Examples 2 in Inverse Functions

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Find the inverses of the functions as follows.

0. $\boldsymbol{y = f(x) = \frac{2}{1-2x} + 1}$ for $\boldsymbol{x \neq \frac{1}{2}}$.

1. $\boldsymbol{y = f(x) = 2(x-3)^3}$ for $\boldsymbol{x}$ real.

2. $\boldsymbol{y = f(x) = (x-1)(x-2)(x-3)}$ for $\boldsymbol{x \geq 0}$.

3. $\boldsymbol{y = f(x) = (x-1)^4}$ for $\boldsymbol{x}$ real.

Suggestions or Solutions
To the Problem in the Example 0

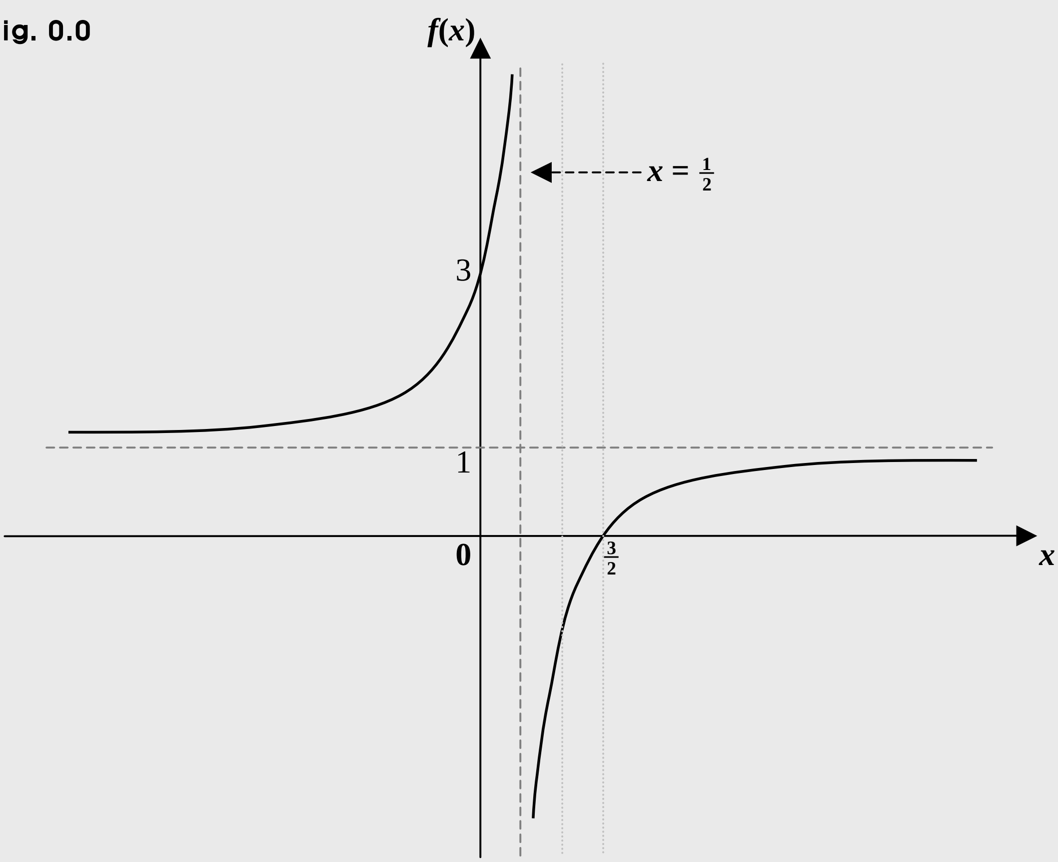
Find the inverse of $y = f(x) = \frac{2}{1-2x} + 1$ for $x \neq \frac{1}{2}$.

Normally, if the expression is a fractional expression as the one above, the function is one-to-one. So the function ***f*** can be one-to-one. And thus, we may want to give it a check.

Using of two arbitrary outputs, and doing a division with the two, we can quickly check to see if it is one-to-one. If the quotient is 1, it is one-to-one.

And in fact, the curve of ***f*** is a hyperbola, so ***f*** will probably be one-to-one.
And putting the curve in a graph, we can put it the way below.

Fig. 0.0



So the domain in $\boldsymbol{f}$ is a set of all real numbers less a number $\frac{1}{2}$.

