Examples 3 in Exponential & Log Functions

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Examples 3 in Exponential & Log Functions

Though these examples are for your skill on functions, they are for your algebra skill, too. So following the processes in each example, pay attention to how expressions get changed at each step, and think about the idea behind the processes.

Once you've understood the idea and the processes, do each example yourself putting your solution processes in your writing. And the same is true, too, of curse, for all the other examples in this book.

Cliché but Always True: knowing is one thing, and Doing it is another.

Find the inverse of each of the functions as follows.

0. $\boldsymbol{f(x) = 6 \cdot 2^{x-1}}$ for $\boldsymbol{x}$ real.

1. $\boldsymbol{g(x) = \dfrac{e^{x} - e^{-x}}{5}}$ where $\boldsymbol{e}$ is a constant greater than 1.

2. $\boldsymbol{h(x) = \log_3 (3x + \sqrt{9x^2 - 1})}$ for $\boldsymbol{x \geq 1/3}$.

3. $\boldsymbol{k(x) = \log_3 (\frac{x}{3} + \sqrt{\frac{x^2 - 1}{9}}) + 1}$ for $\boldsymbol{x \geq 1}$.

Suggestions or Solutions
To the Problem in the Example 0

Find the inverse of $f(x) = 6 \cdot 2^{x-1}$ for x real.

To begin with, $y = f(x) = 6 \cdot 2^{x-1} \Rightarrow \frac{y}{6} = 2^{x-1}$.

So next, by the definition for log functions, we can get $\frac{y}{6} = 2^{x-1} \Leftrightarrow x - 1 = \log_2 \frac{y}{6}$.

Thus we get $x - 1 = \log_2 \frac{y}{6} \Rightarrow x = \log_2 \frac{y}{6} + 1$.

So next, putting it in the x-y system, we get $y = \log_2 \frac{x}{6} + 1$.

And next, we get $6 \cdot 2^{x-1} > 0$ since x is real, and thus, the range of f is $y > 0$.
So the domain of the log function is $x > 0$.
And thus, assuming the log function is g, we get $y = g(x) = \log_2 \frac{x}{6} + 1$ for $x > 0$.

And we can get the same the way below, too.

Simplifying the expression of the exponential function f, we can get
$f(x) = 6 \cdot 2^{x-1} = 3 \cdot 2 \cdot 2^{x-1} = 3 \cdot 2^{x-1+1} = 3 \cdot 2^{x}$. So we can set $\frac{y}{3} = 2^x$.

And next, by the definition for log functions, we get $\frac{y}{3} = 2^x \Leftrightarrow x = \log_2 \frac{y}{3}$.
So next, swapping the variables and assuming g is the log function, we get
$y = g(x) = \log_2 \frac{x}{3}$ for $x > 0$.

If not quite sure of the idea behind the processes above, follow the steps below.

How do we find the inverse of a function?

To get straight to the point, finding the inverse, we put the input variable in terms of the output variable, and then swap the variables.

So finding the inverse of $y = f(x)$, we find an equation where x is set equal to an expression in terms of y, and then, swap x and y.

In short, finding the inverse of $y = f(x)$, we put x in terms of y, and then, swap x and y.

In other words, finding an equation where the input variable is set equal to an expression in terms of the output variable, and then, swapping the variables, we get the inverse.

How then, can we get such an equation?

To begin with, the definition for inverse functions is $\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y)}$.

The definition is saying that taking the inverse of $\boldsymbol{f}$, we get $\boldsymbol{f^{-1}}$, read as $\boldsymbol{f}$ inverse.
So $\boldsymbol{f^{-1}}$ is the inverse of $\boldsymbol{f}$. What then do we mean by $\boldsymbol{f^{-1}(y)}$?

It is <u>an expression in terms of $\boldsymbol{y}$</u>. So for instance, it can be $\boldsymbol{f^{-1}(y) = y^2 + 2y + 5}$.
And we have $\boldsymbol{x = f^{-1}(y)}$, which is the inverse of $\boldsymbol{f}$.

(Note however, $\boldsymbol{f^{-1}}$ is just the name of the inverse, so we can use other letter as the name. For instance, we can use $\boldsymbol{g}$ as the name of the inverse. Then, we have $\boldsymbol{x = g(y)}$, the expression of which is the same as the expression of $\boldsymbol{f^{-1}(y)}$. Thus, $\boldsymbol{g}$ is the inverse of $\boldsymbol{f}$.)

So finding the inverse, what do we need to get?

We need to get the equation where $\boldsymbol{x}$ is equal to <u>an expression in terms of $\boldsymbol{y}$</u>.
And we can put the same idea the way below, too.

We know that in $\boldsymbol{y = f(x)}$, $\boldsymbol{x}$ is the input variable, and $\boldsymbol{y}$ is the output variable. So if finding the inverse of $\boldsymbol{f}$, we want to put the input variable in terms of the output variable. And putting the input variable in terms of the output variable, we get the equation where the input variable is set equal to an expression expressed in terms of the output variable.

How then in this problem, can we get such an equation stated above?

The function given in this problem is an exponential function, and taking the inverse of a function exponential, we get a log function. So the inverse is a log function.
How then, can we get the log function?

We can get it using the definition for logs, because the definition shows how functions of the two kinds are connected to each other. And the definition is as follows.

$\boldsymbol{y = b^x \Leftrightarrow x = \log_b y}$, where $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$.

How then, can we use the definition to get the log function, the solution to this problem?

Expressing the definition above in general, we can put it the way below.

- $\boldsymbol{p(y) = b^{q(x)} \Leftrightarrow q(x) = \log_b p(y)}$, where $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$.

