Examples 5 in Lines

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0. Assuming three points $\boldsymbol{P_1(-2k - 1, 5)}$, $\boldsymbol{P_2(1, k + 3)}$, and $\boldsymbol{P_3(k + 1, k - 1)}$ are in a line, find $\boldsymbol{k}$.

1. Assuming $\boldsymbol{s}$ and $\boldsymbol{t}$ are nonzero constants, find the equation of a line including two points $\boldsymbol{(s, 0)}$ and $\boldsymbol{(0, t)}$ in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane.

2. Find the line passing through two points **(1, 2)** and **(3, 5)**.

3. Find the line of slope 3 passing through a point **(-2, 9)**.

Suggestions or Solutions
To the Problem in the Example 0

Assuming three points $P_1(-2k - 1, 5)$, $P_2(1, k + 3)$, and $P_3(k + 1, k - 1)$ are in a line, find k.

Suppose that the equation of the line is $y = ax + b$, and that $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$.

Then, $x_1 = -2k - 1$, $y_1 = 5$, $x_2 = 1$, $y_2 = k + 3$, $x_3 = k + 1$, and $y_3 = k - 1$.

Next, assuming a is the slope of the line, by the definition for a slope in a line, we get

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1} \Rightarrow (x_3 - x_1)(y_2 - y_1) = (x_2 - x_1)(y_3 - y_1).$$

So putting first, the coordinate values into the left hand side, we get:

$(x_3 - x_1)(y_2 - y_1) = \{(k + 1) - (-2k - 1)\}\{(k + 3) - 5\} = (3k + 2)(k - 2) = 3k^2 - 4k - 4$.

And next, on the right hand side, we get

$(x_2 - x_1)(y_3 - y_1) = \{1 - (-2k - 1)\}\{(k - 1) - 5\} = (2k + 2)(k - 6) = 2k^2 - 10k - 12$.

Thus, we get

$2k^2 - 10k - 12 = 3k^2 - 4k - 4 \Rightarrow k^2 + 6k + 8 = 0 \Rightarrow (k + 2)(k + 4) = 0 \Rightarrow k = -2$ or -4.

If not quite sure of the idea behind the processes above, follow the steps below.

Since a line has the three points given, the points have to satisfy an equation for lines.
So applying the three points to such an equation, we get three equations for k.
How then do we apply those points to the equation?

We put them into the equation. Putting a point into the equation, we put the coordinates of the point into the corresponding variables in the equation. For instance, the variable x gets the x-coordinate of the point, and the variable y gets the y-coordinate.

So putting each point given into an equation for lines, we get an equation for k.
Thus, we get three equations for k, since we have three points to put into.

Next, we can use two forms in equations for lines.

One is $\boldsymbol{ax + by + c = 0}$, often called a general equation of a line.
And the other is $\boldsymbol{y = ax + b}$, often called the slope-intercept form.

Of the two forms then, which one should we use?

Using the general form, we get to solve a system of three equations for four unknowns, which are $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{k}$. Then, the number of equations is less than that of the unknowns.

So we cannot solve the system, and thus, cannot get the value of $\boldsymbol{k}$.

Using the other form, we get a system of three equations for three unknowns, which are $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{k}$, and thus, we can get the $\boldsymbol{k}$-value, since the number of the equations equals the number of the unknowns. So we want to use the other form, put the points into the form, and then, solve the system of equations.

Solving a problem in this kind though, instead of setting up such a system, we may want to take advantage of the definition of the slope of a line.   How?

Suppose now, that the line passing through the three points is $\boldsymbol{y = ax + b}$, and that

$\boldsymbol{P_1 = (x_1, y_1)}$, $\boldsymbol{P_2 = (x_2, y_2)}$, and $\boldsymbol{P_3 = (x_3, y_3)}$.

Then, we get $\boldsymbol{x_1 = -2k - 1}$, $\boldsymbol{y_1 = 5}$, $\boldsymbol{x_2 = 1}$, $\boldsymbol{y_2 = k + 3}$, $\boldsymbol{x_3 = k + 1}$, and $\boldsymbol{y_3 = k - 1}$.

