Examples 3 in Composite Functions

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0. Assuming $\boldsymbol{f}(\frac{2x+1}{3}) = 2x+5$, find $\boldsymbol{f}(7)$, and specify $\boldsymbol{f}(\frac{3x-2}{2})$.

1. Assuming $\boldsymbol{f}(\frac{2x-1}{x}) = (1-2x^2)(3x^2+x+2)$, find $\boldsymbol{f}(8)$.

2. Assuming $\boldsymbol{f}(x) = \frac{2x}{3x+1}$, and $\boldsymbol{g}(x) = \frac{x}{3-2x}$, find the $\boldsymbol{x}$-value that satisfies $(\boldsymbol{f} \bullet \boldsymbol{g})(x) = \boldsymbol{f}(x)$.

Suggestions or Solutions
To the Problem in the Example 0

Assuming $f\left(\frac{2x+1}{3}\right) = 2x+5$, find $f(7)$, and specify $f\left(\frac{3x-2}{2}\right)$.

Assuming first, $y = f(u)$, and $y = g(x) = \frac{2x+1}{3}$, we get $y = f\{g(x)\} = f\left(\frac{2x+1}{3}\right) = 2x + 5$.

So setting $u = \frac{2x+1}{3}$, we can get

$$u = \frac{2x+1}{3} \Rightarrow 2x+1 = 3u \Rightarrow x = \frac{3u-1}{2} \Rightarrow 2x+5 = 2\cdot\frac{3u-1}{2} + 5 = 3u-1+5 = 3u+4.$$

So we get $f(u) = 3u + 4$, and thus, we get $f(7) = 3\cdot7 + 4 = 25$.

And assuming next, $h(x) = \frac{3x-2}{2}$, we get $f\left(\frac{3x-2}{2}\right) = f\{h(x)\}$.

And thus, putting $\frac{3x-2}{2}$ into u in $3u + 4$, we get $3\cdot\frac{3(3x-2)}{2} + 4 = \frac{9x-6+8}{2} = \frac{9x+2}{2}$.

So we get $f\left(\frac{3x-2}{2}\right) = \frac{9x+2}{2}$ for x real.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, what do we mean by $f(7)$?

It's the value of the output for 7 in case of the function f, and 7 is the input, of course.

So knowing the expression of f, we can readily find $f(7)$ putting 7 into the input variable in f, then evaluating the expression of f.

We are given though, this instead: $y = f\left(\frac{2x+1}{3}\right) = 2x+5$ for x real. So what do we have?

The function given is a composite function. Why?

Getting a composite function, we blend a function into another function.

So assuming $\boldsymbol{g(x)} = \frac{2x+1}{3}$, and blending $\boldsymbol{g}$ into $\boldsymbol{f}$, we get $\boldsymbol{y} = \boldsymbol{f}\left(\frac{2x+1}{3}\right) = \mathbf{2}\boldsymbol{x} + \mathbf{5}$. Not sure?

Assuming $\boldsymbol{y = f(u)}$, and blending $\boldsymbol{g}$ into $\boldsymbol{f}$, we get $\boldsymbol{y = f(g)}$. And we can give a name to it.

For instance, we can set $\boldsymbol{k(x) = (f \bullet g)(x) = f(g)}$.

Then, after putting the expression of $\boldsymbol{g}$ into every input variable in the expression of $\boldsymbol{f}$, we get the expression of $\boldsymbol{k}$, which is $\mathbf{2}\boldsymbol{x} + \mathbf{5}$.

In short, putting the expression of $\boldsymbol{g}$ into $\boldsymbol{u}$ in $\boldsymbol{f(u)}$, we get $\boldsymbol{f}\left(\frac{2x+1}{3}\right) = \mathbf{2}\boldsymbol{x} + \mathbf{5}$.

And of curse, the expression of $\boldsymbol{f}$ is an expression in terms of $\boldsymbol{u}$.

So after putting the expression of $\boldsymbol{g}$ into every $\boldsymbol{u}$ in the expression of $\boldsymbol{f}$, then simplifying the expression of $\boldsymbol{f}$, we get $\mathbf{2}\boldsymbol{x} + \mathbf{5}$, which is the expression of $\boldsymbol{k}$.