And $\boldsymbol{p(y)}$ is an expression in terms of $\boldsymbol{y}$ as $\boldsymbol{p(y) = 3y + 1}$, and $\boldsymbol{q(x)}$ is an expression in terms of $\boldsymbol{x}$ as $\boldsymbol{q(x) = 2x + 5}$.

And in the case of the example above, using the definition for logs, we get

$$\boldsymbol{3y + 1 = b^{(2x + 5)} \Leftrightarrow 2x + 5 = \log_b (3y + 1)}.$$

So we can get $\boldsymbol{3y + 1 = b^{(2x + 5)} \Rightarrow 2x + 5 = \log_b (3y + 1)}$, which is a log function.

Given therefore, an exponential function, we can get the inverse using the definition for logs, because the inverse is a log function.

And the same is true for the inverse of a log function, too. So given a log function, we can get the inverse using the definition for logs, because the inverse is an exponential function. And in that case, we can put the definition for logs the way below.

$\boldsymbol{y = \log_b x \Leftrightarrow x = b^y}$, where $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$.

And expressing the definition above in general, we can put it the way below.

$\boldsymbol{p(y) = \log_b q(x) \Leftrightarrow q(x) = b^{p(y)}}$, where $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$.

And of course, $\boldsymbol{p(y)}$ is an expression in terms of $\boldsymbol{y}$, and $\boldsymbol{q(x)}$ is an expression in terms of $\boldsymbol{x}$.

Now, in this problem, since we need to find the inverse of the exponential $\boldsymbol{k}$, we want to find a log function using the function $\boldsymbol{k}$, together with the definition for logs.

In short, finding the solution to this problem, we want to find a log function.
How then, can we find the log function?

Finding the log function, we want to use the definition for log functions.
And then, we swap the variables, because we need to put the inverse in the ***x-y*** system, and not in the ***y-x*** system. That's simply because we usually put a function in the ***x-y*** system, where x is the input variable, and y is the output variable.

How do we use the definition though, in this problem, finding the log function?

Before using the definition, we want to put first, the exponential function f in the form stated above, and the form is $p(y) = b^{q(x)}$, where b is constant, $p(y)$ is an expression in terms of y, and $q(x)$ is an expression in terms of x. So let's now put it in the form.

We have $y = f(x) = 6 \cdot 2^{x-1}$. So putting it in the form of $p(y) = b^{q(x)}$, we get $\frac{y}{6} = 2^{x-1}$.

It's because we can set $p(y) = \frac{y}{6}$, and $q(x) = x - 1$.

So next, by the definition, we get $\frac{y}{6} = 2^{x-1} \Leftrightarrow x - 1 = \log_2 \frac{y}{6}$.

Thus, we get $x - 1 = \log_2 \frac{y}{6} \Rightarrow x = \log_2 \frac{y}{6} + 1$.

Next, swapping the variables, we get $y = \log_2 \frac{x}{6} + 1$. We are not quite there yet, though.

We want to get the domain of the log function. How?

The domain is the range of the exponential function given, because the log function is the inverse of the exponential function.

We know that the domain of the exponential function f is a set of all real numbers.

And we get $6 \cdot 2^{x-1} > 0$ if x can get all real numbers, and thus, the range of f is $y > 0$.

So the domain of the inverse, that is, the domain of the log function is $x > 0$.

And thus, assuming the log function is g, we get $y = g(x) = \log_2 \frac{x}{6} + 1$ for $x > 0$.

And we can put it this way, too: $\boldsymbol{y = g(x) = \log_2 \frac{x}{3}}$ for $\boldsymbol{x > 0}$.

And we can have this, as well: $\mathbf{6{\cdot}2^{x\text{-}1} = 3{\cdot}2^{x}}$.
So we can notice that we can begin this problem with $\boldsymbol{f(x) = 3{\cdot}2^x}$ instead of $\boldsymbol{f(x) = 6{\cdot}2^{x-1}}$.

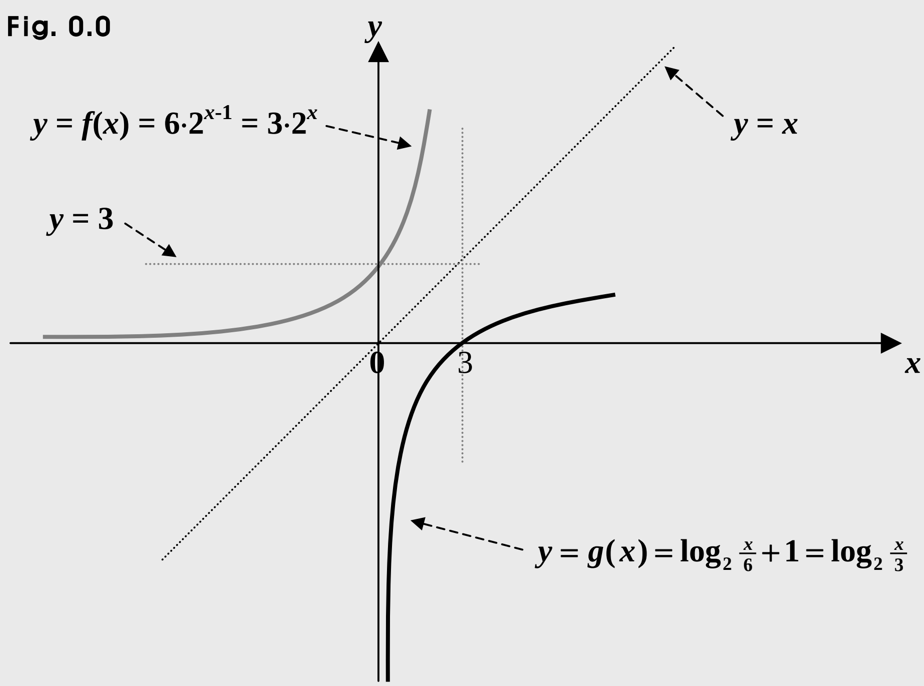
Then, we will get ended up with $\boldsymbol{y = g(x) = \log_2 \frac{x}{3}}$ for $\boldsymbol{x > 0}$.

