arrow[r, "h"] & Y_q \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} X_p \arrow[r, "h \bullet p"] & Y_q \end{tikzcd}$$

In $\boldsymbol{q}$, we have $\boldsymbol{X_q = X_p = X_f}$, of course.

So again, a composite function is defined in the domain of the function blends into. And also, the range of the function blends into has to be a subset of the domain of the other function. We want to check this every time we do blending.

Thus first, $\boldsymbol{p}$ is the function blends into, so $\boldsymbol{q}$ has to be composite function defined in the domain of $\boldsymbol{p}$, where the domain is originally the domain of $\boldsymbol{f}$.

And next, the range of $\boldsymbol{p}$ is $\boldsymbol{y \geq 1}$, and the domain of $\boldsymbol{h}$ is $\boldsymbol{R}$. So we can get the composite function $\boldsymbol{q}$, since the range of $\boldsymbol{p}$ is a subset of the domain of $\boldsymbol{h}$.

So next, we want to get the expression of $\boldsymbol{q}$.

We have $\boldsymbol{y = q(x) = (h \bullet p)(x) = h(p)}$, $\boldsymbol{y = p(x) = 2x^2 + 1}$, and $\boldsymbol{y = h(x) = 3x^3 + 2}$.

So we get $\boldsymbol{q(x) = h(p) = 3p^3 + 2}$, since $\boldsymbol{h(x) = 3x^3 + 2}$. And we have $\boldsymbol{p = 2x^2 + 1}$.

Thus, we get $\boldsymbol{q(x) = 3(2x^2 + 1)^3 + 2}$. And we know $\boldsymbol{q = h \bullet (g \bullet f)}$, and the domain of $\boldsymbol{q}$ is $\boldsymbol{R}$.

So we get $\boldsymbol{y = h \bullet (g \bullet f)(x) = 3(2x^2 + 1)^3 + 2}$ for $\boldsymbol{x}$ real.

In the problem 3, we've found that if $\boldsymbol{x \in R}$, we get $\boldsymbol{3(2x^2 + 1)^3 + 2 \geq 5}$.

So the range of $\boldsymbol{q}$ is $\boldsymbol{y \geq 5}$, which is thus, $\boldsymbol{Y_q}$ in the diagram above.

However, the domain of $\boldsymbol{h}$ is $\boldsymbol{R}$. So $\boldsymbol{q}$ is not a function from the domain of f to the range of g. That's because the range of $\boldsymbol{p}$, the function blends into is not equal to the domain of the other function $\boldsymbol{h}$.

However, $\boldsymbol{q}$ is a composite function perfectly working in the domain of the function blends into. So the solution to this problem is $\boldsymbol{q}$.

In short:

$\boldsymbol{f(x) = x}$, $\boldsymbol{g(x) = 2x^2 + 1}$, and $\boldsymbol{h(x) = 3x^3 + 2}$.

So first, we get $\boldsymbol{g \bullet f = g\{f(x)\} = 2f^2 + 1 = 2x^2 + 1}$.

Thus, $\boldsymbol{h \bullet (g \bullet f) = h \bullet g = h(g) = h(2x^2 + 1) = 3(2x^2 + 1)^3 + 2 = 24x^6 + 36x^4 + 18x^2 + 5}$.

Suggestions or Solutions
To the Problem in the Example 5

Find $(h \bullet g) \bullet f$ using the definition below.
$f: x \longrightarrow x$, $g: x \longrightarrow 2x^2 + 1$, and $h: x \longrightarrow 3x^3 + 2$.

> $f(x) = x$, $g(x) = 2x^2 + 1$, and $h(x) = 3x^3 + 2$.
>
> Thus first, we get $(h \bullet g) \bullet f = h \bullet g$, because f is an identity function.
>
> So $(h \bullet g) \bullet f = h \bullet g = h(g) = 3g^3 + 2 = 3(2x^2 + 1)^3 + 2 = 24x^6 + 36x^4 + 18x^2 + 5$ for x real.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, in $(h \bullet g) \bullet f$, we know $h \bullet g$ is a composite function.

So assuming $p = h \bullet g$, we get $(h \bullet g) \bullet f = p \bullet f$.

And calling it q, we get $q = p \bullet f$, which is of course, equal to $(h \bullet g) \bullet f$.

So we have $q = (h \bullet g) \bullet f$.

And thus, we want to begin with $h \bullet g$, where g is the function mixes into h.

