Examples 2 in Composite Functions

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Using three functions below, check to see if $(h\bullet g)\bullet f = h\bullet(g\bullet f)$.

f: $x \longrightarrow x + 1$, g: $x \longrightarrow x^2 - 1$, and h: $x \longrightarrow x^3 + 2$.

And we will get to see some more examples, which are going to be though, quite similar to the one above.

Suggestions or Solutions
To the Problem in the Example Given

Using three functions below, check to see if $(h \bullet g) \bullet f = h \bullet (g \bullet f)$.

$f: x \longrightarrow x + 1$, $g: x \longrightarrow x^2 - 1$, and $h: x \longrightarrow x^3 + 2$.

To begin with, assuming R is a set of all real numbers, we have

$f(x) = x + 1$ for $x \in R$, $g(x) = x^2 - 1$ for $x \in R$, and $h(x) = x^3 + 2$ for $x \in R$.

So beginning with $h \bullet (g \bullet f)$, we get

First, $g \bullet f = g(f) = f^2 - 1 = (x + 1)^2 - 1$.

And next, setting $p = g \bullet f$, we get $p(x) = (x + 1)^2 - 1$.

So we get $h \bullet (g \bullet f) = h \bullet p = h(p) = p^3 + 2 = \{(x + 1)^2 - 1\}^3 + 2$ for x real.

Next, moving on to $(h \bullet g) \bullet f$, we get

First, $h \bullet g = h(g) = g^3 + 2 = (x^2 - 1)^3 + 2$.

And next, setting $r = h \bullet g$, we get $r(x) = (x^2 - 1)^3 + 2$.

So we get $(h \bullet g) \bullet f = r \bullet f = r(f) = (f^2 - 1)^3 + 2 = \{(x + 1)^2 - 1\}^3 + 2$ for x real.

Therefore, we get $h \bullet (g \bullet f) = (h \bullet g) \bullet f$.

If not quite sure of the idea behind the processes above, follow the steps below.

What we want to do in this example is to check to see if composite-function-operations are associative.

To begin with, finding $(h \bullet g) \bullet f$, we want to find $h \bullet g$ first.

So first, assuming $y = p(x) = (h \bullet g)(x)$, we have

$$\begin{tikzcd} X_g \arrow[r, "g"] \arrow[rr, bend right, "p"'] & Y_g \arrow[r, "h"] & Y_p \end{tikzcd} \quad \Rightarrow \quad X_g \xrightarrow{h \bullet g} Y_p$$

And in p, we have $X_p = X_g$.

Now, the function g blends into the function h, the range of g is $y \geq -1$, and the domain of h is R, a set of all real numbers.

So we can get the composite function p, defined in the domain of g, which is R. Thus next, we want to get the expression of p.

We have $y = p(x) = (h \bullet g)(x) = h(g)$, $y = g(x) = x^2 - 1$, and $y = h(x) = x^3 + 2$.

So we get $p(x) = h(g) = g^3 + 2 = (x^2 - 1)^3 + 2$ for x real.

And we can now move on to $(h \bullet g) \bullet f$, which is equal to $p \bullet f$.

So first, assuming $y = q(x) = (p \bullet f)(x)$, we have

$$\begin{tikzcd} X_f \arrow[r, "f"] \arrow[rr, bend right, "q"'] & Y_f \arrow[r, "p"] & Y_q \end{tikzcd} \quad \Rightarrow \quad X_f \xrightarrow{p \bullet f} Y_q$$

And in q, we have $X_q = X_f$.

Now, the function f blends into the function p, the range of f is R, and so is the domain of p. So we can get the composite function q, defined in the domain of f, which is R.

Thus next, we want to get the expression of q.

We have $y = q(x) = (p \bullet f)(x) = p(f)$, $p(x) = (x^2 - 1)^3 + 2$, and $f(x) = x + 1$.