Why though, is it the same as $\boldsymbol{y = g(x) = \log_2 \frac{x}{6} + 1}$ for $\boldsymbol{x > 0}$?

Simplifying $\boldsymbol{\log_2 \frac{x}{6} + 1}$, we get $\boldsymbol{\log_2 \frac{x}{6} + 1 = \log_2 \frac{x}{6} + \log_2 2 = \log_2(\frac{x}{6} \cdot 2) = \log_2 \frac{x}{3}}$.

If not quite sure of the process above, refer to **POWERS AND LOGARITHMS**.

Fig. 0.0



The two curves are symmetric about the line $\boldsymbol{y = x}$.

Suggestions or Solutions

To the Problem in the Example 1

Find the inverse of $g(x) = \dfrac{e^x - e^{-x}}{5}$ where e is a constant greater than 1.

Setting $t = e^x$, we get $y = g(x) = \frac{t-t^{-1}}{5} = \frac{t^2-1}{5t} \Rightarrow 5ty = t^2 - 1 \Rightarrow t^2 - 5yt - 1 = 0$

$\Rightarrow t^2 - 5yt - 1 = t^2 - 5yt + (\frac{5y}{2})^2 - (\frac{5y}{2})^2 - 1 = (t - \frac{5y}{2})^2 - \frac{25y^2}{4} - 1 = (t - \frac{5y}{2})^2 - \frac{25y^2+4}{4} = 0$

$\Rightarrow (t - \frac{5y}{2})^2 = \frac{25y^2+4}{4} \Rightarrow t - \frac{5y}{2} = \frac{\pm\sqrt{25y^2+4}}{2} \Rightarrow t = \frac{\pm\sqrt{25y^2+4}}{2} + \frac{5y}{2} = \frac{5y\pm\sqrt{25y^2+4}}{2} \Rightarrow t = \frac{5y\pm\sqrt{25y^2+4}}{2}$.

We have $5y \le \sqrt{25y^2} < \sqrt{25y^2+4}$. So we get $t = \frac{5y+\sqrt{25y^2+4}}{2}$ because $t > 0$.

Thus, we get $e^x = \frac{5y+\sqrt{25y^2+4}}{2}$.

So next, by the definition for logs, we get $x = \log_e \frac{5y+\sqrt{25y^2+4}}{2}$.

And thus, assuming h is the log function, and putting it in the x-y system, we get $y = h(x) = \log_e \frac{5x+\sqrt{25x^2+4}}{2}$ for x real.

If not quite sure of the idea behind the processes above, follow the steps below.

Finding the inverse of g, we want to put x in terms of y, and of course, x and y are from the function g. Then, swapping the variables, we get the inverse.

In short, we want to get an expression that is put in terms of the output variable y.
How then, can we get such an expression?

To begin with, what function is g?

It is an exponential function.   So?

We know the inverse of an exponential function is a log function, which is therefore, the solution to this problem. And the definition for logs shows how functions of the two kinds are connected to each other. So we want to use the definition for logs.

And the definition is as follows. $\boldsymbol{y = b^x \Leftrightarrow x = \log_b y}$, where $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$.

And expressing the definition above in general, we can put it the way below.

• $\boldsymbol{p(y) = b^{q(x)} \Leftrightarrow q(x) = \log_b p(y)}$, where $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$.

And $\boldsymbol{p(y)}$ is an expression in terms of $\boldsymbol{y}$ as $\boldsymbol{p(y) = 3y + 1}$, and $\boldsymbol{q(x)}$ is an expression in terms of $\boldsymbol{x}$ as $\boldsymbol{q(x) = 2x + 5}$.

And in the case of the example above, using the definition for logs, we get

$$\boldsymbol{3y + 1 = b^{(2x + 5)} \Leftrightarrow 2x + 5 = \log_b (3y + 1)}.$$

So we can get $\boldsymbol{3y + 1 = b^{(2x + 5)} \Rightarrow 2x + 5 = \log_b (3y + 1)}$, which is a log function.

So in this problem, since we need to find the inverse of the exponential function $\boldsymbol{g}$, we want to find a log function using the function $\boldsymbol{g}$, together with the definition for logs.

In short, finding the solution to this problem, we want to find a log function.

And thus, we want to put first, the exponential function $\boldsymbol{g}$ in the form of $\boldsymbol{p(y) = b^{q(x)}}$.

How then, do we put it in the form?

Putting it in the form is the key to this problem. And in fact, the same is true, too, for all the other problems in this kind.

Now, we have $y = g(x) = \dfrac{e^x - e^{-x}}{5}$. The function g does not seem to get readily put in the form. How then, can we put it in the form?

We've got to do some algebra.
We can try first, taking as an unknown, the power e^x itself.
Then, we can get an equation for the unknown, and can try solving the equation.
Then, the solution will be an equality in a form of $p(y) = e^x$.
And then, we can try solving the equality for x, then swap the variables in the solution.

So to begin with, setting $t = e^x$, we can set $y = g(x) = \frac{t-t^{-1}}{5}$.