That is, if $\boldsymbol{R}$ is a set of all real numbers, the domain is $\boldsymbol{R} - \{\frac{1}{2}\}$.

What about the range in $\boldsymbol{f}$, though?

The range in $\boldsymbol{f}$ is a set of all real numbers less a number 1, so it is $\boldsymbol{R} - \{1\}$.

And we can simply put is this way, too: $\boldsymbol{y \neq 1}$.

That's because $\boldsymbol{y}$ cannot be 1, because $\frac{2}{1-2x}$ cannot be 0 for any value of $\boldsymbol{x}$, so $\frac{2}{1-2x}+\mathbf{1}$ cannot be 1. And the domain cannot include $\frac{3}{2}$ since division by 0 is not allowed.

Since $\boldsymbol{f}$ is one-to-one, we should be able to find the inverse.

Finding the expression first, we want to solve for $\boldsymbol{x}$ the equation made of the expression of $\boldsymbol{f}$, then swap the variables.

So solving the equation for $\boldsymbol{x}$, we get

$$y = \frac{2}{1-2x} + 1 \Rightarrow y - 1 = \frac{2}{1-2x} \Rightarrow 1 - 2x = \frac{2}{y-1} \Rightarrow 2x = 1 - \frac{2}{y-1} \Rightarrow x = \frac{1}{2} - \frac{1}{y-1}.$$

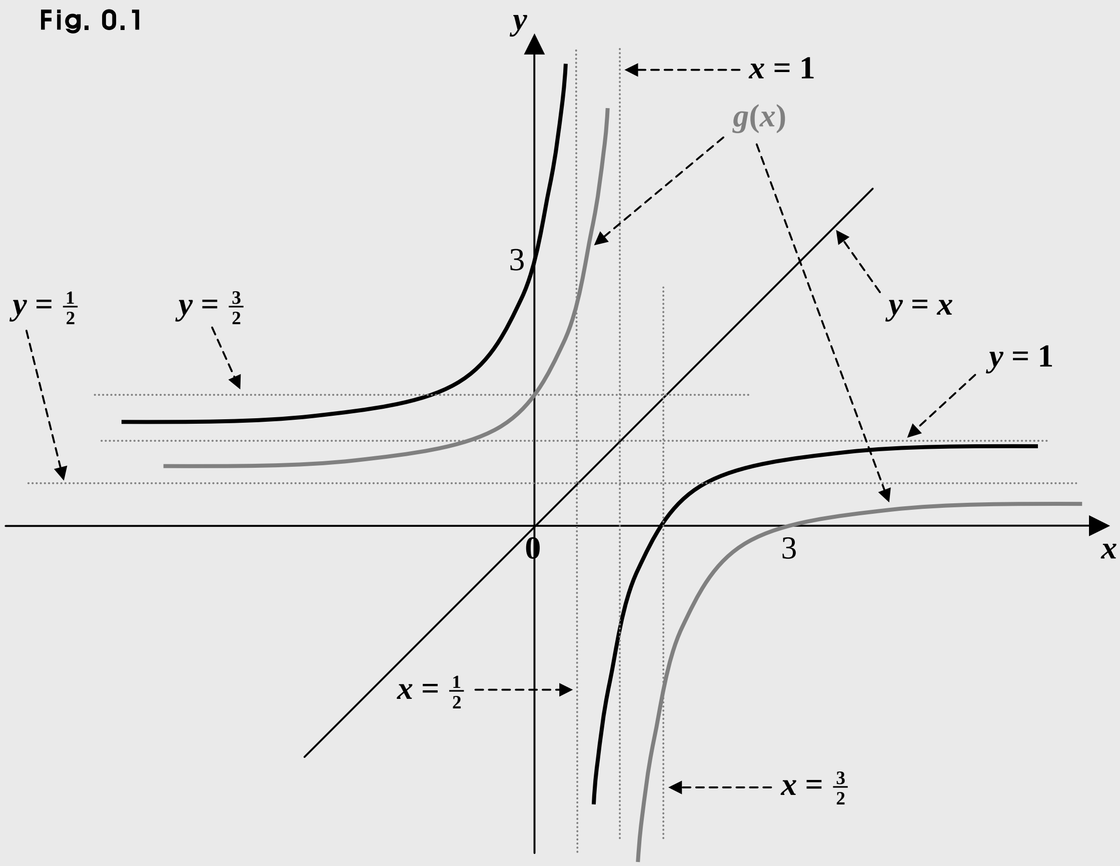
And next, swapping the variables, we get $y = \frac{1}{2} - \frac{1}{x-1}$. What then, is the next?