And thus, we can set $\boldsymbol{k(x) = (f \bullet g)(x) = f\{g(x)\}} = \mathbf{2}\boldsymbol{x} + \mathbf{5}$, where $\boldsymbol{g(x)} = \frac{2x+1}{3}$.

Why though, is $\boldsymbol{u}$ the input variable of $\boldsymbol{f}$?

In $\boldsymbol{f \bullet g}$, $\boldsymbol{f}$ is not the function blends into, so we can use any letter as the input variable in $\boldsymbol{f}$.

So putting the expression of $\boldsymbol{g}$ into the input variable in $\boldsymbol{f}$, we get the expression of $\boldsymbol{k}$.

And we know the expression of $\boldsymbol{k}$ is $\mathbf{2}\boldsymbol{x} + \mathbf{5}$, which is therefore, certainly not the expression of $\boldsymbol{f}$.

So we *cannot* just put 7 into $\boldsymbol{x}$ in $\mathbf{2}\boldsymbol{x} + \mathbf{5}$ to get $\boldsymbol{f}\mathbf{(7)}$.

What then, is the input variable in ***f***? Or rather, what is the expression of ***f***?

We have set $\boldsymbol{y = f(u)}$, where the expression is an expression in terms of ***u***.

So putting the expression of ***g*** into every ***u*** in the expression of ***f***, we get the expression of ***f•g***.

That is, putting $\frac{2x+1}{3}$ into ***u*** in $\boldsymbol{f(u)}$, we get $\boldsymbol{f\left(\frac{2x+1}{3}\right) = 2x + 5}$. So what?

We can set $\boldsymbol{u = \frac{2x+1}{3}}$.
And we have this, too: $\boldsymbol{y = f\left(\frac{2x+1}{3}\right) = 2x + 5}$. So what do we get?

We get this, of course: $\boldsymbol{y = f(u) = 2x + 5}$, in which however, the expression is not an expression in terms of ***u***. So what do we do with the equation, $\boldsymbol{u = \frac{2x+1}{3}}$?

Solving it, we can put $\boldsymbol{2x + 5}$, in terms of ***u***. How?

We have $\boldsymbol{u = \frac{2x+1}{3}}$, too. And ***x*** in $\frac{2x+1}{3}$ is the same as the very ***x*** in $\boldsymbol{2x + 5}$.

So solving for ***x*** the equation $\boldsymbol{u = \frac{2x+1}{3}}$, we get an expression in terms of ***u***, which is of course, the solution to the equation.

And then, putting the expression into ***x*** in $\boldsymbol{2x + 5}$, we get the expression of ***f***.

That is, we get

$$\boldsymbol{u = \frac{2x+1}{3} \Rightarrow 2x + 1 = 3u \Rightarrow x = \frac{3u-1}{2} \Rightarrow 2x + 5 = 2 \cdot \frac{3u-1}{2} + 5 = 3u - 1 + 5 = 3u + 4}.$$

And thus, we get $\boldsymbol{f(u) = 3u + 4}$.

And of course, putting $\frac{2x+1}{3}$ back into $\boldsymbol{u}$ in $\boldsymbol{3u + 4}$, we get this back: $\boldsymbol{2x + 5}$.

That is, blending the expression of $\boldsymbol{g}$ into the expression of $\boldsymbol{f}$, we get the expression of $\boldsymbol{k}$, which is $\boldsymbol{f \bullet g}$.

So now, putting 7 into $\boldsymbol{u}$, we can simply get $\boldsymbol{f(7) = 3 \cdot 7 + 4 = 25}$.

Now, when blending $\boldsymbol{g}$ into $\boldsymbol{f}$, we put $\boldsymbol{g}$ into $\boldsymbol{u}$ in $\boldsymbol{f(u)}$, then we get the composite function $\boldsymbol{f \bullet g = f(g) = 3g + 4}$, that is, putting $\frac{2x+1}{3}$ into $\boldsymbol{u}$ in $\boldsymbol{f(u)}$, we can get the composite function, $\boldsymbol{(f \bullet g)(x) = 2x + 5}$ for $\boldsymbol{x}$ real.

