Examples 1 in Inverse Functions

Click this to start.

Examples 1 in Inverse Functions

Though these examples are for your skill on functions, particularly on inverse functions, they are for your algebra skill, too. And the same is true, too, for all the other examples in this book.

Math skill is about problem solving skill, that is, the skill to get solutions to problems. And what actually connects problems to solutions is algebra. What algebra?

Doing algebra, you get to manipulate expressions, that is, you need to change or alter, convert or transform expressions the way you can get the ones you are looking for. What expressions?

Most often used are numbers of many kinds, variables and constants, monomials and polynomials, powers and logarithms, radicals and fractionals, ratios and rates, and equations and functions.

And what matters is *your* algebra, that is, *your* calculation skill, and more specifically, how to handle those expressions to get to the ones you are busy looking for. And you know what you are busy looking for doing problems. They are the solutions, of course.

0. Find the inverse of $\boldsymbol{y = f(x) = ax + b}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant, and $\boldsymbol{a \neq 0}$.

1. Find the inverses of the functions as follows.

0. $\boldsymbol{y = f(x) = \sqrt{2 - x} + 3}$ for $\boldsymbol{x \leq 2}$.

1. $\boldsymbol{y = f(x) = (2x - 1)(2x + 5)}$ for $\boldsymbol{x \geq 0}$.

Suggestions or Solutions
To the Problem in the Example 0

Find the inverse of $y = f(x) = ax + b$ where a and b are constant, and $a \neq 0$.

$y = ax + b \Rightarrow x = \frac{1}{a}(y - b)$.

So if g is the inverse, we get $y = g(x) = \frac{1}{a}(x - b)$.

If not quite sure of the idea behind the processes above, follow the steps below.

Finding the inverse of a function, what do we need to begin with?

We may want to begin with checking to see if the function is one-to-one.
Usually though, if an inverse is asked, the original function is assumed to be one-to-one.
Anyway, the function f is one-to-one. Why?

The curve of f is a line, so no output can appear more than once.

Checking analytically though, we can do either a division or subtraction with two arbitrary outputs as $f(u)$ and $f(v)$ where u and v are constant, and $u \neq v$, of course.

And if the quotient is 1 or the difference is 0, the function f is not one-to-one.

Otherwise, it is one-to-one. So let's try the testing for practice.

Taking the difference, we get $f(u) - f(v) = (au + b) - (av + b) = a(u - v) \neq 0$.

So $f(u) \neq f(v)$, and thus, f is one-to-one.

What then is the next to do?

We want to find the expression of the inverse. How?

We want to solve for the input variable the equation made of the expression of the original function, then swap the variables.
So first, we want to solve for x the equation $y = ax + b$.

Then, we get $y = ax + b \Rightarrow x = \frac{1}{a}(y - b)$. And next, swapping the variables, we get $y = \frac{1}{a}(x - b)$, which is the expression of the inverse.

So assuming g is the inverse, we get $y = g(x) = \frac{1}{a}(x - b)$ for x real.

Why in the inverse, is the domain a set of all real numbers?

Of the original function f, the range is a set of all real numbers, and the range is the domain of the inverse. And in fact, the domain of f is assumed to be a set of all real numbers since the domain is not specified.

In short:

$y = ax + b \Rightarrow x = \frac{1}{a}(y - b)$.

So assuming g is the inverse, we get $y = g(x) = \frac{1}{a}(x - b)$ for x real.

Suggestions or Solutions
To the Problem 0 in the Example 1

Find the inverse of $y = f(x) = \sqrt{2-x} + 3$ for $x \le 2$.

$\boldsymbol{y = \sqrt{2-x} + 3 \Rightarrow \sqrt{2-x} = y - 3 \Rightarrow 2 - x = (y-3)^2 \Rightarrow x = 2 - (y-3)^2}$.

