Examples 6 in Lines

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0. Suppose a line $\boldsymbol{L}$ includes two points **(2, 1)** and **(3, 5)**, and another line $\boldsymbol{N}$ includes a point **(1, 1)** and is perpendicular to the line $\boldsymbol{L}$. Find the line $\boldsymbol{N}$.

1. Suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are nonzero constants, $\boldsymbol{A}$ is a line $\mathbf{3}\boldsymbol{x} + \frac{\boldsymbol{y}}{\boldsymbol{a}} - \mathbf{2} = \mathbf{0}$,
$\boldsymbol{B}$ is a line $\mathbf{5}\boldsymbol{x} - \boldsymbol{by} + \mathbf{3} = \mathbf{0}$, and $\boldsymbol{C}$ is a line $\mathbf{2}\boldsymbol{x} + \frac{\boldsymbol{y}}{\mathbf{1}-\mathbf{2}\boldsymbol{b}} - \mathbf{5} = \mathbf{0}$.
Suppose also, the line $\boldsymbol{A}$ is perpendicular to the line $\boldsymbol{B}$, but is parallel to the line $\boldsymbol{C}$.

Find $\boldsymbol{a}$ and $\boldsymbol{b}$.

Suggestions or Solutions
To the Problem in the Example 0

Find a line called *N*, which includes a point (1, 1), and is perpendicular to another line called *L* passing through two points (2, 1) and (3, 5).

First, the slope of $L = \dfrac{5-1}{3-2} = \dfrac{4}{1} = 4$. Next, the product of the slopes of L and N is -1.

So the slope of $N = -\dfrac{1}{4}$. Since N includes $(1, 1)$, the equation of N is $y - 1 = -\frac{1}{4}(x-1)$.

If not quite sure of the idea behind the processes above, follow the steps below.

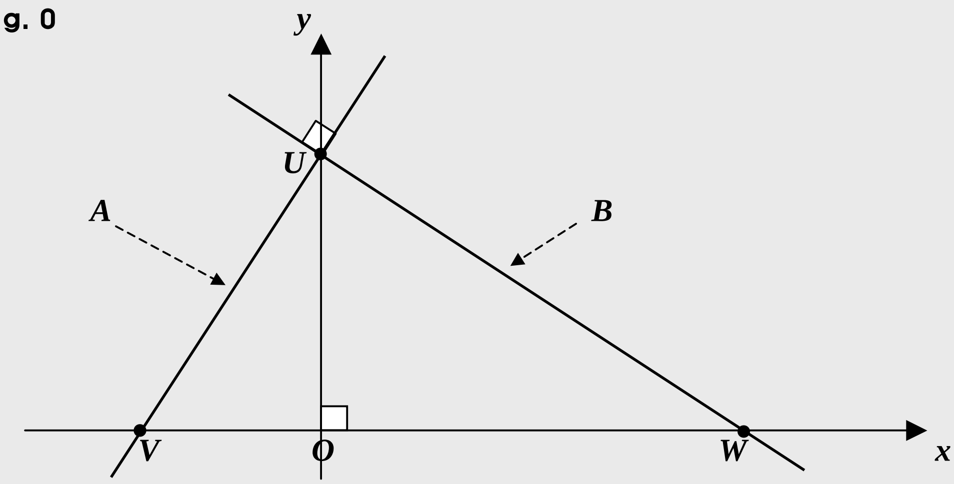
We need to use the fact that the two lines ***L*** and ***N*** are perpendicular to each other.

Under what condition then, two lines are perpendicular to each other?

It is in fact, what we are going to cover in this example.