So first, assuming $y = p(x) = (h \bullet g)(x)$, we have

$$\begin{tikzcd} X_g \arrow[r, "g"] \arrow[rr, bend right, "p"'] & Y_g \arrow[r, "h"] & Y_p \end{tikzcd} \quad \Rightarrow \quad X_g \xrightarrow{h \bullet g} Y_p$$

In p, we have $X_p = X_g$, of course.

And a composite function is defined in the domain of the function blends into. Also, the range of the function blends into has to be a subset of the domain of the other function.

Now, the function g blends into the function h, the range of g is $y \geq 1$, and the domain of h is R, which is a set of all real numbers.

So we can get the composite function $\boldsymbol{p}$, defined in the domain of $\boldsymbol{g}$, and the domain is $\boldsymbol{R}$.

Thus next, we want to get the expression of $\boldsymbol{p}$.

How then, do we get the expression of $\boldsymbol{p}$?

We have $\boldsymbol{y = p(x) = (h \bullet g)(x) = h(g)}$, $\boldsymbol{y = g(x) = 2x^2 + 1}$, and $\boldsymbol{y = h(x) = 3x^3 + 2}$.

Thus, we get $\boldsymbol{p(x) = h(g) = 3g^3 + 2 = 3(2x^2 + 1)^3 + 2}$ for $\boldsymbol{x}$ real.

So next, we want to move on to $\boldsymbol{(h \bullet g) \bullet f}$, which is equal to $\boldsymbol{p \bullet f}$.

Thus first, assuming $\boldsymbol{y = q(x) = (p \bullet f)(x)}$, we have

$$\begin{tikzcd} X_f \arrow[r, "f"] \arrow[rr, bend right, "q"'] & Y_f \arrow[r, "p"] & Y_q \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} X_f \arrow[r, "p \bullet f"] & Y_p \end{tikzcd}$$

In $\boldsymbol{q}$, we have $\boldsymbol{X_q = X_f}$, of course.

Now, the function $\boldsymbol{p}$ blends into the function $\boldsymbol{f}$, the range of $\boldsymbol{f}$ is $\boldsymbol{R}$, a set of all real numbers, and the domain of $\boldsymbol{p}$ is $\boldsymbol{R}$, too. So we can get the composite function $\boldsymbol{q}$, defined in the domain of $\boldsymbol{f}$, and the domain is $\boldsymbol{R}$. And next, we want to get the expression of $\boldsymbol{q}$.

We have $\boldsymbol{y = q(x) = (p \bullet f)(x) = p(f)}$, $\boldsymbol{y = p(x) = 3(2x^2 + 1)^3 + 2}$, and $\boldsymbol{y = f(x) = x}$, which is an identity function.

So we just simply get $\boldsymbol{y = q(x) = p(x) = 3(2x^2 + 1)^3 + 2}$ for $\boldsymbol{x}$ real.

In short:

$\boldsymbol{f(x) = x}$, $\boldsymbol{g(x) = 2x^2 + 1}$, and $\boldsymbol{h(x) = 3x^3 + 2}$.

Thus first, we get $\boldsymbol{(h \bullet g) \bullet f = h \bullet g}$, because $\boldsymbol{f}$ is an identity function.

So $\boldsymbol{(h \bullet g) \bullet f = h \bullet g = h(g) = 3g^3 + 2 = 3(2x^2 + 1)^3 + 2 = 24x^6 + 36x^4 + 18x^2 + 5}$ for $\boldsymbol{x}$ real.

Suggestions or Solutions
To the Problem in the Example 6

Find $h \bullet (g \bullet g)$ using the definition as follows. g: $x \longrightarrow 2x^2 + 1$, and h: $x \longrightarrow 3x^3 + 2$.

$g(x) = 2x^2 + 1$, and $h(x) = 3x^3 + 2$.

So first, $g \bullet g = g\{g(x)\} = 2g^2 + 1 = 2(2x^2 + 1)^2 + 1$.

Next, setting $p = g \bullet g$, we get $h \bullet (g \bullet g) = h \bullet p = h(p) = 3\{2(2x^2 + 1)^2 + 1\}^3 + 2$.

If not quite sure of the idea behind the processes above, follow the steps below.

Making a composite function, we can use one function only, too.
And we can use the same function as many times as needed. It is the case, of course, only if the range of each function blends into is the domain of the next function.