First, since $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ are in the line, we get the slope, $\boldsymbol{a} = \frac{y_2 - y_1}{x_2 - x_1}$ by the definition.
Next, $\boldsymbol{P_1}$ and $\boldsymbol{P_3}$ are in the line, too, so we can get the slope this way, too: $\boldsymbol{a} = \frac{y_3 - y_1}{x_3 - x_1}$.

So we get $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1} \Rightarrow \boldsymbol{(x_3 - x_1)(y_2 - y_1) = (x_2 - x_1)(y_3 - y_1)}$.

So putting the given coordinates into the equality above, we can get the equation for $\boldsymbol{k}$.

So to begin with, putting them into the left hand side, and simplifying it, we get
$\boldsymbol{(x_3 - x_1)(y_2 - y_1) = \{(k + 1) - (-2k - 1)\}\{(k + 3) - 5\} = (3k + 2)(k - 2) = 3k^2 - 4k - 4}$.

Next, doing the same to the right hand side, we get
$(x_2 - x_1)(y_3 - y_1) = \{1 - (-2k - 1)\}\{(k - 1) - 5\} = (2k + 2)(k - 6) = 2k^2 - 10k - 12$.

So we get $2k^2 - 10k - 12 = 3k^2 - 4k - 4 \Rightarrow k^2 + 6k + 8 = 0 \Rightarrow (k + 2)(k + 4) = 0$.

Therefore, we can see that $k = -2$ or -4.

Putting each of the values of k into each of the points, we can see that
(3, 5), **(1, 1)**, and **(-1, -3)** are in a line, and **(7, 5)**, **(1, -1)**, and **(-3, -5)** are in a line, too.

In short:

Assuming first, the line is $y = ax + b$, and $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$, we can set $x_1 = -2k - 1$, $y_1 = 5$, $x_2 = 1$, $y_2 = k + 3$, $x_3 = k + 1$, and $y_3 = k - 1$.

Next, assuming a is the slope, by the definition for slopes, we get

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1} \Rightarrow (x_3 - x_1)(y_2 - y_1) = (x_2 - x_1)(y_3 - y_1).$$

So putting first, the coordinate values into the left hand side, we get
$(x_3 - x_1)(y_2 - y_1) = \{(k + 1) - (-2k - 1)\}\{(k + 3) - 5\} = (3k + 2)(k - 2) = 3k^2 - 4k - 4$.

And next, on the right hand side, we get
$(x_2 - x_1)(y_3 - y_1) = \{1 - (-2k - 1)\}\{(k - 1) - 5\} = (2k + 2)(k - 6) = 2k^2 - 10k - 12$.

Thus, we get
$2k^2 - 10k - 12 = 3k^2 - 4k - 4 \Rightarrow k^2 + 6k + 8 = 0 \Rightarrow (k + 2)(k + 4) = 0 \Rightarrow k = -2$ or -4.

Suggestions or Solutions
To the Problem in the Example 1

Assuming that *s* and *t* are nonzero constants, find the line passing through two points (*s*, 0) and (0, *t*) in the *x*-*y* plane.

Suppose first, the line is $\boldsymbol{y = ax + b}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant.

Then, we get $a = \frac{\Delta y}{\Delta x} = \frac{t-0}{0-s} = -\frac{t}{s} \Rightarrow y = -\frac{t}{s}x + b$.

Since **(*s*, 0)** is in the line, $0 = -\frac{t}{s}s + b = -t + b \Rightarrow b = t$.

Thus, $y = -\frac{t}{s}x + t \Rightarrow \frac{y}{t} = -\frac{x}{s} + 1 \Rightarrow \frac{x}{s} + \frac{y}{t} = 1$.

Therefore, the line is $\frac{x}{s} + \frac{y}{t} = 1$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, what do we know about the point **(*s*, 0)**?

Since the ***y***-coordinate is 0, it is the point where the line meets the ***x***-axis, so what can we say about ***s***?

It is the ***x***-intercept, and similarly, **(0, *t*)** is the point where the line meets the ***y***-axis, so ***t*** is the ***y***-intercept. Then, what form of equation can serve the best?

Using the slope-intercept form, we can readily get both intercepts.

So we may want to use this form: $\boldsymbol{y = ax + b}$, along with the two points given.

Then, first, we can get $a = \frac{\Delta y}{\Delta x} = \frac{t-0}{0-s} = -\frac{t}{s}$, which is the slope of the line passing through the two points **(*s*, 0)** and **(0, *t*)**.