Then, we can put it this way: $y = \frac{1}{5}(t - \frac{1}{t}) = \frac{1}{5}\cdot\frac{t^2-1}{t} = \frac{t^2-1}{5t} \Rightarrow 5ty = t^2 - 1 \Rightarrow t^2 - 5yt - 1 = 0$, which is quadratic, and thus, is not quite offensive. So next, solving it, we can get

$$t^2 - 5yt - 1 = t^2 - 5yt + (\tfrac{5y}{2})^2 - (\tfrac{5y}{2})^2 - 1 = (t - \tfrac{5y}{2})^2 - \tfrac{25y^2}{4} - 1 = (t - \tfrac{5y}{2})^2 - \tfrac{25y^2+4}{4} = 0$$

$$\Rightarrow (t - \tfrac{5y}{2})^2 = \tfrac{25y^2+4}{4} \Rightarrow t - \tfrac{5y}{2} = \tfrac{\pm\sqrt{25y^2+4}}{2} \Rightarrow t = \tfrac{\pm\sqrt{25y^2+4}}{2} + \tfrac{5y}{2} = \tfrac{5y\pm\sqrt{25y^2+4}}{2} \Rightarrow t = \tfrac{5y\pm\sqrt{25y^2+4}}{2}.$$

And next, we know $t = e^x$. So we now have $e^x = \frac{5y\pm\sqrt{25y^2+4}}{2}$, which is therefore, in the form we want. What form? It is the form stated above, $p(y) = e^x$. What then is the next?

We cannot apply the definition for log functions to this: $e^x = \frac{5y\pm\sqrt{25y^2+4}}{2}$.
We've got two equations though.
One is $e^x = \frac{5y+\sqrt{25y^2+4}}{2}$, and the other is $e^x = \frac{5y-\sqrt{25y^2+4}}{2}$.
So we want to choose one from the two above. How?

Using the domain of the given function g, we can get the extent of e^x. Then, comparing the extent of e^x with the two extents of the two expressions, $\frac{5y+\sqrt{25y^2+4}}{2}$ and $\frac{5y-\sqrt{25y^2+4}}{2}$, we can make a choice. What then, is the domain of g?

We have $\boldsymbol{y = g(x)} = \frac{\boldsymbol{e^{x} - e^{-x}}}{\boldsymbol{5}}$, which can be defined for all $\boldsymbol{x}$ real.

So the domain is a set of all real numbers.

Thus, to begin with, for all $\boldsymbol{x}$ real, we get $\boldsymbol{e^x} > \mathbf{0}$. And we have $\boldsymbol{t = e^x}$. So we get $\boldsymbol{t} > \mathbf{0}$.

Thus next, getting first, the extents of the two expressions $\frac{5y\pm\sqrt{25y^2+4}}{2}$, we can begin with $\mathbf{5}\boldsymbol{y} \leq \sqrt{\mathbf{25}\boldsymbol{y}^2} < \sqrt{\mathbf{25}\boldsymbol{y}^2 + \mathbf{4}}$. Why not $\mathbf{5}\boldsymbol{y} = \sqrt{\mathbf{25}\boldsymbol{y}^2}$, though?

Going back to the function $\boldsymbol{g}$ above, we can see that the value of $\boldsymbol{y}$ can be negative, so $\mathbf{5}\boldsymbol{y}$ can be negative, but $\sqrt{\mathbf{25}\boldsymbol{y}^2} \geq \mathbf{0}$ no matter what value $\boldsymbol{y}$ may get.

So we get $\mathbf{5}\boldsymbol{y} \leq \sqrt{\mathbf{25}\boldsymbol{y}^2}$. And we know $\sqrt{\mathbf{25}\boldsymbol{y}^2} < \sqrt{\mathbf{25}\boldsymbol{y}^2 + \mathbf{4}}$.

Thus next, for all values of $\boldsymbol{y}$, we get $\sqrt{\mathbf{25}\boldsymbol{y}^2 + \mathbf{4}} > \mathbf{5}\boldsymbol{y}$.

And thus, we get $\mathbf{5}\boldsymbol{y} + \sqrt{\mathbf{25}\boldsymbol{y}^2 + \mathbf{4}} > \mathbf{0}$, and $\mathbf{5}\boldsymbol{y} - \sqrt{\mathbf{25}\boldsymbol{y}^2 + \mathbf{4}} < \mathbf{0}$.

And we have $\boldsymbol{t} > \mathbf{0}$, and $\boldsymbol{t} = \frac{5y\pm\sqrt{25y^2+4}}{2}$. So we get $\boldsymbol{t} = \frac{5y+\sqrt{25y^2+4}}{2}$.

And we have $\boldsymbol{t = e^x}$. Thus, we get $\boldsymbol{e^x} = \frac{5y+\sqrt{25y^2+4}}{2}$.

And the definition for logs is $\boldsymbol{A = b^c} \Leftrightarrow \boldsymbol{c} = \mathbf{log}_{\boldsymbol{b}}\boldsymbol{A}$.