The domain of $\boldsymbol{f}$ is $\boldsymbol{x \le 2}$. So finding the range, we get

$\boldsymbol{x \le 2 \Rightarrow \sqrt{2-x} \ge 0 \Rightarrow \sqrt{2-x} + 3 \ge 3}$. And thus, the range is $\boldsymbol{y \ge 3}$.

So the domain of the inverse is $\boldsymbol{x \ge 3}$.

And thus, assuming $\boldsymbol{g}$ is the inverse, we get $\boldsymbol{y = g(x) = 2 - (x-3)^2}$ for $\boldsymbol{x \ge 3}$.

If not quite sure of the idea behind the processes above, follow the steps below.

The function $\boldsymbol{f}$ is one-to-one. If not sure though, you may want to give it a check.

Now, what do we need to begin with finding the inverse?

Since $\boldsymbol{f}$ is one-to-one, we should be able to find the inverse. Finding the expression first, we want to solve for $\boldsymbol{x}$ the equation made of the expression of $\boldsymbol{f}$, then swap the variables.

So solving the equation for $\boldsymbol{x}$, we get

$$\boldsymbol{y = \sqrt{2-x} + 3 \Rightarrow \sqrt{2-x} = y - 3 \Rightarrow 2 - x = (y-3)^2 \Rightarrow x = 2 - (y-3)^2}.$$

And next, swapping the variables, we get $\boldsymbol{y = 2 - (x-3)^2}$.

Why swapping though?

The inputs of the inverse are the outputs of the original function, and the outputs of the inverse are the inputs of the original.
Now, is that all?

Not quite, and in fact, a critical factor is missing, and is the domain of the inverse.

What then, is the domain?

The domain is the range of the original function f, so we want to find the range of f.

The domain of f is $x \le 2$. So finding the range, we get

$x \le 2 \Rightarrow \sqrt{2-x} \ge 0 \Rightarrow \sqrt{2-x} + 3 \ge 3$. And thus, the range is $y \ge 3$.

So the domain of the inverse is $x \ge 3$.

And thus, assuming g is the inverse, we get $y = g(x) = 2 - (x-3)^2$ for $x \ge 3$.

What if we use the original function name indicating the inverse?

(Using f^{-1} instead of g, we indicate the inverse, too.)

The definition for inverse functions is $y = f(x) \Leftrightarrow x = f^{-1}(y)$.

And we know the inverse function $x = f^{-1}(y)$ is a function of y.

And also, we have its expression in terms of y, which is $2 - (y-3)^2$, found above.

So we can put the inverse this way, too: $x = f^{-1}(y) = 2 - (y-3)^2$ for $y \ge 3$.

Normally though, we put functions in the x-y system where x is the input variable.

So we usually use x as the input variable, along with another function name as g. And of course, in such a case, we use y as the output variable in g.

In short:

$$y = \sqrt{2-x} + 3 \Rightarrow \sqrt{2-x} = y - 3 \Rightarrow 2 - x = (y-3)^2 \Rightarrow x = 2 - (y-3)^2.$$

The domain of f is $x \le 2$. So finding the range, we get

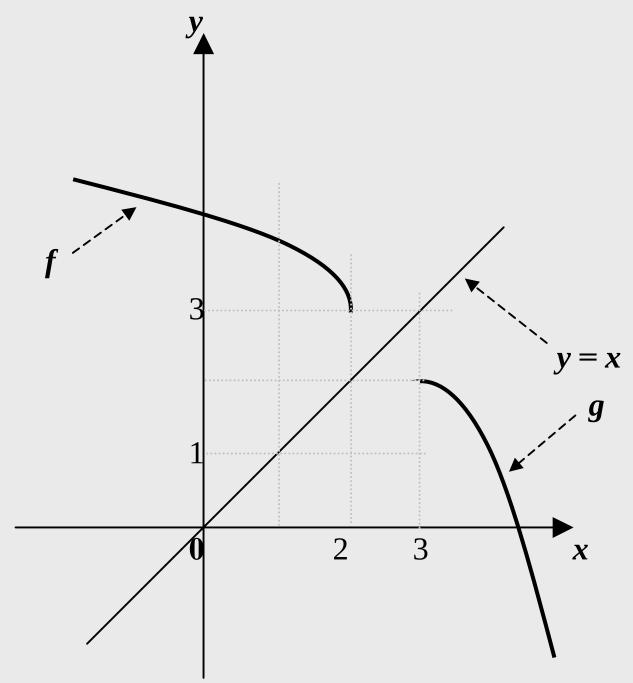
$x \le 2 \Rightarrow \sqrt{2-x} \ge 0 \Rightarrow \sqrt{2-x} + 3 \ge 3$. And thus, the range is $y \ge 3$.

