# Examples in Periodic Functions



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Doing the examples below, assume that $u$, $v$, $t$, $a$, and $b$ are nonzero constants.

0. Assuming $f(x) = f(x + t)$, show that $f(x) = f(x - t)$.

1. Assuming $f(x) = f(x + u) = f(x + v)$, show that $f(x) = f(x + u + v)$.

2. Assuming $f(x) = f(x + t)$, show that $f(ax + b)$ is periodic.

3. Assuming $f(x) = |x - k|$ for $k - 1 \le x \le k + 1$, where $k$ is an even integer, find $f(13.2)$ and $f(-6.7)$.

## Suggestions or Solutions
### To the Problem in the Example 0

**Assuming $f(x) = f(x + t)$ where $t$ is a nonzero constant, show that $f(x) = f(x - t)$.**

> Assuming $s = x + t$ where $t$ is constant, we get $x = s - t$.
>
> So we get $f(x) = f(x + t) \Rightarrow f(s - t) = f(s - t + t) = f(s) \Rightarrow f(s) = f(s - t)$.
>
> And thus, we can get $f(x) = f(x - t)$.

*If not quite sure of the idea behind the processes above, follow the steps below.*

To begin with, setting $s = x + t$, we get $x = s - t$, where $s$ is a variable.

How can we though, just set $s = x + t$?

It's because since $t$ is constant, $(x + t)$ varies as $x$ varies, so $(x + t)$ works like a single variable, and thus, we can replace $(x + t)$ with a letter, and take the letter as a variable. So we can just set $s = x + t$. And we can get $x = s - t$.

Thus next, in the equality $f(x) = f(x + t)$, replacing $x$ with $(s - t)$, we get

$f(x) = f(x + t) \Rightarrow f(s - t) = f(s - t + t) = f(s) \Rightarrow f(s) = f(s - t)$.

And we know $s$ is a variable, and thus, we can use any other letter as the variable $s$ if the other letter is not used as any other variable in the same expression. Why?

A variable is like a box that can contain a number at a time, but keeps changing its content. And we use a letter to name the variable if of course, the letter is not used as any other variable in the same expression.

And we know $s$ is a variable, and $x$ is not used as any other variable in the expression where $f(s) = f(s - t)$. So replacing $s$ with $x$, we can get $f(x) = f(x - t)$.

## Suggestions or Solutions
### To the Problem in the Example 1

**Assuming $f(x) = f(x + u) = f(x + v)$ where $u$ and $v$ are nonzero constants, show that $f(x) = f(x + u + v)$.**

In the example 0 above, we have $f(x) = f(x + u) \Rightarrow f(x) = f(x - u)$.

And assuming $s = x - u$, where $u$ is constant, we get $x = s + u$.

So we get $f(x) = f(x + v) \Rightarrow f(s + u) = f(s + u + v)$.

And thus, we can get $f(x + u) = f(x + u + v)$. And we have $f(x) = f(x + u)$, too.

So we get $f(x) = f(x + u + v)$.

*If not quite sure of the idea behind the processes above, follow the steps below.*

First, what do we mean by $f(x) = f(x + u) = f(x + v)$ where $u$ and $v$ are constant and $\neq 0$?

It means that the function $f$ is a periodic function. What then, is the period?

If we have $0 < u < v$, then $u$ is the period. What then, about $v$?

It is an integer multiple of $u$.
That is to say that we have $f(x) = f(x + u) = f(x + nu)$ where $n$ is an integer.
So $f$ is a periodic function where the period is $u$.

And by the same token, if we have $0 < v < u$, then $v$ is the period.
So the smaller of the two $u$ and $v$ is the period of the function $f$.

And next, in the example 0 above, we have a fact below.

- Assuming $f(x) = f(x + t)$ where $t$ is constant and $\neq \mathbf{0}$, we get $f(x) = f(x - t)$.

So we can get $f(x) = f(x + u) \Rightarrow f(x) = f(x - u)$.

And assuming $s = x - u$, where $u$ is constant, we get $x = s + u$.

And we have $f(x) = f(x + v)$, too.

So since $x = s + u$, if in equality above, replacing $x$ with $(s + u)$, we can get

$f(x) = f(x + v) \Rightarrow f(s + u) = f(s + u + v)$.

And we know $s$ is a variable, and $x$ is not used as any other variable in the expression where $f(s + u) = f(s + u + v)$. And thus, replacing $s$ with $x$, we can get

$f(x + u) = f(x + u + v)$.

