Examples 4 in Composite Functions

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0. Suppose a is constant, $f: x \longrightarrow 9ax - 5$, and $g: x \longrightarrow 7 - 2ax$. Then,

0.0. Find the value of a for which $g \bullet f = f \bullet g$.

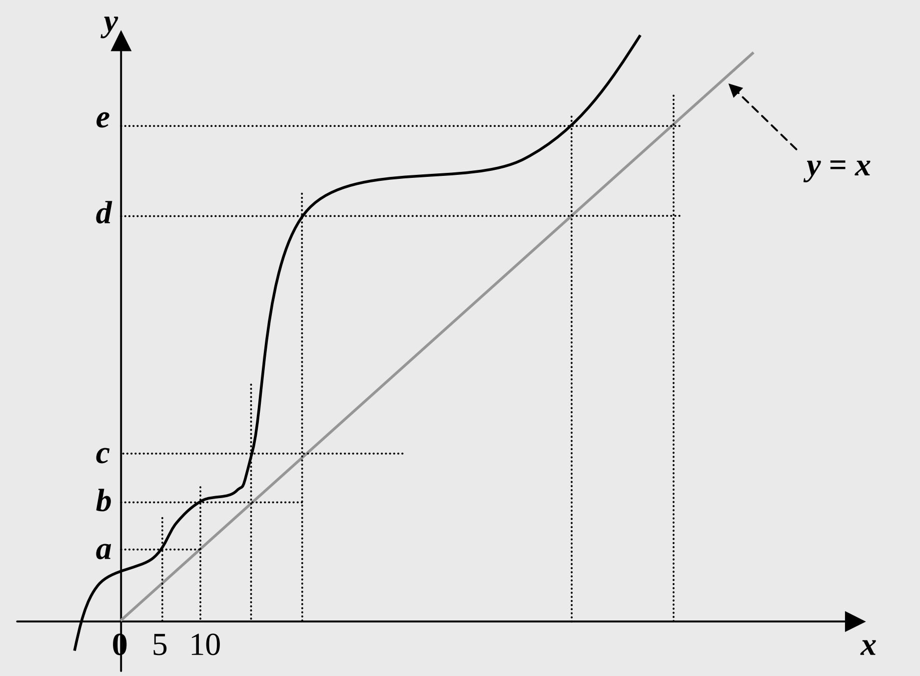
0.1. Find a function h so that $f \bullet h = g$, using the value of a found in the problem 0.

0.2. Find a function h so that $h \bullet g \bullet f = f$, using the value of a found in the problem 0.

1. Suppose $y = f_1(x)$, $f_2 = f_1 \bullet f_1$, $f_3 = f_2 \bullet f_1$, $f_4 = f_3 \bullet f_1$, $\ldots$ $f_k = f_{k-1} \bullet f_1$.

And in the graph below, a, b, c, d, and e are constant, and the curve of f_1 is in black.

Fig. 1.0



Then,

0. Find $f_4(10)$.

1. Find p, q, and r so that we get $f_2(p) = c$, $f_2(q) = b$, and $f_3(r) = c$.

Suggestions or Solutions
To the Problem 0 in the Example 0

Assuming a is constant, f: $x \longrightarrow 9ax - 5$, and g: $x \longrightarrow 7 - 2ax$, find the value of a for which $g \bullet f = f \bullet g$.

This is just another example of algebra practice on composite functions.

Now, we have $y = f(x) = 9ax - 5$ for x real, and $y = g(x) = 7 - 2ax$ for x real.

So to begin with, finding $g \bullet f$, we get

$g \bullet f = g(f) = 7 - 2af = 7 - 2a(9ax - 5) = 7 - 18a^2x + 10a$.

And next, moving on to $f \bullet g$, we get

$f \bullet g = f(g) = 9ag - 5 = 9a(7 - 2ax) - 5 = 63a - 18a^2x - 5$.

Thus next, we get

$g \bullet f = f \bullet g \Rightarrow 7 - 18a^2x + 10a = 63a - 18a^2x - 5 \Rightarrow 7 + 10a = 63a - 5$

$\Rightarrow 53a = 12 \Rightarrow a = \frac{12}{53}$.

