Examples 3 in Inverse Functions

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0. Suppose that $\boldsymbol{f(x) = \frac{2ax+b}{cx-2}}$ has its inverse as follows. $\boldsymbol{f^{-1}(x) = \frac{x+2}{2x-3}}$ for $\boldsymbol{x \neq \frac{3}{2}}$.

Then, find the values of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

1. Assuming $\boldsymbol{f(x) = 2x + u}$ and $\boldsymbol{g(x) = v - 3x}$, find $\boldsymbol{f^{-1}}$, $\boldsymbol{g^{-1}}$, $\boldsymbol{g \bullet f}$, $\boldsymbol{(g \bullet f)^{-1}}$, $\boldsymbol{(g \bullet f) \bullet (g \bullet f)^{-1}}$, and check to see if $\boldsymbol{(g \bullet f)^{-1} = f^{-1} \bullet g^{-1}}$.

2. Assuming $\boldsymbol{f(x) = \sqrt{4x^2 - 1}}$ for $\boldsymbol{x \geq \frac{1}{2}}$, and $\boldsymbol{g(x) = x(x^2 - 2x - 1)}$ for $\boldsymbol{x \geq 2}$, find $\boldsymbol{f^{-1}(1)}$ and $\boldsymbol{g^{-1}(-2)}$.

Suggestions or Solutions
To the Problem in the Example 0

Suppose that $f(x) = \frac{2ax+b}{cx-2}$ has its inverse as follows. $f^{-1}(x) = \frac{x+2}{2x-3}$ for $x \neq \frac{3}{2}$.

Then, find the values of *a*, *b* and *c*.

Assuming $y = f(x)$, we get $y = \frac{2ax+b}{cx-2} \Rightarrow (cx-2)y = 2ax + b \Rightarrow x(cy-2a) = b + 2y$

$\Rightarrow x = \frac{2y+b}{cy-2a}$. And next, swapping the variables, we get $y = \frac{2x+b}{cx-2a}$.

And we have $f^{-1}(x) = \frac{x+2}{2x-3} = \frac{2x+4}{4x-6}$, which equals $\frac{2x+b}{cx-2a}$.

So comparing the numerators, we get $b = 4$, and next, comparing term by term in the denominators, we can simply get $c = 4$, and $a = 3$.

If not quite sure of the idea behind the processes above, follow the steps below.

This is just an example of algebra practice on inverse functions.

Getting the solution to a problem, we want to first, understand the problem, of course.

Understanding is one thing, though. Doing it is another. And doing it, we usually need to do algebra. We can actually get the solution doing algebra.

Now, we have the definition for inverse functions as follows. $y = f(x) \Leftrightarrow x = f^{-1}(y)$, where $f(x)$ is one-to-one, of course.

Why then, do we have $f^{-1}(x)$ instead of $f^{-1}(y)$? That is, why x and not y?

There is nothing wrong with the variable, and it simply means that the letter x has been chosen to be used as the input variable in the inverse of the function f. Why?

Using $\boldsymbol{y}$ as the input variable as in $\boldsymbol{f^{-1}(y)}$, we can just put the expression this way: $\frac{y+2}{2y-3}$. Not quite sure?

Taking the inverse of $\boldsymbol{f}$, we first solve for $\boldsymbol{x}$ the equation made of the expression of $\boldsymbol{f}$, then swap the variables. And then, we usually choose a name for the inverse.

So for instance, we often set $\boldsymbol{y = g(x)}$.

Even after swapping the variables however, if we keep the name of the inverse unchanged, the name will just remain to be $\boldsymbol{f^{-1}}$ instead of $\boldsymbol{g}$ as in the instance above.

So if $\boldsymbol{g}$ is chosen to be the name of the inverse, we can just set $\boldsymbol{y = g(x) =} \frac{x+2}{2x-3}$ for $\boldsymbol{x \neq} \frac{3}{2}$. What can we do about this problem, then?

We just take the inverse of $\boldsymbol{f}$, and compare it to the one given, which is $\frac{x+2}{2x-3}$, of course.