Thus, we get $\boldsymbol{y = -\frac{t}{s}x + b}$. So next, getting $\boldsymbol{b}$, we get the line.

Getting $\boldsymbol{b}$, we can use **(*s*, 0)**, since it is in the line.

Then we can get $\boldsymbol{y = -\frac{t}{s}x + b \Rightarrow 0 = -\frac{t}{s}s + b = -t + b \Rightarrow b = t}$.

So the line we want is $\boldsymbol{y = -\frac{t}{s}x + t}$, which therefore, can be the solution.

We can make the equation look a bit better.

Dividing both sides by $\boldsymbol{t}$ each, we get $\boldsymbol{\frac{y}{t} = -\frac{x}{s} + 1 \Rightarrow \frac{x}{s} + \frac{y}{t} = 1}$, which looks a bit better.

And in fact, the equation above is often called the intercept form for lines.

In the equation above, $\boldsymbol{s}$ is the $\boldsymbol{x}$-intercept, and $\boldsymbol{t}$ is the $\boldsymbol{y}$-intercept.
So given the two intercepts, we can quickly get the line using the form above.

In short:

Assuming first, the line is $\boldsymbol{y = ax + b}$, we get $\boldsymbol{a = \frac{\Delta y}{\Delta x} = \frac{t-0}{0-s} = -\frac{t}{s} \Rightarrow y = -\frac{t}{s}x + b}$.

Next, since **(*s*, 0)** is in the line, we get $\boldsymbol{0 = -\frac{t}{s}s + b = -t + b \Rightarrow b = t}$.

So we get $\boldsymbol{y = -\frac{t}{s}x + t \Rightarrow \frac{y}{t} = -\frac{x}{s} + 1 \Rightarrow \frac{x}{s} + \frac{y}{t} = 1}$.

Therefore, the line is $\boldsymbol{\frac{x}{s} + \frac{y}{t} = 1}$.

Suggestions or Solutions
To the Problem in the Example 2

Find the line passing through two points (1, 2) and (3, 5).

Having a point **(1, 2)** and a slope of $\boldsymbol{a}$, the line is: $\boldsymbol{y - 2 = a(x - 1)}$.

The line has two points **(1, 2)** and **(3, 5)**, so its slope is: $\frac{\Delta y}{\Delta x} = \frac{5-2}{3-1} = \frac{3}{2}$.

Therefore, the line is: $\boldsymbol{y - 2 = \frac{3}{2}(x - 1)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Given the slope and a point in a line, we can readily get the equation of the line using the form of $\boldsymbol{y - t = a(x - s)}$, where $\boldsymbol{a}$ is the slope and $\boldsymbol{(s, t)}$ is the point given.

So the line of slope **2** passing through a point **(2, 3)** is $\boldsymbol{y - 3 = 2(x - 2)}$.

In this problem though, we are not given the slope?

Yes, we are. Where then is it?

It is between the two points given. The slope of the line segment between the two is the slope of the line the two belong to. So we can get the slope from the two, then using one of the two, we can get the equation of the line.

And it's not probably quite hard to remember the slope-intercept form if we give just a little care to the idea. And the form is: $\boldsymbol{y = ax + b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant.

Then, getting the values of $\boldsymbol{a}$ and $\boldsymbol{b}$, we get the equation of the particular line we want.
Also, knowing the line includes a point $\boldsymbol{(s, t)}$, we can get this: $\boldsymbol{y - t = a(x - s)}$ by means of this form: $\boldsymbol{y = ax + b}$.   How?

Putting the point (s, t) into the form: $y = ax + b$, we can get

$y = ax + b \Rightarrow t = as + b \Rightarrow b = t - as$.

So we get $y = ax + b = ax + t - as = a(x - s) + t \Rightarrow y - t = a(x - s)$.

Thus for instance, the line of slope 3 passing through a point $(1, 2)$ is: $y - 2 = 3(x - 1)$.

Now, what have we got in this problem?

We have two points $(1, 2)$ and $(3, 5)$, and need to find the line passing through the two.

Using the two points $(1, 2)$ and $(3, 5)$, we can get the slope, which is $\frac{\Delta y}{\Delta x} = \frac{5-2}{3-1} = \frac{3}{2}$.