And thus, it can be the case where composite-function-operations are commutative even if neither of the two functions is an identity function.
So we can have cases where $p \bullet q = q \bullet p$ even if neither of p and q is an identity function.

In other words, it is *sometimes* the case where $p \bullet q = q \bullet p$ if p and q are functions, of course. So it is not always the case.

- Thus, *in general*, assuming F and G are functions, we have $F \bullet G \neq G \bullet F$.

Suggestions or Solutions

To the Problem 1 in the Example 0

Assuming a is constant, f: $x \longrightarrow 9ax - 5$, and g: $x \longrightarrow 7 - 2ax$, find a function h so that we get $f \bullet h = g$, using the value of a found in the problem 0.0.

To begin with, $f(x) = 9 \cdot \frac{12}{53} x - 5 = \frac{108}{53} x - 5$, and $g(x) = 7 - 2 \cdot \frac{12}{53} x = 7 - \frac{24}{53} x$.

So first, we can get $f(x) = \frac{108}{53} x - 5 \Rightarrow f(h) = \frac{108}{53} h - 5$. And next, we can get

$\frac{108}{53} h - 5 = 7 - \frac{24}{53} x \Rightarrow \frac{108}{53} h = 12 - \frac{24}{53} x \Rightarrow h = \frac{53}{108}(12 - \frac{24}{53} x) = \frac{53}{12 \cdot 9}(12 - \frac{24}{53} x) = \frac{1}{9}(53 - 2x)$.

Therefore, $h(x) = \frac{1}{9}(53 - 2x)$ for x real.

If not quite sure of the idea behind the processes above, follow the steps below.

(Again, this is just another example of algebra practice on composite functions.)

Now, we have $f(x) = 9ax - 5$ for x real, and $g(x) = 7 - 2ax$ for x real.

And also, we found in the problem 1, that $a = \frac{12}{53}$. So now we have

$f(x) = 9 \cdot \frac{12}{53} x - 5 = \frac{108}{53} x - 5$, and $g(x) = 7 - 2 \cdot \frac{12}{53} x = 7 - \frac{24}{53} x$.

Then, to begin with, we can put $f \bullet h = g$ this way, too: $f(h) = g$.

And next, we have $f(x) = \frac{108}{53} x - 5$. So we get $f(h) = \frac{108}{53} h - 5$.

And next, we have $g(x) = 7 - \frac{24}{53} x$. And we have $f(h) = g$, too.

So we can set $\frac{108}{53} h - 5 = 7 - \frac{24}{53} x$. And thus, we get

$\frac{108}{53} h - 5 = 7 - \frac{24}{53} x \Rightarrow \frac{108}{53} h = 12 - \frac{24}{53} x \Rightarrow h = \frac{53}{108}(12 - \frac{24}{53} x) = \frac{53}{12 \cdot 9}(12 - \frac{24}{53} x) = \frac{1}{9}(53 - 2x)$.

Therefore, we get $h(x) = \frac{1}{9}(53 - 2x)$ for x real.

Suggestions or Solutions
To the Problem 2 in the Example 0

Assuming a is constant, f: $x \longrightarrow 9ax - 5$, and g: $x \longrightarrow 7 - 2ax$, find a function h so that we get $h \bullet g \bullet f = f$, using the value of a found in the problem 0.0.

To begin with, $f(x) = \frac{108}{53}x - 5$, and $g(x) = 7 - \frac{24}{53}x$.

And setting $m = \frac{108}{53}$, and $n = -\frac{24}{53}$, we have $f(x) = mx - 5$, and $g(x) = nx + 7$.

So assuming $p = g \bullet f$, we get $p = g \bullet f = g(f) = nf + 7 = n(mx - 5) + 7 = mnx - 5n + 7$.

So $h(g \bullet f) = h(p) = h(mnx - 5n + 7) \Rightarrow h(g \bullet f) = f \Rightarrow h(mnx - 5n + 7) = mx - 5$.

And thus, assuming $y = h(t)$, we can set $t = mnx - 5n + 7$.

So we get $t = mnx - 5n + 7 \Rightarrow x = \frac{1}{mn}(t + 5n - 7)$. And we have $f(x) = mx - 5$.