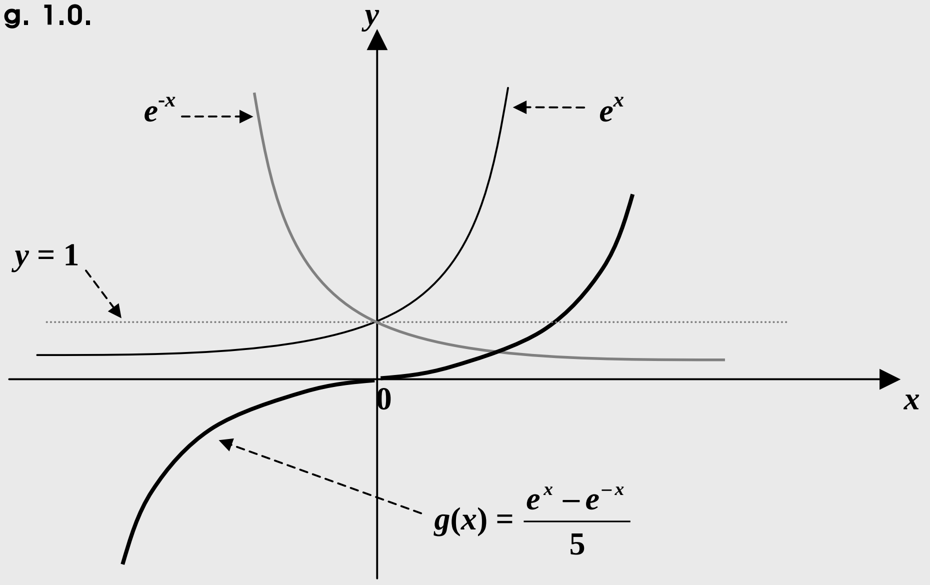
So next, by the definition for logs, we get $\boldsymbol{x} = \mathbf{log}_{\boldsymbol{e}} \frac{5y+\sqrt{25y^2+4}}{2}$.

What then, is the next?

We want to get the domain, which is the range of the exponential function $\boldsymbol{g}$.

We can quickly see the range of $\boldsymbol{g}$ using the graph of $\boldsymbol{g}$. And the graph is as follows.

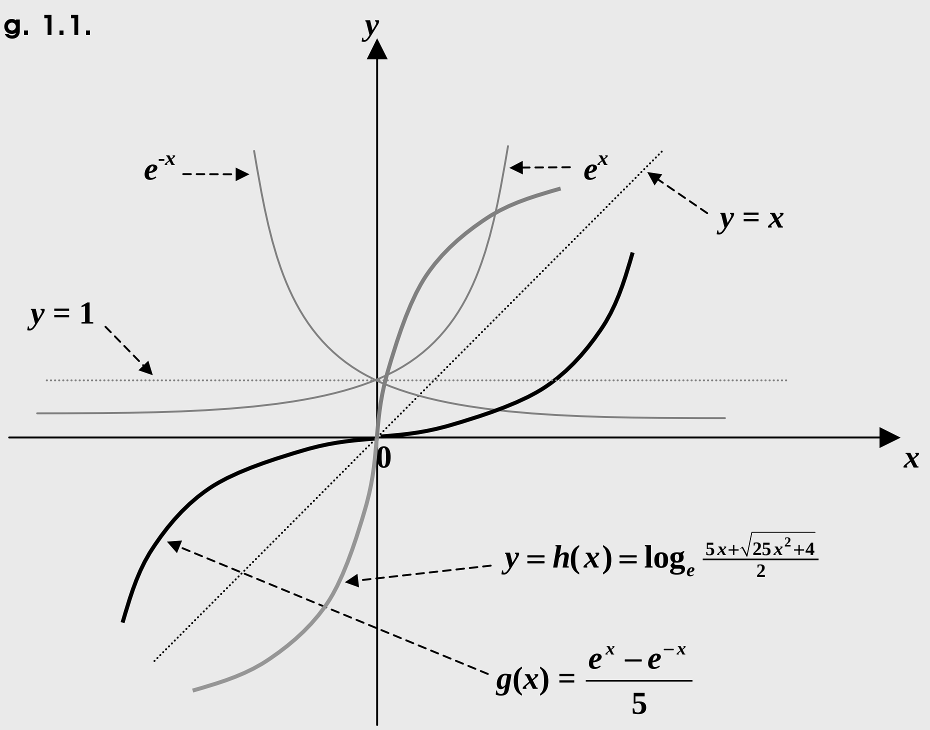
Fig. 1.0.



So we can see the range of ***g*** is a set of all real numbers, which is thus, the domain of the log function. So assuming ***h*** is the log function, and swapping the variables, we get

$y = h(x) = \log_e \frac{5x+\sqrt{25x^2+4}}{2}$ for ***x*** real.

Fig. 1.1.



Suggestions or Solutions
To the Problem in the Example 2

Find the inverse of $h(x) = \log_3(3x + \sqrt{9x^2 - 1})$ for $x \geq 1/3$.

First, $y = h(x) = \log_3(3x + \sqrt{9x^2-1}) \Rightarrow y = \log_3(3x + \sqrt{9x^2-1})$.

Next, by the definition for logs, $y = \log_3(3x + \sqrt{9x^2-1}) \Leftrightarrow 3x + \sqrt{9x^2-1} = 3^y$.

Setting $t = 3^y$, we get $3x + \sqrt{9x^2-1} = t$.

$t - 3x = \sqrt{9x^2-1} \Rightarrow (t-3x)^2 = 9x^2 - 1 \Rightarrow t^2 - 6tx + 9x^2 = 9x^2 - 1 \Rightarrow 6tx = t^2 + 1$
$\Rightarrow x = \frac{t}{6} + \frac{1}{6t}$.

So we get $x = \frac{1}{6}(t + \frac{1}{t}) = \frac{1}{6}(3^y + \frac{1}{3^y}) = \frac{1}{6}(3^y + 3^{-y})$.

$x \geq 1/3 \Rightarrow 9x^2 - 1 \geq 0 \Rightarrow \sqrt{9x^2-1} \geq 0 \Rightarrow 3x + \sqrt{9x^2-1} \geq 1$, since $x \geq 1/3$. Thus, we get

$\log_3(3x + \sqrt{9x^2-1}) \geq \log_3 1 = 0 \Rightarrow \log_3(3x + \sqrt{9x^2-1}) \geq 0$.

And thus, assuming the inverse is g, we get $y = g(x) = \frac{1}{6}(3^x + 3^{-x})$ for $x \geq 0$.