So we get $y = q(x) = p(f) = (f^2 - 1)^3 + 2 = \{(x + 1)^2 - 1\}^3 + 2$, which is OK in R.

And thus, $y = \{(h \bullet g) \bullet f\}(x) = \{(x + 1)^2 - 1\}^3 + 2$ for x real.

And next, finding $\boldsymbol{h\bullet(g\bullet f)}$, we want to find $\boldsymbol{g\bullet f}$ first.

So first, assuming $\boldsymbol{y = r(x) = (g\bullet f)(x)}$, we have

$$\begin{tikzcd} X_f \arrow[r, "f"] \arrow[rr, bend right, "r"'] & Y_f \arrow[r, "g"] & Y_r \end{tikzcd} \quad \Rightarrow \quad X_f \xrightarrow{g\bullet f} Y_r$$

And in $\boldsymbol{r}$, we have $\boldsymbol{X_r = X_f}$.

Now, the function $\boldsymbol{f}$ blends into the function $\boldsymbol{g}$, the range of $\boldsymbol{f}$ is $\boldsymbol{R}$, and so is the domain of $\boldsymbol{g}$. So we can get the composite function $\boldsymbol{r}$, defined in $\boldsymbol{R}$.

Thus next, we want to get the expression of $\boldsymbol{r}$.

We have $\boldsymbol{y = r(x) = (g\bullet f)(x) = g(f)}$, $\boldsymbol{y = g(x) = x^2 + 1}$, and $\boldsymbol{y = f(x) = x + 1}$.

So we get $\boldsymbol{r(x) = g(f) = f^2 + 1 = (x + 1)^2 + 1}$ for $\boldsymbol{x}$ real.

And we can now move on to $\boldsymbol{h\bullet(g\bullet f)}$, which is equal to $\boldsymbol{h\bullet r}$.

So first, assuming $\boldsymbol{y = s(x) = (h\bullet r)(x)}$, we have

$$\begin{tikzcd} X_r \arrow[r, "r"] \arrow[rr, bend right, "s"'] & Y_r \arrow[r, "h"] & Y_s \end{tikzcd} \quad \Rightarrow \quad X_f \xrightarrow{h\bullet r} Y_s$$

And in $\boldsymbol{s}$, we have $\boldsymbol{X_s = X_r = X_f}$.

Now, the function $\boldsymbol{r}$ blends into the function $\boldsymbol{h}$, the range of $\boldsymbol{r}$ is $\boldsymbol{y \geq 1}$, and the domain of $\boldsymbol{h}$ is $\boldsymbol{R}$. So we can get the composite function $\boldsymbol{s}$, defined in the domain $\boldsymbol{R}$.

Thus next, we want to get the expression of $\boldsymbol{s}$.

We have $\boldsymbol{y = s(x) = (h\bullet p)(x) = h(p)}$, $\boldsymbol{h(x) = x^3 + 2}$, and $\boldsymbol{p(x) = (x + 1)^2 + 1}$.

So we get $\boldsymbol{y = s(x) = h(p) = p^3 + 2 = \{(x + 1)^2 + 1\}^3 + 2}$, which is OK in $\boldsymbol{R}$.

And thus, $\boldsymbol{y = \{h\bullet(g\bullet f)\}(x) = \{(x + 1)^2 + 1\}^3 + 2}$ for $\boldsymbol{x}$ real.

Now, putting threads together, we have

$y = \{(h \bullet g) \bullet f\}(x) = \{(x + 1)^2 + 1\}^3 + 2$ for x real.

$y = \{h \bullet (g \bullet f)\}(x) = \{(x + 1)^2 + 1\}^3 + 2$ for x real.

Therefore, we get $h \bullet (g \bullet f) = (h \bullet g) \bullet f$.

In short:

To begin with, assuming R is a set of all real numbers, we have
$f(x) = x + 1$ for $x \in R$, $g(x) = x^2 - 1$ for $x \in R$, and $h(x) = x^3 + 2$ for $x \in R$.