We want to get the domain of the inverse.

The domain is the range of $\boldsymbol{f}$, and we know the range, which is $\boldsymbol{y \neq 1}$, that is, a set of all real numbers less 1.
So the domain in the inverse is $\boldsymbol{x \neq 1}$, which is a set of all real numbers less 1, too.

And thus, assuming $\boldsymbol{g}$ is the inverse, we get $y = g(x) = \frac{1}{2} - \frac{1}{x-1}$ for $\boldsymbol{x \neq 1}$.

Fig. 0.1



The curve of ***f*** is the curve in black, and the curve of ***g*** is the curve in gray.

Each curve is made of two pieces, called branches in the case as above.

Such a curve is called a hyperbola.

And the two curves are symmetric about the line ***y = x***.

Suggestions or Solutions

To the Problem in the Example 1

Find the inverse of $y = f(x) = 2(x-3)^3$ for x real.

Assuming $a \neq b$, a and b are constant, we get $\frac{f(a)}{f(b)} = \frac{2(a-3)^3}{2(b-3)^3} = \frac{(a-3)^3}{(b-3)^3} = (\frac{a-3}{b-3})^3 \neq 1$.

So the function f is one-to-one. Thus next, we can get the inverse, and get first,

$$y = 2(x-3)^3 \Rightarrow (x-3)^3 = \frac{y}{2} \Rightarrow x - 3 = \sqrt[3]{\frac{y}{2}} \Rightarrow x = \sqrt[3]{\frac{y}{2}} + 3.$$

And thus, assuming g is the inverse, we get $y = \sqrt[3]{\frac{x}{2}} + 3$ for x real.

If not quite sure of the idea behind the processes above, follow the steps below.

Assuming $a \neq b$, a and b are constant, and checking to see if f is one-to-one, we get

$\frac{f(a)}{f(b)} = \frac{2(a-3)^3}{2(b-3)^3} = \frac{(a-3)^3}{(b-3)^3} = (\frac{a-3}{b-3})^3$, and $\frac{a-3}{b-3} \neq 1$, because $a \neq b$.

So the function f is one-to-one, and thus, we should be able to find the inverse.

Finding the expression first, we want to solve for x the equation made of the expression of f, then swap the variables. So solving the equation for x, we get

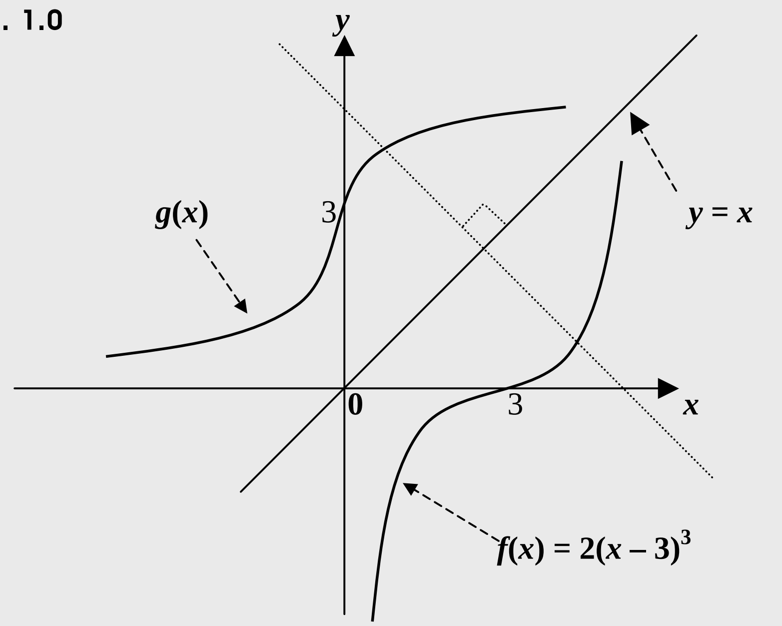
$$y = 2(x-3)^3 \Rightarrow (x-3)^3 = \frac{y}{2} \Rightarrow x - 3 = \sqrt[3]{\frac{y}{2}} \Rightarrow x = \sqrt[3]{\frac{y}{2}} + 3.$$

And next, swapping the variables, we get $y = \sqrt[3]{\frac{x}{2}} + 3 = (\frac{x}{2})^{\frac{1}{3}} + 3$.