So assuming $\boldsymbol{y = f(x)}$, and solving the equation made of the expression of $\boldsymbol{f}$, we get

$$\boldsymbol{y = \frac{2ax+b}{cx-2} \Rightarrow (cx-2)y = 2ax + b \Rightarrow x(cy-2a) = b + 2y \Rightarrow x = \frac{2y+b}{cy-2a}}.$$

And next, swapping the variables, we get $\boldsymbol{y = \frac{2x+b}{cx-2a}}$.

So next, we want to compare the expression above with the one given, which is $\frac{x+2}{2x-3}$.

We now have $\frac{2x+b}{cx-2a}$, and $\frac{x+2}{2x-3}$. Comparing the two though, we can see the coefficients of the $\boldsymbol{x}$-terms in the numerators don't match. So we want to make both the same.

Then, we simply get $\frac{2x+b}{cx-2a}$, and $\frac{2x+4}{4x-6}$.

Now, comparing the constant terms in the numerators, we just get $\boldsymbol{b = 4}$, and next, comparing term by term in the denominators, we simply get $\boldsymbol{c = 4}$, and $\boldsymbol{a = 3}$.

And we can get the same in just another way, too, which is as follows.

We know that if there exists the f^{-1} of an f, we get $(f^{-1})^{-1} = f$, which is a theorem, and is saying that the inverse of the inverse is just the original function, and the idea was covered when we covered the theorems on inverses in the section **Inverse Functions 2**.

So let's use the theorem. That is, we take the inverse of the inverse, then do the comparison. We have $y = f^{-1}(x) = \frac{x+2}{2x-3}$. So finding the expression of the inverse of f^{-1}, we want to solve for x the equation $y = \frac{x+2}{2x-3}$, then swap the variables. And thus, solving it, we get $y = \frac{x+2}{2x-3} \Rightarrow (2x - 3)y = x + 2 \Rightarrow x(2y - 1) = 3y + 2 \Rightarrow x = \frac{3y+2}{2y-1}$.

And next, swapping the variables, we get $y = \frac{3x+2}{2x-1}$.
So the expression $\frac{3x+2}{2x-1}$ is the expression of the original function f.

We have $f(x) = \frac{2ax+b}{cx-2}$.
So the expression in f is $\frac{2ax+b}{cx-2}$, and thus, has to be the same as $\frac{3x+2}{2x-1}$.
And thus, we can now compare the two.

Prior to the comparison though, we may want to set $\frac{3x+2}{2x-1} = \frac{6x+4}{4x-2}$.

Then, we can make the comparison a bit easier. We now have $\frac{2ax+b}{cx-2}$, and $\frac{6x+4}{4x-2}$.

So comparing term by term in the numerators, we just get $a = 3$, and $b = 4$, and next, comparing the x-terms term in the denominators, we simply get $c = 4$.

In short:

Assuming $y = f(x)$, we get $y = \frac{2ax+b}{cx-2} \Rightarrow (cx - 2)y = 2ax + b \Rightarrow x(cy - 2a) = b + 2y$

$\Rightarrow x = \frac{2y+b}{cy-2a}$. And next, swapping the variables, we get $y = \frac{2x+b}{cx-2a}$.

And we have $f^{-1}(x) = \frac{x+2}{2x-3} = \frac{2x+4}{4x-6}$, which equals $\frac{2x+b}{cx-2a}$.

So comparing the numerators, we get $b = 4$, and next, comparing term by term in the denominators, we simply get $c = 4$, and $a = 3$.

Suggestions or Solutions
To the Problem in the Example 1

Assuming $f(x) = 2x + u$ and $g(x) = v - 3x$, find f^{-1}, g^{-1}, $g\bullet f$, $(g\bullet f)^{-1}$, $(g\bullet f)\bullet(g\bullet f)^{-1}$, and check to see if $(g\bullet f)^{-1} = f^{-1}\bullet g^{-1}$.

Let's begin with the inverse function f^{-1}.

Finding the expression first, we want to solve for x the equation made of the expression of f, then swap the variables. So solving the equation for x, we get first,

$y = 2x + u \Rightarrow x = \frac{y-u}{2}$. And next, swapping the variables, we get $y = \frac{x-u}{2}$.