If not quite sure of the idea behind the processes above, follow the steps below.

Taking the inverse of an exponential function, we get a log function.

So what do we get taking the inverse of a log function?

We get an exponential function. So this time, we want to find a function exponential.

Functions in both the kinds are inverse of each other, so we can find it by the definition for logs. And the definition is $\boldsymbol{y = b^x \Leftrightarrow x = \log_b y}$, where $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$.

And we can put it this way, too: $\boldsymbol{y = \log_b x \Leftrightarrow x = b^y}$, where $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$.

And thus, getting the inverse either log or exponential, we can apply the definition above to the function we take the inverse of. Then, swapping the variables, we get the inverse.

The definition above is however, just a prototype if you will. So it's in the simplest form. And expressing the definition in general, we can put it the way as follows.

$\boldsymbol{p(y) = \log_b q(x) \Leftrightarrow q(x) = b^{p(y)}}$, where $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$.

And of course, $\boldsymbol{p(y)}$ is an expression in terms of $\boldsymbol{y}$ as $\boldsymbol{p(y) = 3y + 1}$, and $\boldsymbol{q(x)}$ is an expression in terms of $\boldsymbol{x}$ as $\boldsymbol{q(x) = 2x + 5}$.

And in the case of the example above, using the definition for logs, we get

$\boldsymbol{3y + 1 = \log_b (2x + 5) \Leftrightarrow 2x + 5 = b^{(3y + 1)}}$.

So we can get $\boldsymbol{3y + 1 = \log_b (2x + 5) \Rightarrow 2x + 5 = b^{(3y + 1)}}$, which is an exponential function.

So in this problem, since we need to find the inverse of the log function $\boldsymbol{h}$, we want to find an exponential function using the function $\boldsymbol{h}$, along with the definition for logs.

In short, finding the solution to this problem, we want to find a function exponential.

- Now in this problem, we have $\boldsymbol{y = h(x) = \log_3 (3x + \sqrt{9x^2 - 1})}$ for $\boldsymbol{x \geq 1/3}$.

So the exponential function $\boldsymbol{g}$ is already in the form of $\boldsymbol{p(y) = \log_b q(x)}$.

In this case, $\boldsymbol{p(y) = y}$, and $\boldsymbol{q(x) = 3x + \sqrt{9x^2 - 1}}$.

So next, by the definition, we can set $\boldsymbol{y = \log_3 (3x + \sqrt{9x^2 - 1}) \Leftrightarrow 3x + \sqrt{9x^2 - 1} = 3^y}$.

Thus next, swapping the variables, and naming the inverse, do we get the inverse?

It's not quite the case yet.
That's because just swapping the variables now, we just get $\boldsymbol{3y + \sqrt{9y^2 - 1} = 3^x}$, which is not the form we want, because the output variable $\boldsymbol{y}$ in the inverse is yet to be isolated.

That is, assuming the inverse is $\boldsymbol{g}$, using the equality as is, we cannot put the inverse in a form of this: $\boldsymbol{y = g(x)}$. So we cannot just use this equality: $\boldsymbol{3x + \sqrt{9x^2 - 1} = 3^y}$.

That is, we want to isolate $\boldsymbol{x}$ first. How then, can we do that?

Assuming $\boldsymbol{t = 3^y}$, we get $\boldsymbol{3x + \sqrt{9x^2 - 1} = t}$. Then, we get

$$\boldsymbol{t - 3x = \sqrt{9x^2 - 1} \Rightarrow (t - 3x)^2 = 9x^2 - 1 \Rightarrow t^2 - 6tx + 9x^2 = 9x^2 - 1}$$

$$\boldsymbol{\Rightarrow 6tx = t^2 + 1 \Rightarrow x = \tfrac{t}{6} + \tfrac{1}{6t}}.$$

And we know $\boldsymbol{t = 3^y}$. So we get $\boldsymbol{x = \frac{1}{6}(t + \frac{1}{t}) = \frac{1}{6}(3^y + \frac{1}{3^y}) = \frac{1}{6}(3^y + 3^{-y})}$.

Thus, we get $\boldsymbol{x = \frac{1}{6}(3^y + 3^{-y})}$. We are not quite there yet though.
What then, is the next?

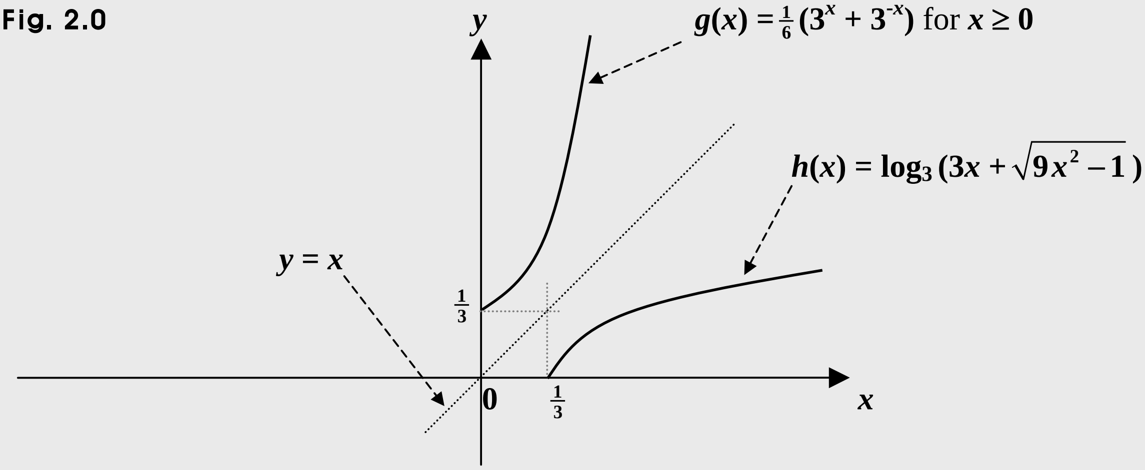
We want to get the domain of the inverse, which is the range of the original function $\boldsymbol{h}$.
And the domain of $\boldsymbol{h}$ is $\boldsymbol{x \geq 1/3}$. So getting the range of the function $\boldsymbol{h}$, we can get first,