So beginning with $h \bullet (g \bullet f)$, we get

First, $g \bullet f = g(f) = f^2 - 1 = (x + 1)^2 - 1$.

And next, setting $p = g \bullet f$, we get $p(x) = (x + 1)^2 - 1$.
So we get $h \bullet (g \bullet f) = h \bullet p = h(p) = p^3 + 2 = \{(x + 1)^2 - 1\}^3 + 2$ for x real.

Next, moving on to $(h \bullet g) \bullet f$, we get

First, $h \bullet g = h(g) = g^3 + 2 = (x^2 - 1)^3 + 2$.

And next, setting $r = h \bullet g$, we get $r(x) = (x^2 - 1)^3 + 2$.
So we get $(h \bullet g) \bullet f = r \bullet f = r(f) = (f^2 - 1)^3 + 2 = \{(x + 1)^2 - 1\}^3 + 2$ for x real.

Therefore, we get $h \bullet (g \bullet f) = (h \bullet g) \bullet f$.

So do we have $h \bullet g \bullet f = h \bullet (g \bullet f) = (h \bullet g) \bullet f$?

(If you are really serious about composite functions, refer to the explanations below.)

Let's now check to see if it is the case using three functions below.

$\boldsymbol{f}$**:** $\boldsymbol{A} \longrightarrow \boldsymbol{B}$, $\boldsymbol{g}$**:** $\boldsymbol{C} \longrightarrow \boldsymbol{D}$, and $\boldsymbol{h}$**:** $\boldsymbol{E} \longrightarrow \boldsymbol{J}$.

To begin with, finding $\boldsymbol{(h \bullet g) \bullet f}$, we want to find $\boldsymbol{h \bullet g}$ first.

So assuming $\boldsymbol{y = p(x) = (h \bullet g)(x)}$, $\boldsymbol{D \subseteq E}$, and this time, the range of $\boldsymbol{p}$ is $\boldsymbol{K}$, we have

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[rr, bend right, "p"'] & D \arrow[r, "h"] & K \end{tikzcd} \quad \Rightarrow \quad C \xrightarrow{h \bullet g} K.$$

We assume $\boldsymbol{D \subseteq E}$ so that the range of $\boldsymbol{g}$, the function blends into, is a subset of the domain of $\boldsymbol{h}$. Then, $\boldsymbol{g}$ can blend into $\boldsymbol{h}$, and the domain of $\boldsymbol{p}$ is $\boldsymbol{C}$. So we get

$\boldsymbol{p}$**:** $\boldsymbol{C} \longrightarrow \boldsymbol{K}$, that is, $\boldsymbol{h \bullet g}$**:** $\boldsymbol{C} \longrightarrow \boldsymbol{K}$.

And next, we want to move on to $\boldsymbol{(h \bullet g) \bullet f}$, which is equal to $\boldsymbol{p \bullet f}$.

So assuming $\boldsymbol{y = q(x) = (p \bullet f)(x)}$, $\boldsymbol{B \subseteq C}$, and this time, the range of $\boldsymbol{q}$ is $\boldsymbol{L}$, we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "q"'] & C \arrow[r, "p"] & L \end{tikzcd} \quad \Rightarrow \quad A \xrightarrow{p \bullet f} L.$$

We assume $\boldsymbol{B \subseteq C}$ so that the range of $\boldsymbol{f}$, the function blends into, is a subset of the domain of $\boldsymbol{p}$. Then, $\boldsymbol{f}$ can blend into $\boldsymbol{p}$, and the domain of $\boldsymbol{q}$ is $\boldsymbol{A}$. So we get

$\boldsymbol{q}$**:** $\boldsymbol{A} \longrightarrow \boldsymbol{L}$, that is, $\boldsymbol{p \bullet f = (h \bullet g) \bullet f}$**:** $\boldsymbol{A} \longrightarrow \boldsymbol{L}$.