And thus, $f^{-1}(x) = \frac{x-u}{2}$ for x real.

Next, moving on to the inverse of g, we get

$y = v - 3x \Rightarrow x = \frac{v-y}{3}$. And next, swapping the variables, we get $y = \frac{v-x}{3}$.

And thus, $g^{-1}(x) = \frac{v-x}{3}$ for x real.

Next, moving on to the composite function $g\bullet f$, we get

$g\bullet f = g\{f(x)\} = g(f) = v - 3f = v - 3(2x + u) = v - 3u - 6x$.

Next, moving on to the inverse of the composite function $g\bullet f$, we get first,

$y = v - 3u - 6x \Rightarrow x = \frac{v-3u-y}{6}$. And next, swapping the variables, we get $y = \frac{v-3u-x}{6}$.

And thus, $(g\bullet f)^{-1}(x) = \frac{v-3u-x}{6}$ for x real.

Let's next, move on to the composite function $(g \bullet f) \bullet (g \bullet f)^{-1}$.

Assuming $h = g \bullet f$, and $z = (g \bullet f)^{-1}$, we get

$(g \bullet f) \bullet (g \bullet f)^{-1} = h \bullet z = h(z) = v - 3u - 6z = v - 3u - 6 \cdot \frac{v-3u-x}{6} = v - 3u - v + 3u + x = x$.

So $(g \bullet f) \bullet (g \bullet f)^{-1}$ is an identity function.

And let's next, move on to $(g \bullet f)^{-1} = f^{-1} \bullet g^{-1}$, which is in fact, a theorem, and thus, can hold.

To begin with, $(f^{-1} \bullet g^{-1})(x) = f^{-1}(g^{-1}) = \frac{g^{-1}-u}{2}$ since $f^{-1}(x) = \frac{x-u}{2}$.

And since $g^{-1}(x) = \frac{v-x}{3}$, we get $f^{-1}(g^{-1}) = \dfrac{\frac{v-x}{3} - u}{2} = \dfrac{\frac{v-x-3u}{3}}{2} = \dfrac{v - 3u - x}{6}$, which is the same as $(g \bullet f)^{-1}(x)$.

Suggestions or Solutions

To the Problem in the Example 2

Assuming $f(x) = \sqrt{4x^2 - 1}$ for $x \geq \frac{1}{2}$, and $g(x) = x(x^2 - 2x - 1)$ for $x \geq 2$, find $f^{-1}(1)$ and $g^{-1}(-2)$.

The value of $f^{-1}(1)$ is the input corresponding to the output 1 in the original function f.

We have $y = f(x) = \sqrt{4x^2 - 1}$. So solving $1 = \sqrt{4x^2 - 1}$, we get

$$1 = \sqrt{4x^2 - 1} \Rightarrow 4x^2 - 1 = 1 \Rightarrow x^2 = \tfrac{1}{2} \Rightarrow x = \pm\tfrac{1}{\sqrt{2}} = \pm\tfrac{\sqrt{2}}{2}.$$

And we know $x \geq \frac{1}{2}$, so we get $x = \frac{\sqrt{2}}{2}$, which is $f^{-1}(1)$.

The value of $g^{-1}(-2)$ is the input corresponding to the output -2 of the original function g.

We have $y = g(x) = x(x^2 - 2x - 1)$. So solving $-2 = x(x^2 - 2x - 1)$, we get

$$x^3 - 2x^2 - x + 2 = x^2(x - 2) - (x - 2) = (x - 2)(x^2 - 1) = (x - 2)(x - 1)(x + 1).$$

And thus, we get $(x - 2)(x - 1)(x + 1) = 0 \Rightarrow x = -1, 1$, or 2.

We know $x \geq 2$. Therefore, we get $g^{-1}(-2) = 2$.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's begin with the inverse of the function f. Finding first, the expression of the inverse, we want to solve for x the equation $y = \sqrt{4x^2 - 1}$, then swap the variables.