So the domain of the inverse is $x \ge 3$.

And thus, assuming g is the inverse, we get $y = g(x) = 2 - (x-3)^2$ for $x \ge 3$.

Let's now put the curves of both functions, and see how they behave.

Fig. 1.0.0



Both curves are symmetric about the line $y = x$.

Suggestions or Solutions

To the Problem 1 in the Example 1

Find the inverse of $y = f(x) = (2x - 1)(2x + 5)$ for $x \geq 0$.

$y = (2x - 1)(2x + 5) = 4x^2 + 8x - 5 = 4(x^2 + 2x) - 5 = 4(x^2 + 2x + 1 - 1) - 5$

$= 4(x^2 + 2x + 1) - 4 - 5 = 4(x + 1)^2 - 9 \Rightarrow y + 9 = 4(x + 1)^2$

$\Rightarrow (x + 1)^2 = \frac{1}{4}(y+9) \Rightarrow x+1 = \pm\frac{1}{2}\sqrt{y+9} \Rightarrow x = \pm\frac{1}{2}\sqrt{y+9} - 1.$

$x \geq 0 \Rightarrow (x + 1)^2 \geq 1 \Rightarrow 4(x + 1)^2 \geq 4 \Rightarrow 4(x + 1)^2 - 9 \geq -5.$

So the range of f is $y \geq -5$. So using the range to check the domain, we get

$y \geq -5 \Rightarrow \sqrt{y+9} \geq 2 \Rightarrow \frac{1}{2}\sqrt{y+9} \geq 1 \Rightarrow \frac{1}{2}\sqrt{y+9} - 1 \geq 0 \Rightarrow x \geq 0$, the domain of f.

$y \geq -5 \Rightarrow \sqrt{y+9} \geq 2 \Rightarrow -\frac{1}{2}\sqrt{y+9} \leq -1 \Rightarrow -\frac{1}{2}\sqrt{y+9} - 1 \leq -2 \Rightarrow x \leq -2$, not the domain.

And thus, assuming g is the inverse, we get $y = g(x) = \frac{1}{2}\sqrt{x+9} - 1$ for $x \geq -5$.

If not quite sure of the idea behind the processes above, follow the steps below.

Normally, if the domain is a set of all real numbers, and the expression is a polynomial of even degree as quadratic, the function is not one-to-one.

In the given function f, the expression is quadratic polynomial, and yet, the domain is not the entire real number space but a set of all nonnegative numbers.

So the function f can be one-to-one. And thus, we may want to give a check.
In this case though, use of two arbitrary outputs is not quite useful because the algebra looks quite messy. How then, can we see if it is one-to-one or not?

Putting the curve in a graph, we can quickly see if it is one-to-one.

The curve of $\boldsymbol{f}$ is a parabola, so using the sign of the coefficient of the quadratic term and the axis of symmetry, we can readily see if it is the case, too.

So let's now check to see if it is one-to-one.

To begin with, expanding (simplifying) the expression of $\boldsymbol{f}$, we get
$\mathbf{(2x - 1)(2x + 5) = 4x^2 + 8x - 5}$.

Thus, the coefficient is 4, which is positive, so the parabola is concave-up.
And the axis of symmetry is a line $\boldsymbol{x = -1}$.   Why?