And next, finding $\boldsymbol{h \bullet (g \bullet f)}$, we want to find $\boldsymbol{g \bullet f}$ first.

So assuming $\boldsymbol{y = r(x) = (g \bullet f)(x)}$, $\boldsymbol{B \subseteq C}$, and this time, the range of $\boldsymbol{r}$ is $\boldsymbol{M}$, we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "r"'] & A \arrow[r, "g"] & M \end{tikzcd} \quad \Rightarrow \quad A \xrightarrow{g \bullet f} M.$$

We have already assume that $B \subseteq C$, so the range of f, the function blends into, is a subset of the domain of g. Thus, f can blend into g, and the domain of r is A. so we get

$$r: A \longrightarrow M, \text{ that is, } g \bullet f: A \longrightarrow M.$$

And next, we want to move on to $h \bullet (g \bullet f)$, which is equal to $h \bullet r$.

So assuming $y = s(x) = (h \bullet r)(x)$, $M \subseteq E$, and this time, the range of s is N, we have

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[rr, bend right, "s"'] & A \arrow[r, "h"] & N \end{tikzcd} \quad \Rightarrow \quad A \xrightarrow{h \bullet r} N.$$

We assume $M \subseteq E$ so that the range of r, the function blends into, is a subset of the domain of h. Then, r can blend into h, and the domain of s is A. So we get

$$s: A \longrightarrow N, \text{ that is, } h \bullet r = h \bullet (g \bullet f): A \longrightarrow N.$$

So putting threads together, we have $(h \bullet g) \bullet f: A \longrightarrow L$, and $h \bullet (g \bullet f): A \longrightarrow N$

Then, it seems that we have ended up with $h \bullet (g \bullet f) \neq (h \bullet g) \bullet f$.

Let's take a look at though, what expression each of the two functions above has to get.

Then first, setting $y = f(x)$, $y = g(x)$, and $y = h(x)$, we get

$y = \{(h \bullet g) \bullet f\}(x) = (p \bullet f)(x) = p\{f(x)\}$, and $p(x) = (h \bullet g)(x) = h\{g(x)\}$.

So next, putting $f(x)$ into x in both of $h\{g(x)\}$ and $p(x)$, we get $p\{f(x)\} = h[g\{f(x)\}]$.

Thus, we get $(h \bullet g) \bullet f = p \bullet f = p\{f(x)\} = h[g\{f(x)\}]$.

More specifically, we can set $\{(h \bullet g) \bullet f\}(x) = h[g\{f(x)\}]$. In short, $(h \bullet g) \bullet f = h(g(f))$.

And making it more legible, we can put it this way, too, of course: $(h \bullet g) \bullet f = h\{g(f)\}$.

And next, moving on to $h\bullet(g\bullet f)$, we have $h\bullet(g\bullet f) = h\bullet r = h(r)$, and $r = g\bullet f = g(f)$.

So we get $h\bullet(g\bullet f) = h\bullet r = h(r) = h(g(f))$ since $r = g(f)$.

Thus, we get $h\bullet(g\bullet f) = h(g(f))$.

And more legibly and specifically, we can set $\{h\bullet(g\bullet f)\}(x) = h[g\{f(x)\}]$.

So we get $\{(h\bullet g)\bullet f\}(x) = h[g\{f(x)\}]$, and $\{h\bullet(g\bullet f)\}(x) = h[g\{f(x)\}]$.

That is, the expressions in both functions are the same.

Now, putting threads together again, we have

$(h\bullet g)\bullet f\colon A \longrightarrow L$, and $h\bullet(g\bullet f)\colon A \longrightarrow N$, together with the fact that the expressions in both functions $(h\bullet g)\bullet f$ and $h\bullet(g\bullet f)$ are the same.

Then, we can see that we get $L = N$.