So to begin with, $y = \sqrt{4x^2 - 1} \Rightarrow 4x^2 - 1 = y^2 \Rightarrow x = \pm\frac{1}{2}\sqrt{y^2 + 1}$.

And thus, we have two expressions to choose from. So which one?

We know the domain of f is $x \geq \frac{1}{2}$, which means x is positive anyway, so the choice is

$x = \frac{1}{2}\sqrt{y^2+1}$, since $\sqrt{y^2+1}$ is positive anyway. More precisely though, we can find the range of f, then see if we get the domain back applying the range to $\pm\frac{1}{2}\sqrt{y^2+1}$.

Finding the range, we get $x \geq \frac{1}{2} \Rightarrow x^2 \geq \frac{1}{4} \Rightarrow 4x^2 \geq 1 \Rightarrow 4x^2 - 1 \geq 0$.

So the range is $y \geq 0$. And thus, we get $y \geq 0 \Rightarrow \sqrt{y^2+1} \geq 1 \Rightarrow \frac{1}{2}\sqrt{y^2+1} \geq \frac{1}{2} \Rightarrow x \geq \frac{1}{2}$.

So we get $x = \frac{1}{2}\sqrt{y^2+1}$, and swapping the variables, we get $y = \frac{1}{2}\sqrt{x^2+1}$.

We know that the range of f is $y \geq 0$. So we get $y = f^{-1}(x) = \frac{1}{2}\sqrt{x^2+1}$ for $x \geq 0$.

And thus, we get $f^{-1}(1) = \frac{\sqrt{2}}{2}$. And we can get the same the way below, too.

• Directly interpreting the definition for inverse functions, we can get the solution, too.

The definition goes $y = f(x) \Leftrightarrow x = f^{-1}(y)$, and we want to get $f^{-1}(1)$. That is, we want to get the output of the inverse function f^{-1} when $y = 1$. What then, do we mean by $y = 1$?

The y-value is an output of the original function f. And we know each output of the inverse function is each input of the original function. So the value of $f^{-1}(1)$ is the very input corresponding to the output 1 of the original function f.

We have $y = f(x) = \sqrt{4x^2-1}$. So finding the x-value in the case where $1 = \sqrt{4x^2-1}$, we get the solution to this problem. And thus, solving $1 = \sqrt{4x^2-1}$, we get

$$1 = \sqrt{4x^2-1} \Rightarrow 4x^2 - 1 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}.$$

And we know $x \geq \frac{1}{2}$, so we get $x = \frac{\sqrt{2}}{2}$, which is $f^{-1}(1)$.

• Now, let's move on to the inverse of the other function g.

We have $g(x) = x(x^2 - 2x - 1)$ for $x \geq 2$. So finding first, the expression of the inverse, we solve for x in the equation $y = x(x^2 - 2x - 1)$, then swap the variables.
How can we however, solve the equation?

It's an equation of degree three in x, so we can hardly get the solution.

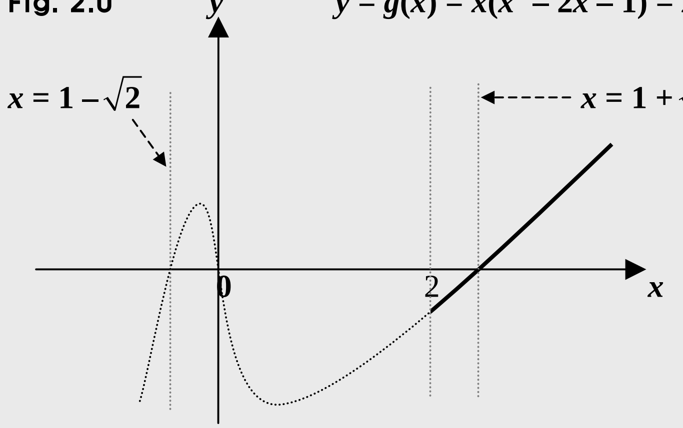
However, as in the case of $f^{-1}(1)$, we don't really have to find the expression of the inverse of g if we just want to get $g^{-1}(-2)$.

Let's see first though, if g is one-to-one.