If a parabola is $\boldsymbol{y = ax^2 + bx + c}$, the axis of symmetry is $\boldsymbol{x = -\frac{b}{2a}}$, which is a line parallel to the $\boldsymbol{y}$-axis. And we have $\boldsymbol{y = 4x^2 + 8x - 5}$, so $\boldsymbol{a = 4 > 0}$, and the axis is $\boldsymbol{x = -1}$.

Now, putting threads together, the coefficient is positive, and the axis of symmetry is outside the domain of $\boldsymbol{f}$, which is $\boldsymbol{x \geq 0}$. So the function $\boldsymbol{f}$ is one-to-one.

(If not sure, refer to **ALGEBRA EXAMPLES CONICS 2**.)

Now, what then, is the next to find the inverse?

Since $\boldsymbol{f}$ is one-to-one, we should be able to find the inverse. Finding the expression first, we want to solve for $\boldsymbol{x}$ the equation made of the expression of $\boldsymbol{f}$, then swap the variables.

So solving the equation for $\boldsymbol{x}$, we get

$$\boldsymbol{y = 4x^2 + 8x - 5 = 4(x^2 + 2x) - 5 = 4(x^2 + 2x + 1 - 1) - 5 = 4(x^2 + 2x + 1) - 4 - 5}$$

$$\boldsymbol{= 4(x + 1)^2 - 9 \Rightarrow y + 9 = 4(x + 1)^2}$$

$$\boldsymbol{\Rightarrow (x + 1)^2 = \tfrac{1}{4}(y + 9) \Rightarrow x + 1 = \pm\tfrac{1}{2}\sqrt{y + 9} \Rightarrow x = \pm\tfrac{1}{2}\sqrt{y + 9} - 1}.$$

And next, swapping the variables, we get $y = \pm\frac{1}{2}\sqrt{x+9} - 1$. So we have two, one is $y = \frac{1}{2}\sqrt{x+9} - 1$, and the other is $y = -\frac{1}{2}\sqrt{x+9} - 1$. Why two, though?

An inverse has, of course, one expression. So we get to choose one of the two. How?

Using the range of the original function, we can make a right choice. Why?

Before swapping variables, we had $x = \pm\frac{1}{2}\sqrt{y+9} - 1$.

And in the expression above, the extent of the values y can take is the range of f, and the extent of the values x can take is the domain of f.

We know the domain of f is $x \geq 0$. And using the domain, we can find the range of f.

When solving the equation made of the expression of f, we have got $y = 4(x + 1)^2 - 9$.

That is, we can set $y = f(x) = 4(x + 1)^2 - 9$ for $x \geq 0$. So finding the range, we get

$$x \geq 0 \Rightarrow (x + 1)^2 \geq 1 \Rightarrow 4(x + 1)^2 \geq 4 \Rightarrow 4(x + 1)^2 - 9 \geq -5.$$

And thus, the range of f is $y \geq -5$. So applying the range to the two expressions, we get

$y \geq -5 \Rightarrow \sqrt{y+9} \geq 2 \Rightarrow \frac{1}{2}\sqrt{y+9} \geq 1 \Rightarrow \frac{1}{2}\sqrt{y+9} - 1 \geq 0 \Rightarrow x \geq 0$, which is the domain of f.

$y \geq -5 \Rightarrow \sqrt{y+9} \geq 2 \Rightarrow -\frac{1}{2}\sqrt{y+9} \leq -1 \Rightarrow -\frac{1}{2}\sqrt{y+9} - 1 \leq -2 \Rightarrow x \leq -2$, which is not.

So the expression of the inverse is $\frac{1}{2}\sqrt{y+9} - 1$.

And of course, putting it in the ***x-y*** system, we swap the variables.

So the expression of the inverse in the ***x-y*** system is $\frac{1}{2}\sqrt{x+9}-1$. What then, is the next?