And the domain is the range of the original function f, which is a set of all real numbers.

And thus, assuming g is the inverse, we get $y = \sqrt[3]{\frac{x}{2}} + 3$ for x real.

Fig. 1.0



In short:

Assuming $\boldsymbol{a \neq b}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant, we get $\frac{f(a)}{f(b)} = \frac{2(a-3)^3}{2(b-3)^3} = \frac{(a-3)^3}{(b-3)^3} = \left(\frac{a-3}{b-3}\right)^3 \neq 1$.

So the function $\boldsymbol{f}$ is one-to-one. Thus next, we can get the inverse, and get first,

$$y = 2(x-3)^3 \Rightarrow (x-3)^3 = \frac{y}{2} \Rightarrow x - 3 = \sqrt[3]{\frac{y}{2}} \Rightarrow x = \sqrt[3]{\frac{y}{2}} + 3.$$

And thus, assuming $\boldsymbol{g}$ is the inverse, we get $y = \sqrt[3]{\frac{x}{2}} + 3$ for $\boldsymbol{x}$ real.

Suggestions or Solutions
To the Problem in the Example 2

Find the inverse of $y = f(x) = (x - 1)(x - 2)(x - 3)$ for $x \geq 0$.

The function f is not one-to-one.   Why?

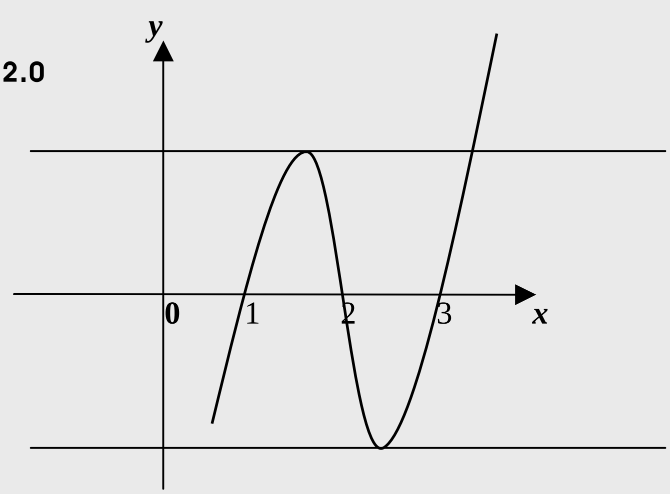
The curve of f passes through the x-axis at $x = 1$, 2, and 3, which are thus, called roots. And the domain includes the roots, so f is not one-to-one.   Why?

If a and b are two different constants, and $f(a) = f(b)$, then f is not one-to-one.

Setting $a = 1$, and $b = 2$, we get $f(1) = f(2) = 0$.   So f is not one-to-one.

Putting in a graph the curve of f, along with two horizontal lines tangent to the curve, we get

Fig. 2.0



Then, finding the two points where the two lines are tangent to the curve, we can partition the domain, and thus, can try finding the inverse for each partition.

There are three partitions, but finding the two points is not quite easy.

Just doing algebra, we will probably see calculations will be quite messy.

Doing calculus though, we can readily find the two points.

However, even if we get the two points, finding the expression is still not quite easy.

That’s because we have to solve an equation made of a polynomial of third degree.

How to get a specific solution algebraically to such an equation is covered in the book, **EQUATIONS**. Such a specific solution is however, in numbers, and not in an expression.

Suggestions or Solutions

To the Problem in the Example 3

Find the inverse of $y = f(x) = (x - 1)^4$ for x real.

Suppose a, b, s and t are constant, $a \neq b$, $s = a - 1$, and $t = b - 1$.

Then, $s \neq t$, and we get $f(a) - f(b) = (a - 1)^4 - (b - 1)^4 = s^4 - t^4 = (s^2 - t^2)(s^2 + t^2)$
$= (s - t)(s + t)(s^2 + t^2)$, which can be 0.