(If taking a function as a system where an expression connects a domain to a range, we can say that if the domains are identical, and the expressions are identical, the functions are all the same, and thus, the ranges have to be identical, too.)

It is the case though, only if $D \subseteq E$, and $B \subseteq C$. We have

$f\colon A \longrightarrow B$, $g\colon C \longrightarrow D$, and $h\colon E \longrightarrow J$.

So more specifically, we need to have $D \subseteq E$, and $B \subseteq C$ so that $h\bullet(g\bullet f)$ and $(h\bullet g)\bullet f$ can exist, and then, we can get $h\bullet(g\bullet f) = (h\bullet g)\bullet f$.

In other words, if $D \subseteq E$, and $B \subseteq C$, we have $h\bullet g\bullet f = h\bullet(g\bullet f) = (h\bullet g)\bullet f$.

That is to say that composite-function-operations are associative if the domains and the ranges can work together.

What if the three functions as follows? $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$, and $h\colon C \longrightarrow D$.

Then, can we get $h\bullet g\bullet f = h\bullet(g\bullet f) = (h\bullet g)\bullet f$ with no additional assumption on domains and ranges?

To begin with, finding $(h\bullet g)\bullet f$, we want to find $h\bullet g$ first.

Lets assume this time, the range of $h\bullet g$ is D, which is the range of h.

Then, assuming $y = p(x) = (h\bullet g)(x)$, we have

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[rr, bend right, "p"'] & C \arrow[r, "h"] & D \end{tikzcd} \quad \Rightarrow \quad B \xrightarrow{h\bullet g} D.$$

Assuming thus, the range of p is D, we get

$$p\colon B \longrightarrow D, \text{ that is, } h\bullet g\colon B \longrightarrow D.$$

And next, we want to move on to $(h\bullet g)\bullet f$, which is equal to $p\bullet f$.

Let's assume again, the range of $p\bullet f$ is D, too, which is the range of h, of course.

Then, assuming $y = q(x) = (p\bullet f)(x)$, we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "q"'] & B \arrow[r, "p"] & D \end{tikzcd} \quad \Rightarrow \quad A \xrightarrow{p\bullet f} D.$$

Assuming thus, the range of q is D, we get $q\colon A \longrightarrow D$, that is, $(h\bullet g)\bullet f\colon A \longrightarrow D$.

And next, finding $h\bullet(g\bullet f)$, we want to find $g\bullet f$ first.

This time, lets assume that the range of $g\bullet f$ is C, which is the range of g.

Then, assuming $y = r(x) = (g\bullet f)(x)$, we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "r"'] & B \arrow[r, "g"] & C \end{tikzcd} \quad \Rightarrow \quad A \xrightarrow{g\bullet f} C.$$

Assuming therefore, the range of r is C, we get

$$r\colon A \longrightarrow C, \text{ that is, } g\bullet f\colon A \longrightarrow C.$$

And next, we want to move on to $h\bullet(g\bullet f)$, which is equal to $h\bullet r$.

Let's assume again though, the range of $h\bullet r$ is D, which is the range of h, of course.

Then, assuming $y = s(x) = (h\bullet r)(x)$, we have

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[rr, bend right, "s"'] & C \arrow[r, "h"] & D \end{tikzcd} \quad \Rightarrow \quad A \xrightarrow{h\bullet r} D.$$

So assuming the range of s is D, we get $q\colon A \longrightarrow D$, that is, $h\bullet(g\bullet f)\colon A \longrightarrow D$.

And thus, putting threads together, we get $h\bullet g\bullet f = h\bullet(g\bullet f) = (h\bullet g)\bullet f$.

It is the case however, only if we make all the assumptions below.

The range of $h\bullet g$ is D, which is the range of h.

If $p = h\bullet g$, the range of $p\bullet f$ is D, too, which is the range of h.

The range of $g\bullet f$ is C, which is the range of g.

And if $r = g\bullet f$, the range of $h\bullet r$ is D, too, which is the range of h.