And we know in the composite function $\boldsymbol{f \bullet g}$, $\boldsymbol{g}$ is the function blends into.

Where then, 7 in $\boldsymbol{f(7)}$ is from?

The 7 in $\boldsymbol{f(7)}$ is an output in the function $\boldsymbol{g}$.

Then in $\boldsymbol{g}$, what is the input, for which the output is 7?
In other words, for what value of $\boldsymbol{x}$ in $\boldsymbol{g(x)}$, the output is 7?

We know $\boldsymbol{u} = \frac{2x+1}{3}$. So we get $\boldsymbol{7} = \frac{2x+1}{3} \Rightarrow \boldsymbol{2x = 20} \Rightarrow \boldsymbol{x = 10}$.

And thus, we should get $\boldsymbol{g(10) = 7}$.

Checking to see if it really is the case, we simply get $\boldsymbol{g(x)} = \frac{2x+1}{3} \Rightarrow \boldsymbol{g(10)} = \frac{2 \cdot 10+1}{3} = \boldsymbol{7}$.

So in the blending process, through the expressions in $\boldsymbol{g}$ and $\boldsymbol{f}$, data flow the way below.

10 ⟶ $\boldsymbol{x}$ in $\boldsymbol{g(x)}$ ⟶ $\boldsymbol{g(10) = 7}$ ⟶ $\boldsymbol{t}$ in $\boldsymbol{f(t)}$ ⟶ $\boldsymbol{f(7) = 25}$.

• Now, let's next, move on to $\boldsymbol{f\left(\frac{3x-2}{2}\right)}$. What do we mean by $\boldsymbol{f\left(\frac{3x-2}{2}\right)}$, though?

It is another composite function, and in the function blends into, the expression is $\frac{3x-2}{2}$.

And we have found this: $\boldsymbol{y = f(u) = 3u + 4}$.

So if $\boldsymbol{h(x) = \frac{3x-2}{2}}$, $\boldsymbol{h}$ is the function blends into, and thus, putting the expression of $\boldsymbol{h}$ into every $\boldsymbol{u}$ in the expression of $\boldsymbol{f}$, we get the expression of the composite function $\boldsymbol{f\left(\frac{3x-2}{2}\right)}$, which is $\boldsymbol{f \bullet h}$.

So putting $\boldsymbol{h}$ into $\boldsymbol{u}$ in $\boldsymbol{f(u)}$, we get $\boldsymbol{f(h)}$, which is the composite function, $\boldsymbol{(f \bullet h)(x)}$.

That is, we get $\boldsymbol{f\left(\frac{3x-2}{2}\right) = (f \bullet h)(x) = f\{h(x)\}}$.

And we have already secured the expression of $\boldsymbol{f}$, which is $\boldsymbol{3u + 4}$, so we can just put the expression of $\boldsymbol{h}$ into $\boldsymbol{u}$ in $\boldsymbol{3u + 4}$, then we get the expression of the composite function $\boldsymbol{f\left(\frac{3x-2}{2}\right)}$.

And thus, putting $\boldsymbol{\frac{3x-2}{2}}$ into $\boldsymbol{u}$ in $\boldsymbol{3u + 4}$, we get $\boldsymbol{3 \cdot \frac{3(3x-2)}{2} + 4 = \frac{9x-6+8}{2} = \frac{9x+2}{2}}$.

So we get $\boldsymbol{f\left(\frac{3x-2}{2}\right) = \frac{9x+2}{2}}$ for $\boldsymbol{x}$ real.

Note:

If not specified with a function, the domain is assumed to be a set of all real numbers or the largest set of real numbers the function can be defined.

Suggestions or Solutions

To the Problem in the Example 1

Assuming $f(\frac{2x-1}{x}) = (1-2x^2)(3x^2+x+2)$**, find** $f(8)$**.**

First, we can get $\mathbf{8} = \frac{2x-1}{x} \Rightarrow 8x = 2x - 1 \Rightarrow 6x = -1 \Rightarrow x = -\frac{1}{6}$.