Thus, f is not one-to-one. However, we can partition f into two functions as below.
One is $y = f_1(x) = (x - 1)^4$ for $x \geq 1$. And the other is $y = f_2(x) = (x - 1)^4$ for $x < 1$.

If $a \geq 1$, $b > 1$, and $a = 2 - b$, we get $a \geq 1 \Rightarrow 2 - b \geq 1 \Rightarrow b \leq 1$, but we have $b > 1$.
So consequently, we get $f(a) - f(b) \neq 0$. And thus, $f_1(x)$ is one-to-one.

If $a < 1$, $b < 1$, and $a = 2 - b$, we get $a < 1 \Rightarrow 2 - b < 1 \Rightarrow b > 1$, but we have $b < 1$.
So consequently, we get $f(a) - f(b) \neq 0$. And thus, $f_2(x)$ is one-to-one.

So to begin with, $y = (x - 1)^4 \Rightarrow x - 1 = \pm\sqrt[4]{y} \Rightarrow x = \pm\sqrt[4]{y} + 1$.

Next finding the range of f_1, we get $x \geq 1 \Rightarrow x - 1 \geq 0 \Rightarrow (x - 1)^4 \geq 0$.

And next, finding the range of f_2, we get $x < 1 \Rightarrow x - 1 < 0 \Rightarrow (x - 1)^4 > 0$.

So next, checking the domains, we get
$y \geq 0 \Rightarrow \sqrt[4]{y} \geq 0 \Rightarrow \sqrt[4]{y} + 1 \geq 1$, which is the domain of f_1.
$y > 0 \Rightarrow \sqrt[4]{y} > 0 \Rightarrow -\sqrt[4]{y} < 0 \Rightarrow -\sqrt[4]{y} + 1 < 1$, which is the domain of f_2.

And we know the range of f_1 is $y \geq 0$, and the range of f_2 is $y > 0$.
And thus, assuming g_1 is the inverse of f_1, we get $y = g_1(x) = \sqrt[4]{x} + 1$ for $x \geq 0$.
And assuming g_2 is the inverse of f_2, we get $y = g_2(x) = -\sqrt[4]{x} + 1$ for $x > 0$.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's check first, to see if the function $\boldsymbol{f}$ is one-to-one.
Then, assuming $\boldsymbol{a}$ and $\boldsymbol{b}$ are two different constants, we can check to see if we get $\boldsymbol{f(a) - f(b) \neq 0}$.

To begin with, we get $\boldsymbol{f(a) - f(b) = (a - 1)^4 - (b - 1)^4}$, which looks a bit bulky.

So setting $\boldsymbol{s = a - 1}$, and $\boldsymbol{t = b - 1}$, we get first, $\boldsymbol{s \neq t}$, so $\boldsymbol{s - t \neq 0}$, since $\boldsymbol{a \neq b}$.

And next, we get $\boldsymbol{f(a) - f(b) = s^4 - t^4 = (s^2 - t^2)(s^2 + t^2) = (s - t)(s + t)(s^2 + t^2)}$.

We know $\boldsymbol{s - t \neq 0}$, and also, $\boldsymbol{(s^2 + t^2) \neq 0}$ because $\boldsymbol{s}$ and $\boldsymbol{t}$ cannot be 0 at the same time since $\boldsymbol{s \neq t}$.

However, we can get $\boldsymbol{s + t = 0}$ if $\boldsymbol{s = -t}$ or $\boldsymbol{-s = t}$, of course.

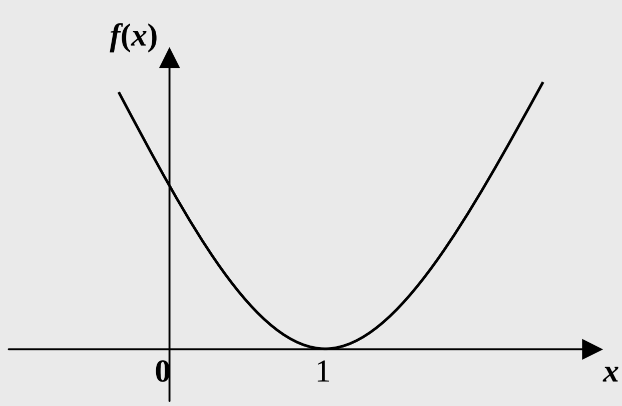
We have $\boldsymbol{s = a - 1}$, and $\boldsymbol{t = b - 1}$, so we get $\boldsymbol{s + t = a + b - 2 = 0}$ if $\boldsymbol{a = 2 - b}$ or $\boldsymbol{b = 2 - a}$.