And we have $f(x) = f(x + u)$, too. So we get $f(x) = f(x + u + v)$.

**In short:**

In the example 0, we have $f(x) = f(x + u) \Rightarrow f(x) = f(x - u)$.

And assuming $s = x - u$, where $u$ is constant, we get $x = s + u$.

So we get $f(x) = f(x + v) \Rightarrow f(s + u) = f(s + u + v)$.

And thus, we can get $f(x + u) = f(x + u + v)$. And we have $f(x) = f(x + u)$, too.

So we get $f(x) = f(x + u + v)$.

## Suggestions or Solutions
### To the Problem in the Example 2

**Assuming $a$, $b$, and $t$ are nonzero constants, and $f(x) = f(x + t)$, show that $f(ax + b)$ is periodic.**

Suppose $g(x) = f(ax + b)$, and $w = ax + b$.

Then, we can get $g(x + \frac{t}{a}) = f\{a(x + \frac{t}{a}) + b\} = f(ax + b + t) = f(w + t)$.

So we get $g(x + \frac{t}{a}) = f(w + t)$. And we have $f(x) = f(x + t)$. So we get $f(w + t) = f(w)$.

Thus, we get $g(x + \frac{t}{a}) = f(w)$.

And we have $w = ax + b$. So we get $f(w) = f(ax + b)$.

And also, we have $g(x) = f(ax + b)$. So we get $g(x) = f(w)$.

Besides, we have this, too: $g(x + \frac{t}{a}) = f(w)$. So we get $g(x + \frac{t}{a}) = g(x)$.

And thus, $g(x)$ is a periodic function where the period is $\frac{t}{a}$.

Now, we have $g(x) = f(ax + b)$. So $f(ax + b)$ is periodic, too, since $g$ is periodic.

*If not quite sure of the idea behind the processes above, follow the steps below.*

To begin with, we know $f$ is periodic, since we have $f(x) = f(x + t)$ where $t$ is constant.

Suppose now, $g(x) = f(ax + b)$, and $w = ax + b$. Why setting $g(x) = f(ax + b)$, though?

That's the trick. In this example, we want to show that $f(ax + b)$ is periodic.
So showing $g(x)$ is periodic, we show $f(ax + b)$ is periodic, too.

Then, we can get $\boldsymbol{g(x+\frac{t}{a}) = f\{a(x+\frac{t}{a})+b\} = f(ax + b + t) = f(w + t)}$.

Why setting $\boldsymbol{g(x+\frac{t}{a}) = f\{a(x+\frac{t}{a})+b\}}$, though?

That's another trick. We will get to use $\boldsymbol{f(x) = f(x + t)}$, which means $\boldsymbol{f}$ is periodic.

So now, getting back to $\boldsymbol{g(x+\frac{t}{a}) = f\{a(x+\frac{t}{a})+b\} = f(ax + b + t) = f(w + t)}$, we get

$\boldsymbol{g(x+\frac{t}{a}) = f(w + t)}$. And we have $\boldsymbol{f(x) = f(x + t)}$. So we get $\boldsymbol{f(w + t) = f(w)}$.

Thus, we get $\boldsymbol{g(x+\frac{t}{a}) = f(w)}$.

And we have $\boldsymbol{w = ax + b}$. So we get $\boldsymbol{f(w) = f(ax + b)}$.

And also, we have $\boldsymbol{g(x) = f(ax + b)}$. So we get $\boldsymbol{g(x) = f(w)}$.

Besides, we have this, too: $\boldsymbol{g(x+\frac{t}{a}) = f(w)}$. So we get $\boldsymbol{g(x+\frac{t}{a}) = g(x)}$.

And we know $\boldsymbol{\frac{t}{a}}$ is constant, since $\boldsymbol{a}$ and $\boldsymbol{t}$ are constant.

So $\boldsymbol{g(x)}$ is periodic, because from $\boldsymbol{g}$, the same output repeats at every $\boldsymbol{\frac{t}{a}}$.

That is, $\boldsymbol{g(x)}$ is a periodic function where the period is $\boldsymbol{\frac{t}{a}}$.

Now, we have $\boldsymbol{g(x) = f(ax + b)}$. So $\boldsymbol{f(ax + b)}$ is periodic, too, since $\boldsymbol{g}$ is periodic.

In fact, if $\boldsymbol{f(ax) = f\{a(x+\frac{b}{a})\}}$, then $\boldsymbol{f(ax)}$ is periodic, and the period can be said to be $\boldsymbol{\frac{b}{a}}$.