So next, we can get

$$x = -\tfrac{1}{6} \Rightarrow (1-2x^2)(3x^2+x+2) = \{1-2\cdot(-\tfrac{1}{6})^2\}\{3\cdot(-\tfrac{1}{6})^2+(-\tfrac{1}{6})+2\}$$

$= (\frac{36}{36}-\frac{2}{36})(\frac{3}{36}-\frac{6}{36}+\frac{72}{36}) = \frac{34}{36}\cdot\frac{69}{36} = \frac{17}{18}\cdot\frac{13}{12} = \frac{221}{216}$. And thus, we get $f(8) = \frac{221}{216}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Suppose first, $y = f(u)$, and $y = g(x) = \frac{2x-1}{x}$. Then, we get

$y = (f\bullet g)(x) = f\{g(x)\} = f(\frac{2x-1}{x}) = (1 - 2x^2)(3x^2 + x + 2)$, which is a composite function.

So g is the function blends into, and blends into the function f.

And therefore next, we put x in terms of u solving for x the equation where $u = \frac{2x-1}{x}$, and then, we want to get the expression of f, don't we?

Not necessarily. Why not?

To begin with, we know **8** in $f(8)$ is an output in the function g, since g is the function blends into the function f. And thus, of course, **8** is an input in the function f.

So when $\boldsymbol{u} = \mathbf{8}$ in $\boldsymbol{f}(\boldsymbol{u})$, the output is $\boldsymbol{f}(\mathbf{8})$. And thus also, $\boldsymbol{f}(\mathbf{8})$ itself is an output of the composite function $\boldsymbol{f \bullet g}$. Why?

In the expression $\mathbf{(1 - 2\boldsymbol{x}^2)(3\boldsymbol{x}^2 + \boldsymbol{x} + 2)}$, where is the $\boldsymbol{x}$ from?

The expression above is the expression of the composite function $\boldsymbol{f \bullet g}$, so the $\boldsymbol{x}$ is the very $\boldsymbol{x}$ in $\mathbf{g}(\boldsymbol{x})$, because $\boldsymbol{g}$ is the function blends into $\boldsymbol{f}$, and we get the function $\boldsymbol{f \bullet g}$.

So after such blending, we get the expression of $\boldsymbol{f \bullet g}$, which is the expression above.
So finding the $\boldsymbol{x}$-value that makes the expression of $\boldsymbol{g}$ produce 8 as the output, what can we do with the $\boldsymbol{x}$-value?

Suppose now, $\boldsymbol{s}$ is the $\boldsymbol{x}$-value, that is, an input of $\boldsymbol{g}$, and $\boldsymbol{t}$ is the output for the input $\boldsymbol{s}$.

Then, we get a flow as below.

$\boldsymbol{s} \dashrightarrow x$ in $\boldsymbol{g}(\boldsymbol{x}) \dashrightarrow \boldsymbol{g}(\boldsymbol{s}) = \mathbf{8} \dashrightarrow \boldsymbol{u}$ in $\boldsymbol{f}(\boldsymbol{u}) \dashrightarrow \boldsymbol{f}(\mathbf{8}) = \boldsymbol{t}$.

And also, $\boldsymbol{f}(\mathbf{8})$ is an output of the composite function $\boldsymbol{f \bullet g}$.

For what input then, does the expression of $\boldsymbol{f \bullet g}$ produce the output equal to the value of $\boldsymbol{f}(\mathbf{8})$?
That is, for what $\boldsymbol{x}$-value does $\mathbf{(1 - 2\boldsymbol{x}^2)(3\boldsymbol{x}^2 + \boldsymbol{x} + 2)}$ produce the value the same as $\boldsymbol{f}(\mathbf{8})$?

It's $\boldsymbol{s}$, of course, which is the $\boldsymbol{x}$-value used in $\boldsymbol{g}$ when the expression of $\boldsymbol{g}$ produces the output 8.

So finding $\boldsymbol{s}$, then putting $\boldsymbol{s}$ into $\boldsymbol{x}$ in the expression $\mathbf{(1 - 2\boldsymbol{x}^2)(3\boldsymbol{x}^2 + \boldsymbol{x} + 2)}$, we get $\boldsymbol{t}$, which is the very value of $\boldsymbol{f}(\mathbf{8})$. How can we find the $\boldsymbol{x}$-value $\boldsymbol{s}$, though?