So $g \bullet g$ is a composite function, and calling it p, we get $h \bullet (g \bullet g) = h \bullet p$.

And thus, to begin with, assuming $y = p(x) = (g \bullet g)(x)$, we have

$$\begin{tikzcd} X_g \arrow[r, "g"] \arrow[rr, bend right, "p"'] & Y_g \arrow[r, "g"] & Y_g \end{tikzcd} \quad \Rightarrow \quad X_g \xrightarrow{g \bullet g} Y_p$$

In p, we have $X_p = X_g$, of course.

Now, the function g blends into itself, the range of g is $y \geq 1$, and the domain of g is R, a set of all real numbers. So we can get the composite function p, defined in the domain of g, and the domain is R. And next, we want to get the expression of p. How?

We have $y = p(x) = (g \bullet g)(x) = g(g)$, and $y = g(x) = 2x^2 + 1$.

So we get $p(x) = g(g) = 2g^2 + 1 = 2(2x^2 + 1)^2 + 1$ for x real. And the range is $y \geq 3$.

So next, we want to move on to $\boldsymbol{h \bullet (g \bullet g)}$, which is equal to $\boldsymbol{h \bullet p}$.

Thus first, assuming $\boldsymbol{y = q(x) = (h \bullet p)(x)}$, we have

$$\begin{tikzcd} X_p \arrow[r, "p"] \arrow[rr, bend right, "q"'] & Y_p \arrow[r, "h"] & Y_q \end{tikzcd} \quad \Rightarrow \quad X_g \xrightarrow{h \bullet p} Y_q$$ In $\boldsymbol{q}$, we have $\boldsymbol{X_q = X_p = X_g}$, of course.

Now, the function $\boldsymbol{p}$ blends into the function $\boldsymbol{h}$, the range of $\boldsymbol{p}$ is $\boldsymbol{y \geq 3}$, and the domain of $\boldsymbol{h}$ is $\boldsymbol{R}$. So we can get the composite function $\boldsymbol{q}$, defined in the domain of $\boldsymbol{p}$, which is originally the domain of $\boldsymbol{g}$, which is $\boldsymbol{R}$. And next, we want to get the expression of $\boldsymbol{q}$.

We have $\boldsymbol{y = q(x) = (h \bullet p)(x) = h(p)}$, $\boldsymbol{p(x) = 2(2x^2 + 1)^2 + 1}$, and $\boldsymbol{h(x) = 3x^3 + 2}$.

So we get $\boldsymbol{y = q(x) = h(p) = 3p^3 + 2 = 3\{2(2x^2 + 1)^2 + 1\}^3 + 2}$ for $\boldsymbol{x}$ real.

And checking the range of $\boldsymbol{q}$, we get $\boldsymbol{2x^2 + 1 \geq 1 \Rightarrow (2x^2 + 1)^2 \geq 1 \Rightarrow 2(2x^2 + 1)^2 \geq 2}$

$\boldsymbol{\Rightarrow 2(2x^2 + 1)^2 + 1 \geq 3 \Rightarrow \{2(2x^2 + 1)^2 + 1\}^3 \geq 3^3 \Rightarrow 3\{2(2x^2 + 1)^2 + 1\}^3 \geq 3^4}$

$\boldsymbol{\Rightarrow 3\{2(2x^2 + 1)^2 + 1\}^3 + 2 \geq 3^4 + 2 \Rightarrow y \geq 83}$.

In short:

$\boldsymbol{g(x) = 2x^2 + 1}$, and $\boldsymbol{h(x) = 3x^3 + 2}$.

So first, $\boldsymbol{g \bullet g = g\{g(x)\} = 2g^2 + 1 = 2(2x^2 + 1)^2 + 1}$.

Next, setting $\boldsymbol{p = g \bullet g}$, we get $\boldsymbol{h \bullet (g \bullet g) = h \bullet p = h(p) = 3\{2(2x^2 + 1)^2 + 1\}^3 + 2}$ for $\boldsymbol{x}$ real.

Suggestions or Solutions
To the Problem in the Example 7

Find $(h \bullet g) \bullet g$ using the definition as follows. g: $x \longrightarrow 2x^2 + 1$, and h: $x \longrightarrow 3x^3 + 2$.

$f(x) = x$, $g(x) = 2x^2 + 1$, and $h(x) = 3x^3 + 2$.