We want to get the domain of the inverse, which is the range of ***f***, which has been though, already found, and is $y \geq -5$.

And thus, assuming ***g*** is the inverse, we get $y = g(x) = \frac{1}{2}\sqrt{x+9}-1$ for $x \geq -5$.

In short:

$$y = (2x-1)(2x+5) = 4x^2 + 8x - 5 = 4(x^2 + 2x) - 5 = 4(x^2 + 2x + 1 - 1) - 5$$

$$= 4(x^2 + 2x + 1) - 4 - 5 = 4(x+1)^2 - 9 \Rightarrow y + 9 = 4(x+1)^2$$

$$\Rightarrow (x+1)^2 = \frac{1}{4}(y+9) \Rightarrow x+1 = \pm\frac{1}{2}\sqrt{y+9} \Rightarrow x = \pm\frac{1}{2}\sqrt{y+9}-1.$$

$$x \geq 0 \Rightarrow (x+1)^2 \geq 1 \Rightarrow 4(x+1)^2 \geq 4 \Rightarrow 4(x+1)^2 - 9 \geq -5.$$

So the range of ***f*** is $y \geq -5$. So using the range to check the domain, we get

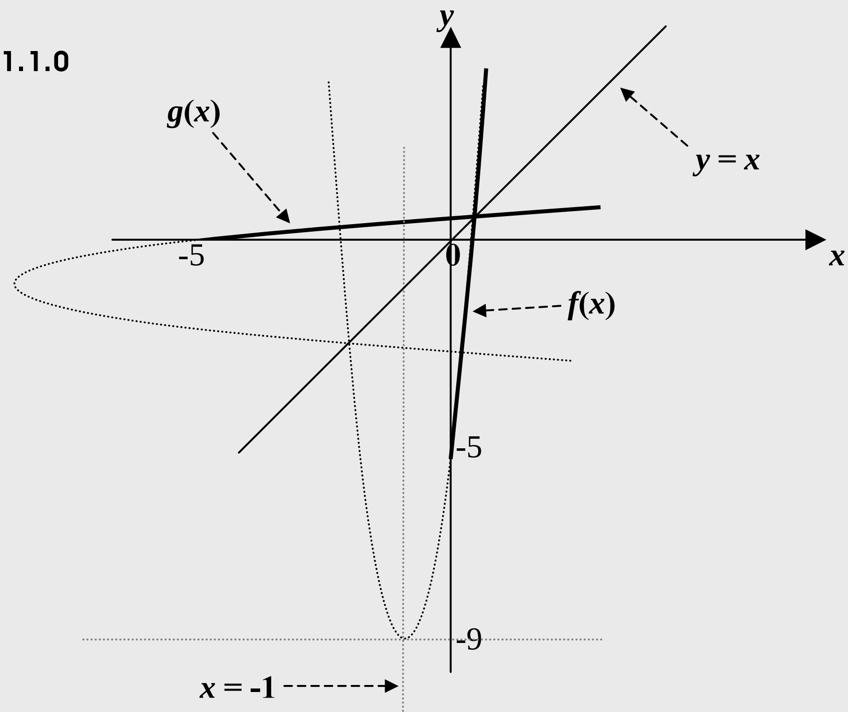
$y \geq -5 \Rightarrow \sqrt{y+9} \geq 2 \Rightarrow \frac{1}{2}\sqrt{y+9} \geq 1 \Rightarrow \frac{1}{2}\sqrt{y+9}-1 \geq 0 \Rightarrow x \geq 0$, which is the domain of ***f***.

$y \geq -5 \Rightarrow \sqrt{y+9} \geq 2 \Rightarrow -\frac{1}{2}\sqrt{y+9} \leq -1 \Rightarrow -\frac{1}{2}\sqrt{y+9}-1 \leq -2 \Rightarrow x \leq -2$, which is not.

And thus, assuming ***g*** is the inverse, we get $y = g(x) = \frac{1}{2}\sqrt{x+9}-1$ for $x \geq -5$.