And the equation of a line of slope a passing through (s, t) is $y - t = a(x - s)$.

We can set $a = \frac{3}{2}$, and $(s, t) = (1, 2)$, which is one of the two points given.

Therefore, the line is $y - 2 = \frac{3}{2}(x - 1)$.

We can put it this way, too: $y - 2 = \frac{3}{2}(x - 1) \Rightarrow y = \frac{3}{2}(x - 1) + 2 = \frac{3}{2}x + \frac{1}{2} \Rightarrow y = \frac{3}{2}x + \frac{1}{2}$.

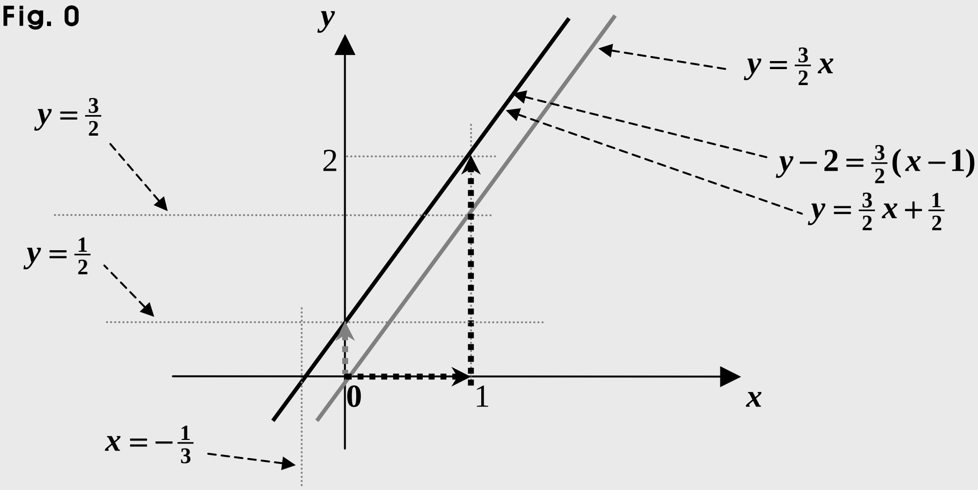
Note that the slope-intercept form $y = ax + b$ is from this form: $y - t = a(x - s)$, often called the slope-point form.

And moving a line $y = ax$ by s along the x-axis, and by t along the y-axis, we can get the line $y - t = a(x - s)$, too.

So moving a line $y = \frac{3}{2}x$ by 1 along the x-axis and by 2 along the y-axis, we get the line $y - 2 = \frac{3}{2}(x - 1)$.

Such a movement is called a parallel transformation, which is covered in the book **GRAPH OPERATIONS**.

Fig. 0



Moving the line in gray by $\frac{1}{2}$ in the direction of the y-axis, we get the line in black, which is $y=\frac{3}{2}x+\frac{1}{2}$.

Also, moving the line in gray by 1 along the x-axis, and by 2 along the y-axis, we get the same line, which is the line in black, which is $y-2=\frac{3}{2}(x-1)$, which is $y=\frac{3}{2}x+\frac{1}{2}$.

In short:

Having a point **(1, 2)** and a slope of a, the line is $y-2=a(x-1)$.

The line has two points **(1, 2)** and **(3, 5)**, so its slope is $\frac{\Delta y}{\Delta x}=\frac{5-2}{3-1}=\frac{3}{2}$.

Therefore, the line is $y-2=\frac{3}{2}(x-1)$.

Suggestions or Solutions
To the Problem in the Example 3

Find the line of slope 3 passing through a point (-2, 9).

Having a point at (s, t) and a slope of a, we get a line, $y - t = a(x - s)$.

So having a point at **(-2, 9)** and a slope of **3**, we get

$y - 9 = 3\{x - (-2)\} = 3(x + 2) \Rightarrow y - 9 = 3(x + 2) \Rightarrow y = 3x + 15$.

If not quite sure of the idea behind the processes above, follow the steps below.

Getting a point and a slope, we can find a particular line.   How?

Infinitely many lines can share one slope, and they are said to be parallel to each other.

Also, infinitely many lines can share one point, but all their slopes have to be different.

So of all lines sharing the same one slope, one only can pass through one particular point.