So first, we get $h \bullet g = h\{g(x)\} = 3g^3 + 2 = 3(2x^2 + 1)^3 + 2$. And setting $p = h \bullet g$, we get

$(h \bullet g) \bullet g = p \bullet g = p(g) = 3(2g^2 + 1)^3 + 2 = 3\{2(2x^2 + 1)^2 + 1\}^3 + 2$ for x real.

If not quite sure of the idea behind the processes above, follow the steps below.

Finding $(h \bullet g) \bullet g$, we want to find $h \bullet g$ first.

So to begin with, assuming $y = p(x) = (h \bullet g)(x)$, we have

$$\begin{tikzcd} X_g \arrow[r, "g"] \arrow[rr, bend right, "p"'] & Y_g \arrow[r, "h"] & Y_p \end{tikzcd} \quad \Rightarrow \quad X_g \xrightarrow{h \bullet g} Y_p$$

In p, we have $X_p = X_g$, of course.

Now, the function g blends into the function h, the range of g is $y \geq 1$, and the domain of h is R, a set of all real numbers. So we can get the composite function p, defined in the domain of g, which is R. Thus next, we want to get the expression of p. How?

We have $y = p(x) = (h \bullet g)(x) = h(g)$, $y = g(x) = 2x^2 + 1$, and $y = h(x) = 3x^3 + 2$.

So we get $p(x) = h(g) = 3g^3 + 2 = 3(2x^2 + 1)^3 + 2$, because $g = 2x^2 + 1$.

And thus, $p(x) = 3(2x^2 + 1)^3 + 2$ for $x \in R$.

So we can now move on to $(h \bullet g) \bullet g$, which is equal to $p \bullet g$.
Thus first, assuming $y = q(x) = (p \bullet g)(x)$, we have

$$\begin{tikzcd} \boldsymbol{X_g} \arrow[r, "g"] \arrow[rr, bend right, "\boldsymbol{q}"'] & \boldsymbol{Y_g} \arrow[r, "p"] & \boldsymbol{Y_q} \end{tikzcd} \quad \Rightarrow \quad \boldsymbol{X_g} \xrightarrow{p\bullet g} \boldsymbol{Y_q}$$

In $\boldsymbol{q}$, we have $\boldsymbol{X_q = X_g}$, of course.

Now, the function $\boldsymbol{g}$ blends into the function $\boldsymbol{p}$, the range of $\boldsymbol{g}$ is $\boldsymbol{y \geq 1}$, and the domain of $\boldsymbol{p}$ is $\boldsymbol{R}$. So we can get the composite function $\boldsymbol{q}$, defined in the domain of $\boldsymbol{g}$, which is $\boldsymbol{R}$.

Thus next, we want to get the expression of $\boldsymbol{q}$.

We have $\boldsymbol{y = q(x) = (p\bullet g)(x) = p(g)}$, $\boldsymbol{p(x) = 3(2x^2 + 1)^3 + 2}$, and $\boldsymbol{g(x) = 2x^2 + 1}$.

So we get $\boldsymbol{y = q(x) = p(g) = 3(2g^2 + 1)^3 + 2 = 3\{2(2x^2 + 1)^2 + 1\}^3 + 2}$, which is OK with the domain $\boldsymbol{R}$.

Therefore, $\boldsymbol{y = \{(h\bullet g)\bullet g\}(x) = 3\{2(2x^2 + 1)^2 + 1\}^3 + 2}$ for $\boldsymbol{x}$ real.

In the problem 6 though, we found this: $\boldsymbol{\{h\bullet(g\bullet g)\}(x) = 3\{2(2x^2 + 1)^2 + 1\}^3 + 2}$ for $\boldsymbol{x}$ real.

So do we have $\boldsymbol{f\bullet g\bullet h = f\bullet(g\bullet h) = (f\bullet g)\bullet h}$?

We will see if it is the case in **Examples 2 in Composite Functions**.

In short:

$\boldsymbol{f(x) = x}$, $\boldsymbol{g(x) = 2x^2 + 1}$, and $\boldsymbol{h(x) = 3x^3 + 2}$.

So first, we get $\boldsymbol{h\bullet g = h\{g(x)\} = 3g^3 + 2 = 3(2x^2 + 1)^3 + 2}$. And setting $\boldsymbol{p = h\bullet g}$, we get

$\boldsymbol{(h\bullet g)\bullet g = p\bullet g = p(g) = 3(2g^2 + 1)^3 + 2 = 3\{2(2x^2 + 1)^2 + 1\}^3 + 2}$.