So we get $m\frac{1}{mn}(t + 5n - 7) - 5 = \frac{1}{n}(t + 5n - 7) - 5 = \frac{1}{n}(t - 7) = \frac{24}{53}(7 - t)$, since $n = -\frac{24}{53}$.

Thus, we get $h(t) = \frac{24}{53}(7 - t)$ for t real.

And putting it in the x-y system, we can just set $y = h(x) = \frac{24}{53}(7 - x)$ for x real.

If not quite sure of the idea behind the processes above, follow the steps below.

When we get solutions, understanding matters, of course, but is just one thing. Doing it is another. Doing it, we do algebra. So algebra matters. It actually connects problems to solutions. And thus, this is another example of algebra practice on composite functions.

Now, doing this problem, we may want to use a tool as follows.

$h \bullet (g \bullet f) = (h \bullet g) \bullet f = h \bullet g \bullet f$, which is formally called a theorem.

So we may want to take advantage of it.

And we have $\boldsymbol{f(x) = \frac{108}{53}x - 5}$, and $\boldsymbol{g(x) = 7 - \frac{24}{53}x}$.

Making it a bit simpler though, we can set $\boldsymbol{m = \frac{108}{53}}$, and $\boldsymbol{n = -\frac{24}{53}}$.

Then, we have $\boldsymbol{f(x) = mx - 5}$, and $\boldsymbol{g(x) = nx + 7}$.

So assuming $\boldsymbol{p = g \bullet f}$, we get $\boldsymbol{p = g \bullet f = g(f) = nf + 7 = n(mx - 5) + 7 = mnx - 5n + 7}$.

Thus, we get $\boldsymbol{p(x) = mnx - 5n + 7}$.

So next, we get $\boldsymbol{h(g \bullet f) = h(p) = h(mnx - 5n + 7)}$.

And thus, we get $\boldsymbol{h(g \bullet f) = f \Rightarrow h(mnx - 5n + 7) = mx - 5}$.

Now, what are we after, though?

We are in fact, looking for the expression of $\boldsymbol{h}$.

Then, we can notice that this is quite similar to the ones in **Examples 3**.

Assuming $\boldsymbol{y = h(t)}$, we can set $\boldsymbol{t = mnx - 5n + 7}$.

That's because we have $\boldsymbol{h \bullet p = h(p) = f}$, and $\boldsymbol{p(x) = mnx - 5n + 7}$.

And thus, solving $\boldsymbol{t = mnx - 5n + 7}$ for $\boldsymbol{x}$, we can put $\boldsymbol{x}$ in terms of $\boldsymbol{t}$, that is, we get an expression in terms of $\boldsymbol{t}$, and the expression equals to $\boldsymbol{x}$.

Then, putting the expression into $\boldsymbol{x}$ in the expression of $\boldsymbol{f}$, which is $\boldsymbol{mx - 5}$, we get the expression in the function $\boldsymbol{h}$. If not quite sure, you may want to refer to **Examples 3**.

So now, solving $\boldsymbol{t = mnx - 5n + 7}$ for $\boldsymbol{x}$, we can simply get

$$\boldsymbol{t = mnx - 5n + 7 \Rightarrow x = \frac{1}{mn}(t + 5n - 7)}.$$

Thus next, putting $\frac{1}{mn}(t+5n-7)$ into x in the expression of f, we get

$m\frac{1}{mn}(t+5n-7)-5=\frac{1}{n}(t+5n-7)-5=\frac{1}{n}(t-7)=\frac{24}{53}(7-t)$ since $n=-\frac{24}{53}$.

So we get $h(t)=\frac{24}{53}(7-t)$ for t real.

And if we want to put it in the x-y system, we can just set $y=h(x)=\frac{24}{53}(7-x)$.

We can use any letters as variables insofar as their relation is maintained.

In short:

We have $f(x)=\frac{108}{53}x-5$, and $g(x)=7-\frac{24}{53}x$.

And setting $m=\frac{108}{53}$, and $n=-\frac{24}{53}$, we have $f(x)=mx-5$, and $g(x)=nx+7$.

So assuming $p=g\bullet f$, we get $p=g\bullet f=g(f)=nf+7=n(mx-5)+7=mnx-5n+7$.