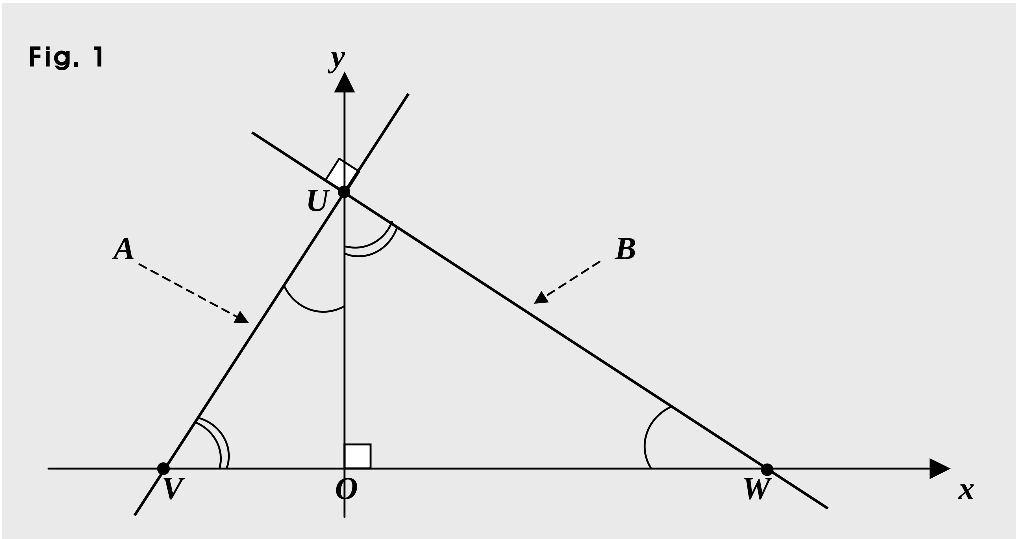
So suppose first, two lines ***A*** and ***B*** are perpendicular to each other, meet at a point ***U***, and meet the ***x***-axis at two points ***V*** and ***W*** respectively as shown in the figure below.

Fig. 0



Next, examining the two triangles ***VOU*** and ***UOW*** in the figure above, we can notice that both share the same set of three angles.

Fig. 1



So we can say that the two triangles ***VOU*** and ***UOW*** are similar triangles.
And it means that the ratios between corresponding sides are the same.

Thus, we get $\left|\dfrac{\overline{OW}}{\overline{OU}}\right| = \left|\dfrac{\overline{OU}}{\overline{OV}}\right|$. (Note that $|x|$ is the absolute value (the magnitude) of x.)

So what does it have to do with the lines perpendicular to each other?

The slope of the line ***B*** is $-\left|\dfrac{\overline{OU}}{\overline{OW}}\right|$, because the slope is negative.

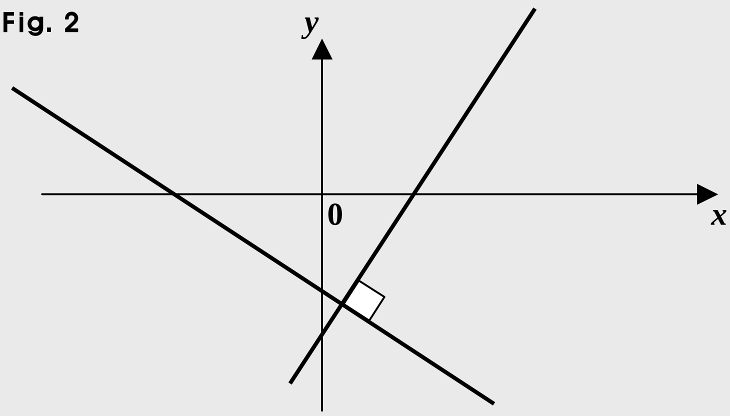
And the slope of the line ***A*** is $\left|\dfrac{\overline{OU}}{\overline{OV}}\right|$. And we have $\left|\dfrac{\overline{OW}}{\overline{OU}}\right| = \left|\dfrac{\overline{OU}}{\overline{OV}}\right|$, too.

So we get $\left|\dfrac{\overline{OU}}{\overline{OV}}\right| \cdot \left(-\left|\dfrac{\overline{OU}}{\overline{OW}}\right|\right) = -\left|\dfrac{\overline{OW}}{\overline{OU}}\right| \cdot \left|\dfrac{\overline{OU}}{\overline{OW}}\right| = -\mathbf{1}$, which is the product of the two slopes.

And thus, the product of the slopes of two lines perpendicular to each other is **-1**.

So using the fact above, we can solve many problems where lines are perpendicular to each other. What if we are given lines as shown below and need to prove the fact above?

Fig. 2



We can relocate the lines above so that the work can be easier.

We can do so without loss of generality if we translate the lines above.

Translating lines, we don’t change the slopes.

Since a line means a slope, if the slope remains the same, we don’t lose the generality of the fact we want to show. Translating a line, we change its position without turning it.