So the period of $\boldsymbol{f(ax + b)}$ can be said to be $\boldsymbol{\frac{b}{a}}$, too.

That's because we can get $\boldsymbol{f(ax + b) = f\{a(x+\frac{b}{a})\}}$.

Besides, if $\boldsymbol{f(x)}$ is periodic, then $\boldsymbol{f(ax)}$ is periodic, too. And vice versa.

## Suggestions or Solutions
### To the Problem in the Example 3

**Assuming $f(x)$ is periodic, the period is 2, and $f(x) = |x - k|$ for $k - 1 \le x \le k + 1$, where $k$ is an even integer, find $f(13.2)$ and $f(-6.7)$.**

What do we mean by this: $f(x) = |x - k|$ for $k - 1 \le x \le k + 1$, where $k$ is even?

When $k = -4$, $f(x) = |x + 4|$ for $-5 \le x \le -3$.

When $k = -2$, $f(x) = |x + 2|$ for $-3 \le x \le -1$.

When $k = 0$, $f(x) = |x|$ for $-1 \le x \le 1$.

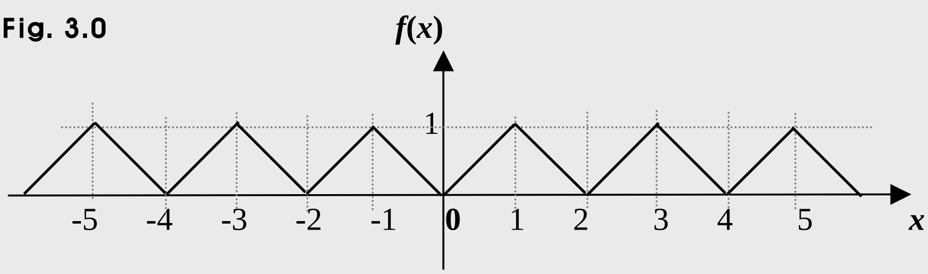
When $k = 2$, $f(x) = |x - 2|$ for $1 \le x \le 3$.

When $k = 4$, $f(x) = |x - 4|$ for $3 \le x \le 5$.

And so forth.

So putting $f(x)$ in a graph, we can put it the way below.

Fig. 3.0




In the graph above, we can see that the period is 2.

Now, if we need to get $f(13.2)$, the interval has to be $13 \le x \le 15$.

And we have $\boldsymbol{k - 1 \le x \le k + 1}$, where $\boldsymbol{k}$ is even. So we get $\boldsymbol{k - 1 = 13 \Rightarrow k = 14}$.

Then, we get $\boldsymbol{f(x) = |x - 14|}$ for $\boldsymbol{13 \le x \le 15}$.

So we get $\boldsymbol{f(13.2) = |13.2 - 14| = 0.8}$.

And next, if we need to get $\boldsymbol{f(-6.7)}$, the interval has to be $\boldsymbol{-7 \le x \le -5}$.

So we get $\boldsymbol{k - 1 = -7 \Rightarrow k = -6}$.

Then, we get $\boldsymbol{f(x) = |x + 6|}$ for $\boldsymbol{-7 \le x \le -5}$.

So we get $\boldsymbol{f(-6.7) = |-6.7 + 6| = 0.7}$.

And we can get the solution the way below, too.

Since $\boldsymbol{f}$ is periodic and the period is 2, we can set $\boldsymbol{f(x) = f(x + 2)}$.

And also, we can set $\boldsymbol{f(x) = f(x - 2)}$, since $\boldsymbol{f}$ is periodic and the period is 2.

And since the period is 2, we can get $\boldsymbol{f(x) = f(x + 2n)}$ for $\boldsymbol{n}$ integer.

And in this case, we can use the interval as follows. $\boldsymbol{-1 \le x \le 1}$.

Then, we need to use $\boldsymbol{f(x) = |x|}$ for $\boldsymbol{-1 \le x \le 1}$.

Then, we can get

$\boldsymbol{f(x) = f(x - 2\cdot 7) = f(x - 14) \Rightarrow f(13.2) = f(13.2 - 14) = f(-0.8) = |-0.8| = 0.8}$.

$\boldsymbol{f(x) = f(x + 2\cdot 3) = f(x + 6) \Rightarrow f(-6.7) = f(-6.7 + 6) = f(-0.7) = |-0.7| = 0.7}$.