So we get $h(g\bullet f)=h(p)=h(mnx-5n+7)\Rightarrow h(g\bullet f)=f\Rightarrow h(mnx-5n+7)=mx-5$.

And thus, assuming $y=h(t)$, we can set $t=mnx-5n+7$.

So solving the equation for x, we get $t=mnx-5n+7\Rightarrow x=\frac{1}{mn}(t+5n-7)$.

Thus next, putting $\frac{1}{mn}(t+5n-7)$ into x in the expression of f, we get

$m\frac{1}{mn}(t+5n-7)-5=\frac{1}{n}(t+5n-7)-5=\frac{1}{n}(t-7)=\frac{24}{53}(7-t)$ since $n=-\frac{24}{53}$.

So we get $h(t)=\frac{24}{53}(7-t)$ for t real.

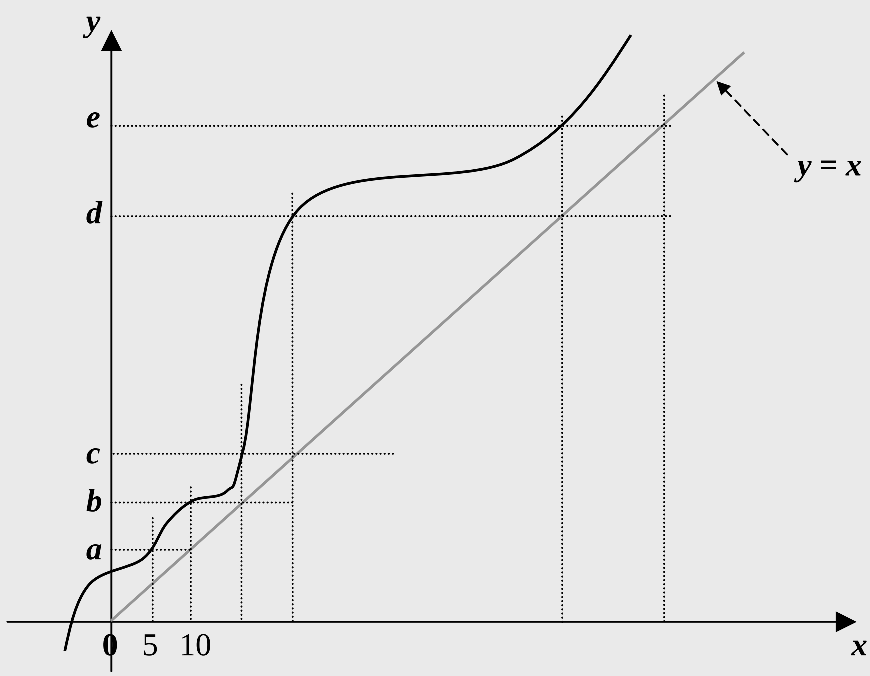
And putting it in the x-y system, we can just set $y=h(x)=\frac{24}{53}(7-x)$ for x real.

Suggestions or Solutions
To the Problem 0 in the Example 1

Suppose $y = f_1(x)$, $f_2 = f_1 \bullet f_1$, $f_3 = f_2 \bullet f_1$, $f_4 = f_3 \bullet f_1$, . . . $f_k = f_{k-1} \bullet f_1$.

And in the graph below, *a, b, c, d,* and *e* are constant, and the curve of f_1 is in black.

Fig. 1.0.0



Then, find $f_4(10)$.

$$f_4(10) = f_3(b) = f_2\{f_1(b)\} = f_2(c) = f_1\{f_1(c)\} = f_1(d) = e \Rightarrow f_4(10) = e.$$

If not quite sure of the idea behind the processes above, follow the steps below.

In this problem, we are given a sequence of composite functions.
So each term in the sequence is a composite function.
Each composite function is however, a blend of the same functions.
And the same function is f_1, used over and over again.
The first term is not though, a composite function but just a function $y = f_1(x)$.