And putting both curves in one graph, we get

Fig. 1.1.0



And both curves are symmetric about the line $y = x$.

What if the function f is $y = f(x) = (2x - 1)(2x + 5)$ for x real?

That is, the domain of f is the entire real number space, that is, a set of all real numbers.

Then, f is not one-to-one, since the same output appears twice as we can see it in the graph above. So can we not get the inverse?

Normally, we can't. We can find however, two functions, each of which is the inverse of f for a particular part of the domain that is a set of all real numbers.

So we want to partition the domain into two parts, one is $x \geq -1$, and the other is $x < -1$.

So to speak, we get to partition the function f into two functions, then find each inverse.

And if the two functions are $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$, the two are as follows.

$\boldsymbol{y = f_1(x) = (2x - 1)(2x + 5)}$ for $\boldsymbol{x \geq -1}$.

$\boldsymbol{y = f_2(x) = (2x - 1)(2x + 5)}$ for $\boldsymbol{x < -1}$.

Then, each is one-to-one, so we should be able to find the inverse of each.

Finding the expression first, we want to solve for $\boldsymbol{x}$ the equation made of the expression of $\boldsymbol{f}$, then swap the variables. So solving the equation for $\boldsymbol{x}$, we get

$$\boldsymbol{y = (2x - 1)(2x + 5) = 4x^2 + 8x - 5 = 4(x^2 + 2x) - 5 = 4(x^2 + 2x + 1 - 1) - 5}$$

$$\boldsymbol{= 4(x^2 + 2x + 1) - 4 - 5 = 4(x + 1)^2 - 9 \Rightarrow y + 9 = 4(x + 1)^2}$$

$$\boldsymbol{\Rightarrow (x + 1)^2 = \tfrac{1}{4}(y+9) \Rightarrow x+1 = \pm\tfrac{1}{2}\sqrt{y+9} \Rightarrow x = \pm\tfrac{1}{2}\sqrt{y+9} - 1}.$$

And next, swapping the variables, we get $\boldsymbol{y = \pm\tfrac{1}{2}\sqrt{x+9} - 1}$.
So we've got two expressions. Which one is which, though?

Using the range of each original function, we can make a right choice. How?

Before swapping variables, we had $\boldsymbol{x = \pm\tfrac{1}{2}\sqrt{y+9} - 1}$.

And if we begin with $\boldsymbol{f_1}$, in the expression above, the extent of the values $\boldsymbol{y}$ can take is the range of $\boldsymbol{f_1}$, and the extent of the values $\boldsymbol{x}$ can take is the domain of $\boldsymbol{f_1}$.

We know the domain of $\boldsymbol{f_1}$ is $\boldsymbol{x \geq 0}$. And using the domain, we can find the range of $\boldsymbol{f_1}$.

And the same is true for $\boldsymbol{f_2}$, also.

When solving the equation made of the expression of f, we have got $y = 4(x+1)^2 - 9$.

That is, we can set $y = f_1(x) = 4(x+1)^2 - 9$ for $x \geq -1$. So finding the range, we get

$x \geq -1 \Rightarrow (x+1)^2 \geq 0 \Rightarrow 4(x+1)^2 \geq 0 \Rightarrow 4(x+1)^2 - 9 \geq -9$.

And also, we can set $y = f_2(x) = 4(x+1)^2 - 9$ for $x < -1$. So finding the range, we get

$x < -1 \Rightarrow (x+1)^2 > 0 \Rightarrow 4(x+1)^2 > 0 \Rightarrow 4(x+1)^2 - 9 > -9$.

And thus, the range of f_1 is $y \geq -9$, and the range of f_2 is $y > -9$.
So applying the ranges to the expressions of the inverses, we get

$y \geq -9 \Rightarrow \sqrt{y+9} \geq 0 \Rightarrow \frac{1}{2}\sqrt{y+9} \geq 0 \Rightarrow \frac{1}{2}\sqrt{y+9} - 1 \geq -1 \Rightarrow x \geq -1$, the domain of f_1.