For instance, if $\boldsymbol{a = 0}$ and $\boldsymbol{b = 2}$, we get $\boldsymbol{s + t = a + b - 2 = 0}$.

And thus, we can get $\boldsymbol{f(a) - f(b) = 0}$ for some values of $\boldsymbol{a}$ and $\boldsymbol{b}$, that is, we cannot get $\boldsymbol{f(a) - f(b) \neq 0}$ for all values of $\boldsymbol{a}$ and $\boldsymbol{b}$. So $\boldsymbol{f}$ is not one-to-one.

And putting the curve of $\boldsymbol{f}$ in a graph, we can get one as follows.

Fig. 3.0



We can try partitioning though, the domain of $\boldsymbol{f}$ so that we can try getting the inverse for each partition. And partitioning, we in fact, partition the function $\boldsymbol{f}$.
We can partition $\boldsymbol{f}$ into two functions as below.

One is $\boldsymbol{y = f_1(x) = (x - 1)^4}$ for $\boldsymbol{x \geq 1}$. And the other is $\boldsymbol{y = f_2(x) = (x - 1)^4}$ for $\boldsymbol{x < 1}$.

Then, both functions $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$ are one-to-one. Let's see though, how it is the case.

We have already done much of the work, and the question is Can we get $\boldsymbol{a + b - 2 \neq 0}$?
That is, we want check to see if we can get this: $\boldsymbol{a \neq 2 - b}$.

So beginning with the function $\boldsymbol{f_1}$, we have $\boldsymbol{a \geq 1}$, and $\boldsymbol{b > 1}$ since the domain is $\boldsymbol{x \geq 1}$.
Why not $\boldsymbol{b \geq 1}$, though?

That's because we need to have $\boldsymbol{a \neq b}$.
So we can put it this way, too: $\boldsymbol{a > 1}$, and $\boldsymbol{b \geq 1}$.

Now, suppose $\boldsymbol{a \geq 1}$, $\boldsymbol{b > 1}$, we can get $\boldsymbol{a = 2 - b}$, and $\boldsymbol{a = 1}$. What then, is $\boldsymbol{b}$?

If $\boldsymbol{a = 1}$, we get $\boldsymbol{a = 2 - b \Rightarrow 1 = 2 - b \Rightarrow b = 1}$, which is however, not allowed since we have $\boldsymbol{a \neq b}$. So there is no value for $\boldsymbol{b}$.

And in fact, we cannot get $\boldsymbol{a = 2 - b}$ if $\boldsymbol{a \geq 1}$, and $\boldsymbol{b > 1}$. Why?

Assuming $\boldsymbol{a \geq 1}$, $\boldsymbol{b > 1}$, and $\boldsymbol{a = 2 - b}$, we get

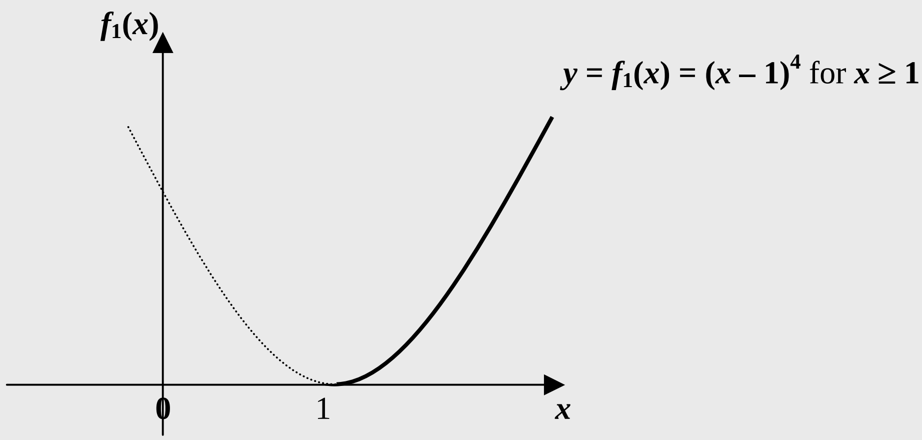
$\boldsymbol{a \geq 1 \Rightarrow 2 - b \geq 1 \Rightarrow b \leq 1}$, which is not possible because we have $\boldsymbol{b > 1}$.