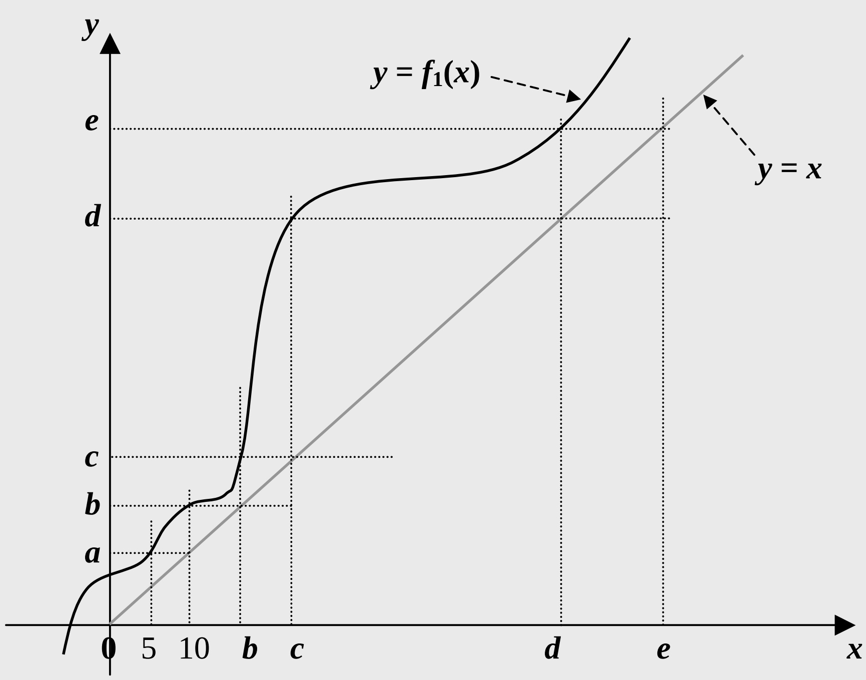
Then, the second term is $\boldsymbol{f_2 = f_1 \bullet f_1}$, the third is $\boldsymbol{f_3 = f_1 \bullet f_1 \bullet f_1}$, and so forth.

Now, at each and every point the line gray above, both coordinates are the same.
That is, $\boldsymbol{x}$-coordinate = $\boldsymbol{y}$-coordinate at every point in the line.

So we can notice that $\boldsymbol{a = 10}$, $\boldsymbol{f_1(10) = b}$, $\boldsymbol{f_1(b) = c}$, $\boldsymbol{f_1(c) = d}$, and $\boldsymbol{f_1(d) = e}$.

And thus, specifying in the graph the values above, we can get

Fig. 1.0.1



What we want to find is though, the output of the composite function $\boldsymbol{f_4(x)}$ when $\boldsymbol{x = 10}$.
The curve and the values in the graph above are however, for the function $\boldsymbol{f_1}$.

How then, can we get the output for the input 10 of the function $\boldsymbol{f_4(x)}$?

We can use the sequence given. Then, first, we can see that $\boldsymbol{f_4 = f_3 \bullet f_1}$.

So we get $\boldsymbol{f_4(x) = (f_3 \bullet f_1)(x) = f_3\{f_1(x)\} \Rightarrow f_4(10) = f_3\{f_1(10)\}}$.

And in the graph above, we have $\boldsymbol{f_1(10) = b}$. So we get $\boldsymbol{f_4(10) = f_3\{f_1(10)\} = f_3(b)}$.

What then, about $\boldsymbol{f_3(b)}$?

Going back to the sequence, we have $\boldsymbol{f_3 = f_2 \bullet f_1}$.

So we get $\boldsymbol{f_3(x) = (f_2 \bullet f_1)(x) = f_2\{f_1(x)\} \Rightarrow f_3(b) = f_2\{f_1(b)\}}$. What then, about $\boldsymbol{f_1(b)}$?

Going back to the graph, we can see this: $\boldsymbol{f_1(b) = c}$.

So we get $\boldsymbol{f_3(b) == f_2\{f_1(b)\} = f_2(c)}$.

And next, going back to the sequence, we get $\boldsymbol{f_2 = f_1 \bullet f_1 \Rightarrow f_2(c) = f_1\{f_1(c)\}}$.

Then again, backing to the graph, we can get $\boldsymbol{f_1(c) = d}$.

So we get $\boldsymbol{f_2(c) = f_1\{f_1(c)\} = f_1(d)}$.

Then again, backing to the graph, we can get $\boldsymbol{f_1(d) = e}$.