Factorizing then, $x^2 - 2x - 1$, we get $(x - 1 - \sqrt{2})(x - 1 + \sqrt{2})$.

So we get $g(x) = x(x - 1 - \sqrt{2})(x - 1 + \sqrt{2})$. And thus, $g = 0$ at $x = 0$, $1 - \sqrt{2}$, or $1 + \sqrt{2}$.

Fig. 2.0 $y = g(x) = x(x^2 - 2x - 1) = x(x - 1 - \sqrt{2})(x - 1 + \sqrt{2})$ for $x \geq 2$.



We have $g(x) = x(x^2 - 2x - 1) = x(x - 1 - \sqrt{2})(x - 1 + \sqrt{2})$ for $x \geq 2$.

So we can actually see that g has its inverse since it is one-to-one.

Let's get the solution now, which is the value of $g^{-1}(-2)$.

To begin with, the definition for inverse functions is $\boldsymbol{y = g(x) \Leftrightarrow x = g^{-1}(y)}$.

So we can see that -2 in $\boldsymbol{g^{-1}(-2)}$ is an output of the original function $\boldsymbol{g}$.

And thus, $\boldsymbol{g^{-1}(-2)}$ is the input for which the expression of $\boldsymbol{g}$ produces the output, -2.

That is, when $\boldsymbol{x = g^{-1}(-2)}$, $\boldsymbol{g(x) = -2}$. So in short, $\boldsymbol{g(g^{-1}(-2)) = -2}$.

And thus, solving for $\boldsymbol{x}$ the equation $\boldsymbol{-2 = x(x^2 - 2x - 1)}$, we get the solution to this problem. So solving it, we get first,

$$-2 = x(x^2 - 2x - 1) \Rightarrow -2 = x^3 - 2x^2 - x \Rightarrow x^3 - 2x^2 - x + 2 = 0.$$

Luckily in this case, we can solve the equation factorizing the polynomial in it.

So factorizing it, we get

$$x^3 - 2x^2 - x + 2 = x^2(x - 2) - (x - 2) = (x - 2)(x^2 - 1) = (x - 2)(x - 1)(x + 1).$$

And thus, we get $\boldsymbol{(x - 2)(x - 1)(x + 1) = 0 \Rightarrow x = -1, 1}$, or $\boldsymbol{2}$.

We know $\boldsymbol{x \geq 2}$. Therefore, we get $\boldsymbol{g^{-1}(-2) = 2}$.

In short:

The value of $\boldsymbol{f^{-1}(1)}$ is the input corresponding to the output 1 of the original function $\boldsymbol{f}$.

We have $\boldsymbol{y = f(x) = \sqrt{4x^2 - 1}}$. So solving $\boldsymbol{1 = \sqrt{4x^2 - 1}}$, we get

$$1 = \sqrt{4x^2 - 1} \Rightarrow 4x^2 - 1 = 1 \Rightarrow x^2 = \tfrac{1}{2} \Rightarrow x = \pm\tfrac{1}{\sqrt{2}} = \pm\tfrac{\sqrt{2}}{2}.$$

And we know $\boldsymbol{x \geq \tfrac{1}{2}}$, so we get $\boldsymbol{x = \tfrac{\sqrt{2}}{2}}$, which is $\boldsymbol{f^{-1}(1)}$.

The value of $\boldsymbol{g^{-1}(-2)}$ is the input corresponding to the output -2 of the original function $\boldsymbol{g}$.

We have $\boldsymbol{y = g(x) = x(x^2 - 2x - 1)}$. So solving $\boldsymbol{-2 = x(x^2 - 2x - 1)}$, we get

$$x^3 - 2x^2 - x + 2 = x^2(x - 2) - (x - 2) = (x - 2)(x^2 - 1) = (x - 2)(x - 1)(x + 1).$$

And thus, we get $\boldsymbol{(x - 2)(x - 1)(x + 1) = 0 \Rightarrow x = -1, 1}$, or $\boldsymbol{2}$.

We know $\boldsymbol{x \geq 2}$. Therefore, we get $\boldsymbol{g^{-1}(-2) = 2}$.