So we cannot get $\boldsymbol{a = 2 - b}$ if $\boldsymbol{a \geq 1}$, and $\boldsymbol{b > 1}$.

And consequently, we get $\boldsymbol{f(a) - f(b) \neq 0}$. And thus, $\boldsymbol{f_1(x)}$ is one-to-one.

Putting $\boldsymbol{f_1}$ in a graph, we can get one as follows.

Fig. 3.1



And next moving on to $\boldsymbol{f_2}$, we have $\boldsymbol{a < 1}$, and $\boldsymbol{b < 1}$ since the domain is $\boldsymbol{x < 1}$.

Now, suppose $\boldsymbol{a < 1}$, $\boldsymbol{b < 1}$, we can get $\boldsymbol{a = 2 - b}$, and $\boldsymbol{a = 0}$.   What then, is $\boldsymbol{b}$?

If $\boldsymbol{a = 0}$, we get $\boldsymbol{a = 2 - b \Rightarrow 0 = 2 - b \Rightarrow b = 2}$, which is not allowed since $\boldsymbol{b < 1}$.

So there is no value for $\boldsymbol{b}$.

And in fact, we cannot get $\boldsymbol{a = 2 - b}$ if $\boldsymbol{a < 1}$ and $\boldsymbol{b < 1}$.   Why?

Assuming $\boldsymbol{a < 1}$, $\boldsymbol{b < 1}$, and $\boldsymbol{a = 2 - b}$, we get

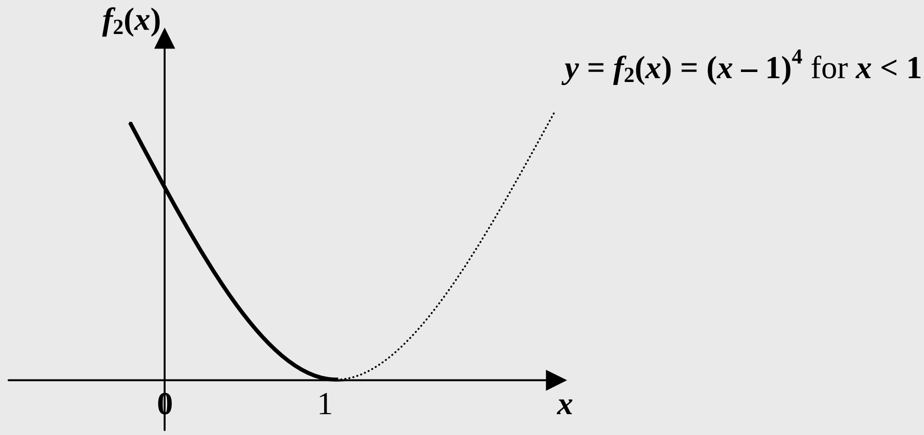
$\boldsymbol{a < 1 \Rightarrow 2 - b < 1 \Rightarrow b > 1}$, which is not possible because we have $\boldsymbol{b < 1}$.

So we cannot get $\boldsymbol{a = 2 - b}$ if $\boldsymbol{a < 1}$, and $\boldsymbol{b < 1}$.

And consequently, we get $\boldsymbol{f(a) - f(b) \neq 0}$. And thus, $\boldsymbol{f_2(x)}$ is one-to-one.

Putting $\boldsymbol{f_2}$ in a graph, we can get one as follows.

Fig. 3.2



So we should be able to find the inverse of each of the two functions f_1 and f_2.

To begin with, we want to find the expression of each.

Finding the expression though, we just want to solve for x the equation made of the expression of f, then swap the variables.

That's because the two original functions f_1 and f_2 share the same expression. So solving the equation for x, we get

$$y = (x-1)^4 \Rightarrow x - 1 = \pm\sqrt[4]{y} \Rightarrow x = \pm\sqrt[4]{y} + 1.$$

And next, swapping the variables, we get $y = \pm\sqrt[4]{x} + 1$.

So we get two expressions, one is the expression in the inverse of f_1, and the other is the expression in the inverse of f_2. How can we see which is which, though?