$y > -9 \Rightarrow \sqrt{y+9} > 0 \Rightarrow -\frac{1}{2}\sqrt{y+9} < 0 \Rightarrow -\frac{1}{2}\sqrt{y+9} - 1 < -1 \Rightarrow x < -1$, the domain of f_2.

So the expression of the inverse of f_1 is $\frac{1}{2}\sqrt{y+9} - 1$.

And the expression of the inverse of f_2 is $-\frac{1}{2}\sqrt{y+9} - 1$.

And of course, putting them in the x-y system, we swap the variables.

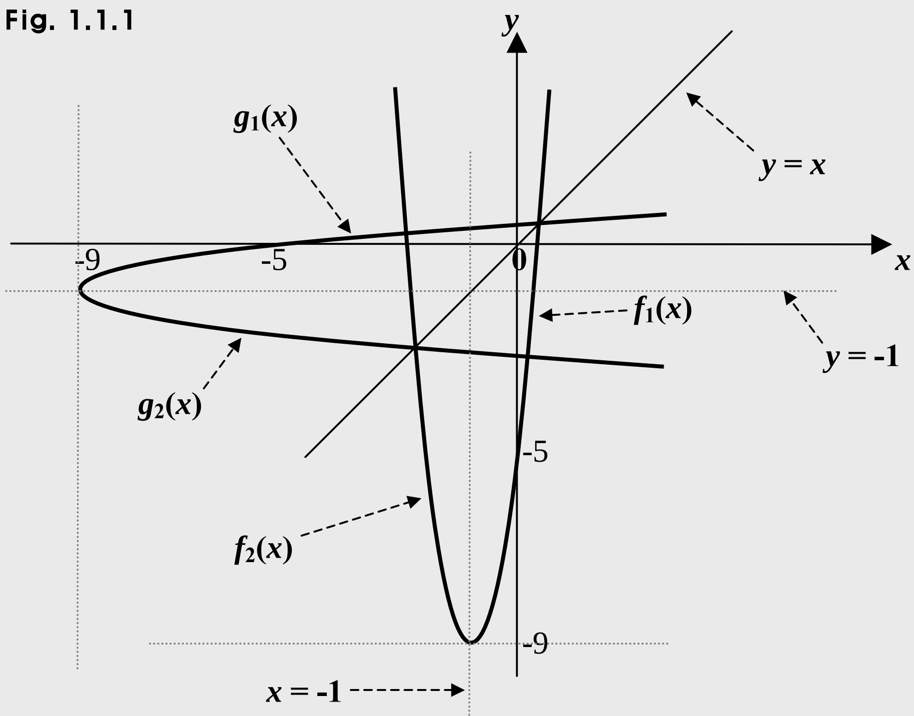
So the expression of the inverse of f_1 in the x-y system is $\frac{1}{2}\sqrt{x+9} - 1$.

And the expression of the inverse of f_2 in the x-y system is $-\frac{1}{2}\sqrt{x+9} - 1$.

And we know the range of f_1 is $y \geq -9$, and the range of f_2 is $y > -9$.

And thus, assuming g_1 is the inverse of f_1, we get $y = g_1(x) = \frac{1}{2}\sqrt{x+9} - 1$ for $x \geq -9$.
And assuming g_2 is the inverse of f_2, we get $y = g_2(x) = -\frac{1}{2}\sqrt{x+9} - 1$ for $x > -9$.
Putting both curves in one graph, we get

Fig. 1.1.1



Each curve is a half parabola.

The curve of $\boldsymbol{g_1}$ is symmetric to the curve of $\boldsymbol{f_1}$ about the line $\boldsymbol{y = x}$.

And also, the curve of $\boldsymbol{g_2}$ is symmetric to the curve of $\boldsymbol{f_2}$ about the line $\boldsymbol{y = x}$, too.