So putting threads together, we get

$\boldsymbol{f_4(10) = f_3(b) = f_2\{f_1(b)\} = f_2(c) = f_1\{f_1(c)\} = f_1(d) = e}$. And thus, $\boldsymbol{f_4(10) = e}$.

In short:

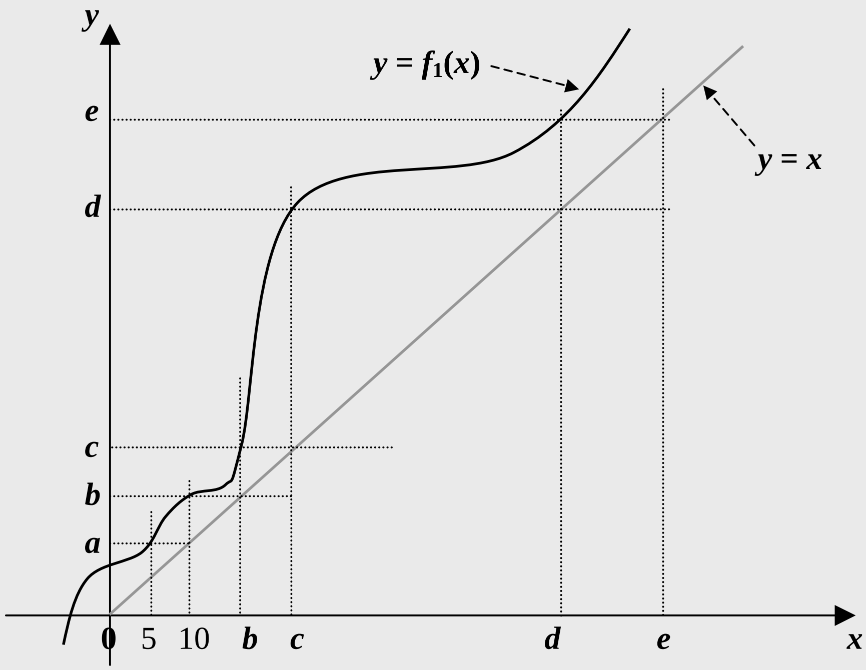
$\boldsymbol{f_4(10) = f_3(b) = f_2\{f_1(b)\} = f_2(c) = f_1\{f_1(c)\} = f_1(d) = e}$.

Suggestions or Solutions
To the Problem 1 in the Example 1

Suppose $y = f_1(x)$, $f_2 = f_1 \bullet f_1$, $f_3 = f_2 \bullet f_1$, $f_4 = f_3 \bullet f_1$, . . . $f_k = f_{k\text{-}1} \bullet f_1$.

And in the graph below, a, b, c, d, and e are constant, and the curve of f_1 is in black.

Fig. 1.1.0



Then, find p, q, and r so that we get $f_2(p) = c$, $f_2(q) = b$, and $f_3(r) = c$.

$f_2 = f_1 \bullet f_1 = f_1(f_1) \Rightarrow f_2(p) = f_1\{f_1(p)\} = c \Rightarrow f_1(p) = b$. And $f_1(10) = b$. So $p = 10$.

$f_2 = f_1 \bullet f_1 = f_1(f_1) \Rightarrow f_2(q) = f_1\{f_1(q)\} = b \Rightarrow f_1(q) = 10$, because $f_1(10) = b$.

And also, we get $a = 10$, and we have $f_1(5) = a = 10$. So $q = 5$.

$f_3 = f_2 \bullet f_1 = f_1 \bullet f_1 \bullet f_1 \Rightarrow f_3(r) = f_1[f_1\{f_1(r)\}] = c \Rightarrow f_1\{f_1(r)\} = b \Rightarrow f_1(r) = 10 \Rightarrow r = 5$.

If not quite sure of the idea behind the processes above, follow the steps below.

Beginning with the equation $f_2(p) = c$, we can use $f_2 = f_1 \bullet f_1$, so we can put it this way:

$f_2(p) = (f_1 \bullet f_1)(p) = c$. What does this equation mean, though?

The equation is saying that from the composite function $f_2(x) = (f_1 \bullet f_1)(x)$, the output c gets produced for a particular value of x in f_2, and the equation is asking us to find *the particular value*.

We don't even know though, the expression of the function $f_2(x)$.