$$\boldsymbol{x \geq 1/3 \Rightarrow 9x^2 - 1 \geq 0 \Rightarrow \sqrt{9x^2 - 1} \geq 0 \Rightarrow 3x + \sqrt{9x^2 - 1} \geq 1}, \text{since } \boldsymbol{x \geq 1/3}.$$

Thus, we get $\boldsymbol{\log_3 (3x + \sqrt{9x^2 - 1}) \geq \log_3 1 = 0 \Rightarrow \log_3 (3x + \sqrt{9x^2 - 1}) \geq 0}$.
So we can see that the range of $\boldsymbol{h}$ is $\boldsymbol{y \geq 0}$. And thus, the domain of the inverse is $\boldsymbol{x \geq 0}$.

So assuming the inverse is g and putting it in the x-y system, so swapping the variables, we get $y = g(x) = \frac{1}{6}(3^x + 3^{-x})$ for $x \geq 0$.

Fig. 2.0



In short:

First, $y = h(x) = \log_3(3x + \sqrt{9x^2 - 1}) \Rightarrow y = \log_3(3x + \sqrt{9x^2 - 1})$.

Next, by the definition for logs, $y = \log_3(3x + \sqrt{9x^2 - 1}) \Leftrightarrow 3x + \sqrt{9x^2 - 1} = 3^y$.

Setting $t = 3^y$, we get $3x + \sqrt{9x^2 - 1} = t$.

$t - 3x = \sqrt{9x^2 - 1} \Rightarrow (t - 3x)^2 = 9x^2 - 1 \Rightarrow t^2 - 6tx + 9x^2 = 9x^2 - 1 \Rightarrow 6tx = t^2 + 1$

$\Rightarrow x = \frac{t}{6} + \frac{1}{6t}$.

So we get $x = \frac{1}{6}(t + \frac{1}{t}) = \frac{1}{6}(3^y + \frac{1}{3^y}) = \frac{1}{6}(3^y + 3^{-y})$.

$x \geq 1/3 \Rightarrow 9x^2 - 1 \geq 0 \Rightarrow \sqrt{9x^2 - 1} \geq 0 \Rightarrow 3x + \sqrt{9x^2 - 1} \geq 1$, since $x \geq 1/3$. Thus, we get

$\log_3(3x + \sqrt{9x^2 - 1}) \geq \log_3 1 = 0 \Rightarrow \log_3(3x + \sqrt{9x^2 - 1}) \geq 0$.

And thus, assuming the inverse is g, we get $y = g(x) = \frac{1}{6}(3^x + 3^{-x})$ for $x \geq 0$.

Suggestions or Solutions
To the Problem in the Example 3

Find the inverse of $k(x) = \log_3 \left(\frac{x}{3} + \sqrt{\frac{x^2-1}{9}} \right) + 1$ for $x \geq 1$.

First, $y = k(x) = \log_3 \left(\frac{x}{3} + \sqrt{\frac{x^2-1}{9}} \right) + 1 \Rightarrow y - 1 = \log_3 \left(\frac{x}{3} + \sqrt{\frac{x^2-1}{9}} \right)$.

Next, by the definition for logs, we get $y - 1 = \log_3 \left(\frac{x}{3} + \sqrt{\frac{x^2-1}{9}} \right) \Leftrightarrow \frac{x}{3} + \sqrt{\frac{x^2-1}{9}} = 3^{y-1}$.

Setting next, $t = 3^{y-1}$, we get $\frac{x}{3} + \sqrt{\frac{x^2-1}{9}} = t$. Then, we can get

$$t - \frac{x}{3} = \sqrt{\frac{x^2-1}{9}} \Rightarrow \left(t - \frac{x}{3}\right)^2 = \frac{x^2-1}{9} \Rightarrow t^2 - \frac{2tx}{3} + \frac{x^2}{9} = \frac{x^2-1}{9} \Rightarrow \frac{2tx}{3} = t^2 + \frac{1}{9} \Rightarrow x = \frac{3t}{2} + \frac{1}{6t}.$$

So we get $x = \frac{1}{2}\left(3t + \frac{1}{3t}\right) = \frac{1}{2}\left(3 \cdot 3^{y-1} + \frac{1}{3 \cdot 3^{y-1}}\right) = \frac{1}{2}(3^y + 3^{-y})$. Next, getting the range of k, we get first, $x \geq 1 \Rightarrow x^2 - 1 \geq 0 \Rightarrow \sqrt{\frac{x^2-1}{9}} \geq 0 \Rightarrow \frac{x}{3} + \sqrt{\frac{x^2-1}{9}} \geq \frac{1}{3}$, since $x \geq 1$, so next, we get $\log_3 \left(\frac{x}{3} + \sqrt{\frac{x^2-1}{9}} \right) \geq \log_3 \frac{1}{3} = -1 \Rightarrow \log_3 \left(\frac{x}{3} + \sqrt{\frac{x^2-1}{9}} \right) \geq -1 \Rightarrow \log_3 \left(\frac{x}{3} + \sqrt{\frac{x^2-1}{9}} \right) + 1 \geq 0$.