The s is the x-value for which the output is 8 in the function g.

So the solution to $\mathbf{8} = \frac{2x-1}{x}$ is the x-value we want. And solving the equation, we get

$\mathbf{8} = \frac{2x-1}{x} \Rightarrow \mathbf{8x = 2x - 1} \Rightarrow \mathbf{6x = -1} \Rightarrow x = -\frac{1}{6}$, which is the x-value s we want,

Therefore, we get $g = \mathbf{8}$ when $x = -\frac{1}{6}$.

That is, $g(-\frac{1}{6}) = \mathbf{8}$. So next, 8 is an input of $f(u)$, which is the function g blends into.

Thus, as explained above, we get $f(8) = (f \bullet g)(x)$ when $x = -\frac{1}{6}$. That is, $f(8) = (f \bullet g)(-\frac{1}{6})$.

So putting the x-value, $-\frac{1}{6}$ into x in the expression of $f \bullet g$, we get the value of $f(8)$.

And we have $(f \bullet g)(x) = (1 - 2x^2)(3x^2 + x + 2)$. So now getting the value of $f(8)$, we get

$$x = -\tfrac{1}{6} \Rightarrow (1 - 2x^2)(3x^2 + x + 2) = \{1 - 2\cdot(-\tfrac{1}{6})^2\}\{3\cdot(-\tfrac{1}{6})^2 + (-\tfrac{1}{6}) + 2\} = (\tfrac{36}{36} - \tfrac{2}{36})(\tfrac{3}{36} - \tfrac{6}{36} + \tfrac{72}{36})$$
$$= \tfrac{34}{36}\cdot\tfrac{69}{36} = \tfrac{17}{18}\cdot\tfrac{13}{12} = \tfrac{221}{216}.$$

And thus, we get $f(8) = \frac{221}{216}$.

So finding the solution to an equation where $\mathbf{8} =$ the expression of g, that is, solving the equation $\mathbf{8} = \frac{2x-1}{x}$, and then, putting the solution into x in the expression of $f \bullet g$, which is $(1 - 2x^2)(3x^2 + x + 2)$, we get the value of $f(8)$.

Suggestions or Solutions
To the Problem in the Example 2

Assuming $f(x) = \frac{2x}{3x+1}$, and $g(x) = \frac{x}{3-2x}$, **find the** x**-value that satisfies** $(f\bullet g)(x) = f(x)$**.**

To begin with, $(f\bullet g)(x)$ is a composite function, where g blends into f.

Finding thus, $h = f\bullet g$ first, we get $f\bullet g = f(g) = \frac{2g}{3g+1}$, where $g = \frac{x}{3-2x}$.

Meanwhile, $\frac{2g}{3g+1} = 2g \cdot \frac{1}{3g+1} = 2g \cdot (3g+1)^{-1}$.

So we get

$2g = \frac{2x}{3-2x}$, $3g+1 = \frac{3x}{3-2x} + 1 = \frac{3x+3-2x}{3-2x} = \frac{x+3}{3-2x}$, and $\frac{2g}{3g+1} = \frac{2x}{3-2x} \cdot \frac{3-2x}{x+3} = \frac{2x}{x+3}$.

And thus, $(f\bullet g)(x) = \frac{2x}{x+3}$.

So next, $(f\bullet g)(x) = f(x) \Rightarrow \frac{2x}{x+3} = \frac{2x}{3x+1}$.

Thus, first, if $x = 0$, the equality holds. And next, assuming $x \neq 0$, and dividing both sides by $2x$, we get $x + 3 = 3x + 1 \Rightarrow 2x = 2 \Rightarrow x = 1$.

$\therefore$ the x-value is 0 or 1.

If not quite sure of the idea behind the processes above, follow the steps below.

This is nothing but an example of algebra practice on composite functions.

To begin with, assuming $h(x) = (f\bullet g)(x)$, we can say $h(x)$ is a composite function, where g blends into f.

Finding thus, $h = f\bullet g$ first, we get $f\bullet g = f(g) = \frac{2g}{3g+1}$, where $g(x) = \frac{x}{3-2x}$.