We know however, $f_2 = f_1 \bullet f_1 = f_1(f_1)$, so we can put it this way, too: $f_2(p) = f_1\{f_1(p)\} = c$.

And setting $u = f_1(p)$, we can put it this way, as well: $f_2(p) = f_1(u) = c$.

Then, looking at the graph above, we can see that $u = b$.

So we get $b = f_1(p)$. Thus again getting back to the graph above, we can see $b = f_1(10)$.

That is, we get $b = f_1(p) = f_1(10)$.   So we get $p = 10$.

In short:

$f_2 = f_1 \bullet f_1 = f_1(f_1) \Rightarrow f_2(p) = f_1\{f_1(p)\} = c \Rightarrow f_1(p) = b$. And $f_1(10) = b$. So $p = 10$.

Let's next, move on to the equation where $f_2(q) = b$.

Then again, we have $f_2 = f_1 \bullet f_1$, so we can put it this way: $f_2(q) = (f_1 \bullet f_1)(q) = b$.

And the equation is saying that from the composite function $f_2(x) = (f_1 \bullet f_1)(x)$, the output b gets produced for a particular value of x in f_2, and the equation is asking us to find *the particular value*.

Then again, since we have $\boldsymbol{f_2 = f_1 \bullet f_1 = f_1(f_1)}$, we can set $\boldsymbol{f_2(q) = f_1\{f_1(q)\} = b}$.

And setting $\boldsymbol{v = f_1(q)}$, we can set $\boldsymbol{f_2(q) = f_1\{f_1(q)\} = f_1(v) = b}$.

Then, looking at the graph above, we can see that $\boldsymbol{f_1(10) = b}$, so we get $\boldsymbol{v = 10}$.

So we get $\boldsymbol{v = f_1(q) \Rightarrow 10 = f_1(q)}$. Then again in the graph, we can see that $\boldsymbol{a = 10}$.
That's because at every point in the line $\boldsymbol{y = x}$, both coordinates are the same.
Thus, we get $\boldsymbol{f_1(q) = a}$.

Then again, backing to the graph, we can see $\boldsymbol{f_1(5) = a}$.
Then, we get $\boldsymbol{f_1(q) = f_1(5)}$. So we get $\boldsymbol{x = 5}$.

In short:

$\boldsymbol{f_2 = f_1 \bullet f_1 = f_1(f_1) \Rightarrow f_2(q) = f_1\{f_1(q)\} = b \Rightarrow f_1(q) = 10}$, because $\boldsymbol{f_1(10) = b}$.

And also, we get $\boldsymbol{a = 10}$, and we have $\boldsymbol{f_1(5) = a = 10}$. So $\boldsymbol{q = 5}$.

And moving on to the next, we have to solve this: $\boldsymbol{f_3(r) = c}$.

First, we have $\boldsymbol{f_3 = f_2 \bullet f_1 = f_1 \bullet f_1 \bullet f_1}$, so we can put it this way: $\boldsymbol{f_3(r) = f_1[f_1\{f_1(r)\}] = c}$.

And examining the graph, we can see that $\boldsymbol{f_1(b) = c}$. So we get $\boldsymbol{f_1\{f_1(r)\} = b}$.
That's because $\boldsymbol{f_3(r) = f_1[f_1\{f_1(r)\}] = f_1(b) = c}$.

Then again, in the graph, we can see $\boldsymbol{f_1(10) = b}$. So we get $\boldsymbol{f_1(r) = 10}$.
That's because $\boldsymbol{f_1\{f_1(r)\} = f_1(10) = b}$.

And backing to the graph again, we can see $\boldsymbol{f_1(5) = a}$.
So we get $\boldsymbol{f_1(5) = 10}$, because $\boldsymbol{a = 10}$. Thus, we get $\boldsymbol{f_1(5) = f_1(r)}$. So we get $\boldsymbol{r = 5}$.

In short:

$\boldsymbol{f_3 = f_2 \bullet f_1 = f_1 \bullet f_1 \bullet f_1 \Rightarrow f_3(r) = f_1[f_1\{f_1(r)\}] = c \Rightarrow f_1\{f_1(r)\} = b \Rightarrow f_1(r) = 10 \Rightarrow r = 5}$.