Using the range in each original function, we can make a right choice.

Before swapping variables, we had $x = \pm\sqrt[4]{y} + 1$.

And if we begin with f_1, in the expression above, the extent of the values y can take is the range in f_1, and the extent of the values x can take is the domain in f_1.

We know the domain in $\boldsymbol{f_1}$ is $\boldsymbol{x \geq 1}$. And using the domain, we can find the range in $\boldsymbol{f_1}$.

And the same is true for $\boldsymbol{f_2}$, also.

So finding the range beginning with $\boldsymbol{y = f_1(x) = (x - 1)^4}$ for $\boldsymbol{x \geq 1}$, we get

$\boldsymbol{x \geq 1 \Rightarrow x - 1 \geq 0 \Rightarrow (x - 1)^4 \geq 0}$.

And next, finding the range in $\boldsymbol{y = f_2(x) = (x - 1)^4}$ for $\boldsymbol{x < 1}$, we get

$\boldsymbol{x < 1 \Rightarrow x - 1 < 0 \Rightarrow (x - 1)^4 > 0}$.

And thus, the range in $\boldsymbol{f_1}$ is $\boldsymbol{y \geq 0}$, and the range in $\boldsymbol{f_2}$ is $\boldsymbol{y > 0}$.

So applying the ranges to the expressions in the inverses, we get

$\boldsymbol{y \geq 0 \Rightarrow \sqrt[4]{y} \geq 0 \Rightarrow \sqrt[4]{y} + 1 \geq 1}$, which is the domain in $\boldsymbol{f_1}$.

$\boldsymbol{y > 0 \Rightarrow \sqrt[4]{y} > 0 \Rightarrow -\sqrt[4]{y} < 0 \Rightarrow -\sqrt[4]{y} + 1 < 1}$, which is the domain in $\boldsymbol{f_2}$.

So the expression in the inverse of $\boldsymbol{f_1}$ is $\boldsymbol{\sqrt[4]{y} + 1}$.

And the expression in the inverse of $\boldsymbol{f_2}$ is $\boldsymbol{-\sqrt[4]{y} + 1}$.

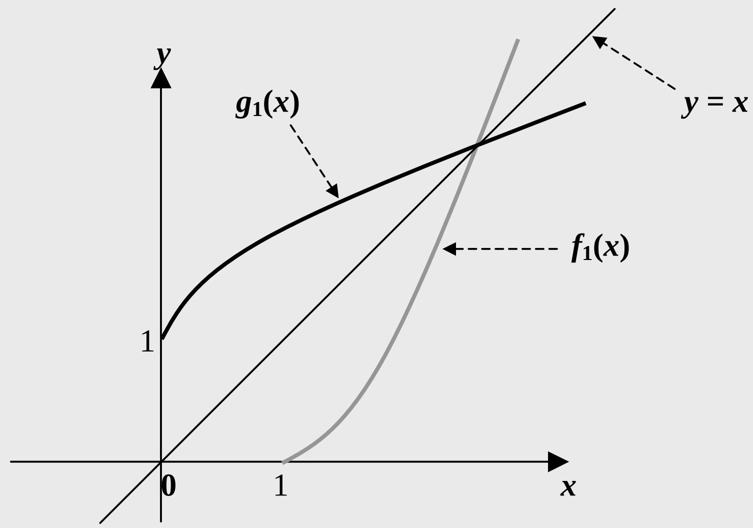
And we know the range in $\boldsymbol{f_1}$ is $\boldsymbol{y \geq 0}$, and the range in $\boldsymbol{f_2}$ is $\boldsymbol{y > 0}$.

And thus, assuming $\boldsymbol{g_1}$ is the inverse of $\boldsymbol{f_1}$, we get $\boldsymbol{y = g_1(x) = \sqrt[4]{x} + 1}$ for $\boldsymbol{x \geq 0}$.

And assuming $\boldsymbol{g_2}$ is the inverse of $\boldsymbol{f_2}$, we get $\boldsymbol{y = g_2(x) = -\sqrt[4]{x} + 1}$ for $\boldsymbol{x > 0}$.

Putting $\boldsymbol{g_1}$ and $\boldsymbol{f_1}$ in one graph, we can get

Fig. 3.3



And putting g_2 and f_2 in one graph, we can get

Fig. 3.4