Meanwhile, $\frac{2g}{3g+1} = \mathbf{2g} \cdot \frac{1}{3g+1} = \mathbf{2g} \cdot (\mathbf{3g+1})^{-1}$.

And thus, we have $\mathbf{2g} = \frac{2x}{3-2x}$, and $\mathbf{3g+1} = \frac{3x}{3-2x} + \mathbf{1} = \frac{3x+3-2x}{3-2x} = \frac{x+3}{3-2x}$.

So we get $\frac{2g}{3g+1} = \frac{2x}{3-2x} \cdot \frac{3-2x}{x+3} = \frac{2x}{x+3}$. And thus, we get $\boldsymbol{h(x) = (f \bullet g)(x)} = \frac{2x}{x+3}$.

And next, we want to find the $\boldsymbol{x}$-value satisfying $\boldsymbol{(f \bullet g)(x) = f(x)}$.

That is to say that we want to solve the equation above.

First, we can set $\frac{2x}{x+3} = \frac{2x}{3x+1}$. And thus, solving the equation, we get the $\boldsymbol{x}$-value for which the equality can hold.

Solving it though, we may not want to just begin with dividing both sides by $\boldsymbol{2x}$.

That's because $\boldsymbol{x}$ can be 0, and no division by 0 is allowed.

And in fact, if $\boldsymbol{x = 0}$, the equality holds. So anyway, 0 is a solution.

Thus next, assuming $\boldsymbol{x \neq 0}$, and dividing both sides by $\boldsymbol{2x}$, we get

$\boldsymbol{x + 3 = 3x + 1 \Rightarrow 2x = 2 \Rightarrow x = 1}$.

And thus, 0 and 1 can be the $\boldsymbol{x}$-value. That is, the $\boldsymbol{x}$-value is 0 or 1.

And again, the domain matters. So we may want to note two cases below.

One is that in $\boldsymbol{h}$, $\boldsymbol{x \neq -3}$, since $\boldsymbol{h(x) = (f \bullet g)(x)} = \frac{2x}{x+3}$, and the denominator cannot be 0.

And the other is that in $\boldsymbol{h}$, $\boldsymbol{x}$ cannot be $\frac{3}{2}$, either.

That's because the domain of $\boldsymbol{h}$ is the domain of $\boldsymbol{g}$ since $\boldsymbol{h = f \bullet g}$, so $\boldsymbol{g}$ is the function blends into, and the domain of $\boldsymbol{g}$ does not have $\frac{3}{2}$, since $\boldsymbol{g(x)} = \frac{x}{3-2x}$, and the denominator cannot be 0.

So doing problems, we may want to check to see if the solution is in the domain.

And in particular, working with fractional expressions and radicals as $\sqrt[n]{x-1}$ where the index n is even as 2, 4, or 6, we want to make sure the denominator is not 0 or what's inside the radical sign is $\geq \mathbf{0}$.

In short:

To begin with, $(f \bullet g)(x)$ is a composite function, where g blends into f.

Finding thus, $h = f \bullet g$ first, we get $f \bullet g = f(g) = \frac{2g}{3g+1}$, where $g = \frac{x}{3-2x}$.

Meanwhile, $\frac{2g}{3g+1} = 2g \cdot \frac{1}{3g+1} = 2g \cdot (3g+1)^{-1}$.

So we get

$2g = \frac{2x}{3-2x}$, $3g+1 = \frac{3x}{3-2x} + 1 = \frac{3x+3-2x}{3-2x} = \frac{x+3}{3-2x}$, and $\frac{2g}{3g+1} = \frac{2x}{3-2x} \cdot \frac{3-2x}{x+3} = \frac{2x}{x+3}$.

And thus, $(f \bullet g)(x) = \frac{2x}{x+3}$.

So next, $(f \bullet g)(x) = f(x) \Rightarrow \frac{2x}{x+3} = \frac{2x}{3x+1}$.

Thus, first, if $x = 0$, the equality holds. And next, assuming $x \neq 0$, and dividing both sides by $2x$, we get $x + 3 = 3x + 1 \Rightarrow 2x = 2 \Rightarrow x = 1$.

$\therefore$ the x-value is **0** or **1**. (The symbol $\therefore$ means therefore.)