And thus, assuming the inverse is s, we get $y = s(x) = \frac{1}{2}(3^x + 3^{-x})$ for $x \geq 0$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, taking the inverse of an exponential function, what do we get?

We get an exponential function. So this time too, we want to find a function exponential. Functions in both the kinds are inverse of each other, so we can find it by the definition for logs. And the definition is $y = b^x \Leftrightarrow x = \log_b y$, where $b > 0$, but $b \neq 1$.

And we can put it this way, too: $y = \log_b x \Leftrightarrow x = b^y$, where $b > 0$, but $b \neq 1$.

Getting thus, the inverse either log or exponential, we can apply the definition above to the function we take the inverse of. Then, swapping the variables, we get the inverse.

The definition above is however, just a prototype if you will. So it's in the simplest form. And expressing the definition in general, we can put it the way below.

$\boldsymbol{p(y) = \log_b q(x) \Leftrightarrow q(x) = b^{p(y)}}$, where $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$.

And of course, $\boldsymbol{p(y)}$ is an expression in terms of $\boldsymbol{y}$, and $\boldsymbol{q(x)}$ is an expression in terms of $\boldsymbol{x}$.

And for instance, using the definition for logs, we can get

$\mathbf{3y + 1 = \log_b (2x + 5) \Leftrightarrow 2x + 5 = b^{(3y + 1)}}$.

So we can get $\mathbf{3y + 1 = \log_b (2x + 5) \Rightarrow 2x + 5 = b^{(3y + 1)}}$, which is an exponential function.

So in this problem, since we need to find the inverse of the log function $\boldsymbol{k}$, we want to find an exponential function using the function $\boldsymbol{k}$, along with the definition for logs.

In short, finding the solution to this problem, we want to find a function exponential.

- Now in this problem, we have $\mathbf{y = k(x) = \log_3 \left(\frac{x}{3} + \sqrt{\frac{x^2-1}{9}}\right) + 1}$ for $\mathbf{x \geq 1}$.

Then first, we want to put the log function $\boldsymbol{k}$ in the form of $\boldsymbol{p(y) = \log_b q(x)}$.
We can simply do so putting $\boldsymbol{k}$ the way below.

$\mathbf{y = \log_3 \left(\frac{x}{3} + \sqrt{\frac{x^2-1}{9}}\right) + 1 \Rightarrow y - 1 = \log_3 \left(\frac{x}{3} + \sqrt{\frac{x^2-1}{9}}\right)}$, which is now in the form above,

since we can take $\mathbf{y - 1}$ as $\boldsymbol{p(y)}$, and can take $\mathbf{\frac{x}{3} + \sqrt{\frac{x^2-1}{9}}}$ as $\boldsymbol{q(x)}$.

Thus, next, by the definition, we can set $\mathbf{y - 1 = \log_3 \left(\frac{x}{3} + \sqrt{\frac{x^2-1}{9}}\right) \Leftrightarrow \frac{x}{3} + \sqrt{\frac{x^2-1}{9}} = 3^{y-1}}$.

So we now have $\mathbf{\frac{x}{3} + \sqrt{\frac{x^2-1}{9}} = 3^{y-1}}$. What then, is the next?

Assuming $\boldsymbol{r}$ is the inverse, we need to put the equation above in this form: $\boldsymbol{x = r(y)}$, where $\boldsymbol{r(y)}$ is an expression in terms of $\boldsymbol{y}$. So we want to isolate $\boldsymbol{x}$ now. How?

Assuming $t = 3^{y-1}$, we can get $\frac{x}{3} + \sqrt{\frac{x^2-1}{9}} = t$. Then, we can get

$$t - \frac{x}{3} = \sqrt{\frac{x^2-1}{9}} \Rightarrow (t - \frac{x}{3})^2 = \frac{x^2-1}{9} \Rightarrow t^2 - \frac{2tx}{3} + \frac{x^2}{9} = \frac{x^2-1}{9} \Rightarrow \frac{2tx}{3} = t^2 + \frac{1}{9} \Rightarrow x = \frac{3t}{2} + \frac{1}{6t}.$$

And we know $t = 3^{y-1}$. So we get $x = \frac{1}{2}(3t + \frac{1}{3t}) = \frac{1}{2}(3 \cdot 3^{y-1} + \frac{1}{3 \cdot 3^{y-1}}) = \frac{1}{2}(3^y + 3^{-y})$.

That is, we now have $x = \frac{1}{2}(3^y + 3^{-y})$. What then, is the next?

We want to get the range of the original function k so that we can get the domain of the inverse. And getting the range, we can begin with the domain of k, which is $x \geq 1$.

Then, we get $x \geq 1 \Rightarrow x^2 - 1 \geq 0 \Rightarrow \sqrt{\frac{x^2-1}{9}} \geq 0 \Rightarrow \frac{x}{3} + \sqrt{\frac{x^2-1}{9}} \geq \frac{1}{3}$, since $x \geq 1$.

And taking next, the log of both sides, we get

$$\log_3 (\frac{x}{3} + \sqrt{\frac{x^2-1}{9}}) \geq \log_3 \frac{1}{3} = -1 \Rightarrow \log_3 (\frac{x}{3} + \sqrt{\frac{x^2-1}{9}}) \geq -1 \Rightarrow \log_3 (\frac{x}{3} + \sqrt{\frac{x^2-1}{9}}) + 1 \geq 0.$$

So the range of k is $x \geq 0$, and thus the domain of the inverse is $y \geq 0$.

And thus, assuming the inverse is r, we get $y = r(x) = \frac{1}{2}(3^x + 3^{-x})$ for $x \geq 0$.

Fig. 3.0