Thus, there has to be a line of one particular slope passing through one particular point. And it is the only line that can be so.

Given a point in a line and the slope, we can readily get the line by the standard equation as follows: $y - t = a(x - s)$, where a is the slope, and (s, t) is the point.

Of course, the equation above is not quite the standard as $y = ax + b$, so we might want to call it the quasi or semi standard.

So the line of slope a passing through a point (s, t) is $y - t = a(x - s)$, which is therefore, a convenient tool that can give us instantly the line having a point and a slope specified.

Now, the line we want is of slope 3, and has a point at **(-2, 9)**, so using the point-slope form above, we can get $y - 9 = 3\{x - (-2)\} = 3(x + 2)$.

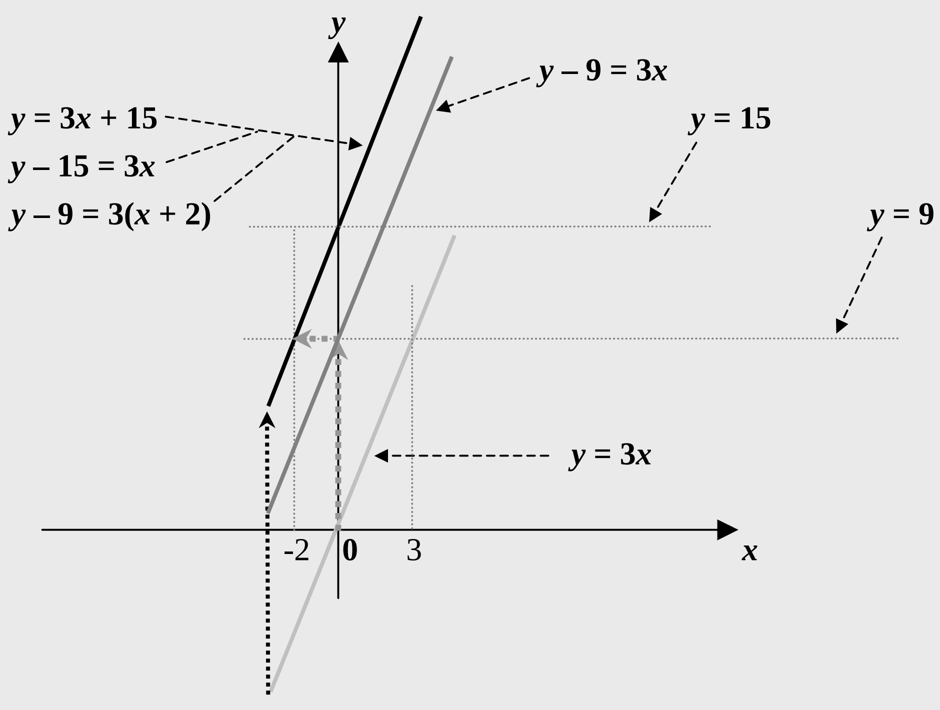
Thus, the line is $y - 9 = 3(x + 2)$, which can be of course, put this way, too: $y = 3x + 15$.

There is another way, too, where we can get the line above.

• The line $y = 3x + 15$ can be taken as the result of translating a line $y = 3x$ by **15** in the direction of the *y*-axis.

• Also, translating the line $y = 3x$ by **-2** along the *x*-axis, and by **9** along the *y*-axis, we can get the line $y - 9 = 3(x + 2)$, which is the same as $y = 3x + 15$, of course.

Fig. 0



In the graph above, we can see two translations.

One is that the line in light gray can get translated directly to the position of the line in black. The dotted arrow in black shows the translation path.

$y = 3x$ ··························▶ $y - 9 = 3\{x - (-2)\}$

The other is that the line in light gray gets translated to the same position via the position of the line in dark gray. The arrows dotted in gray show the translation paths.

$y = 3x$ ··········▶ $y - 9 = 3x$ ··········▶ $y - 9 = 3\{x - (-2)\}$

In short:

Having a point at (s, t) and a slope of a, we get a line, $y - t = a(x - s)$.

So having a point at **(-2, 9)** and a slope of **3**, we get

$y - 9 = 3\{x - (-2)\} = 3(x + 2) \Rightarrow y - 9 = 3(x + 2) \Rightarrow y = 3x + 15$.