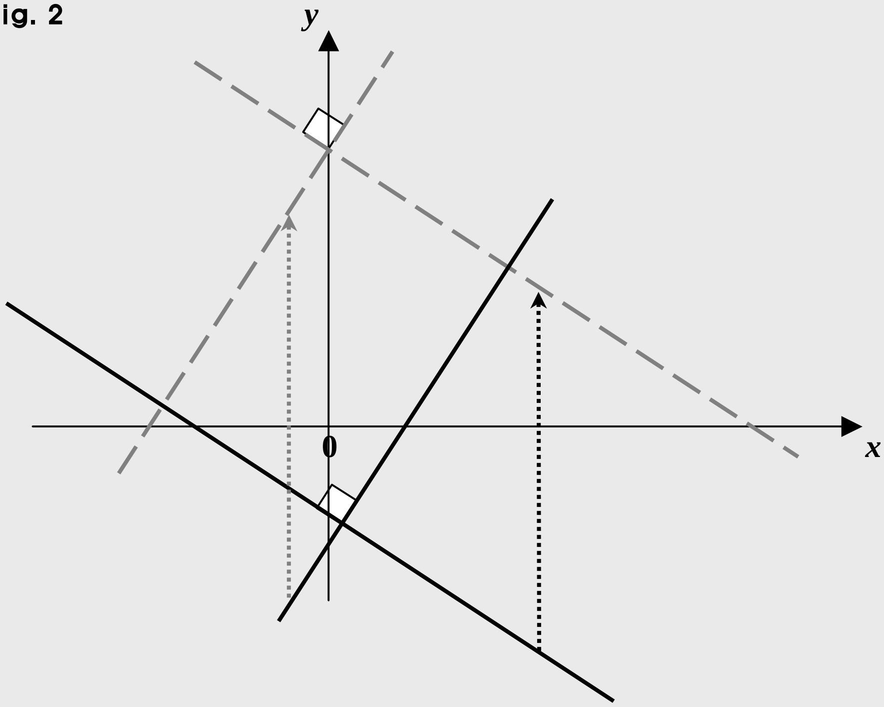
For instance, the line moves either up, down, forward, backward, to the left, or right with neither spinning nor turning at all.

So if we just translate lines, their slopes remain the same.

Thus, the lines perpendicular to each other stay perpendicular to each other during and after the translation.

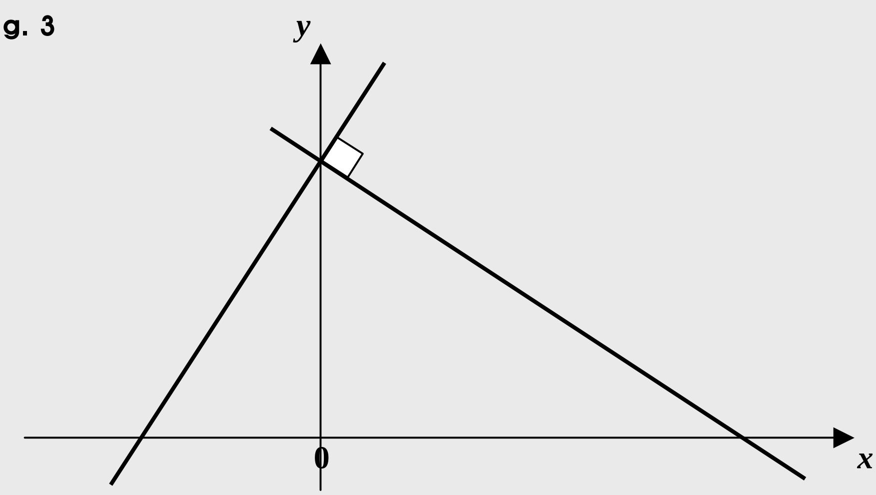
So we can translate the two lines the way shown below.

Fig. 2



The amount of translation is indicated by the length of the dotted arrow.
The lines dashed in the figure above are the lines resulting from the translation.
Then, with the figure below, we can begin the proving processes as the way done above.

Fig. 3



Now, getting back to the problem in this example, we want to find the line ***N***.

Given a slope and a point in a line, we can find the line.
We know that the line ***N*** has a point **(1, 1)**, but not the slope.
Thus, finding the slope, we can get the line ***N***.

Since the line ***N*** is perpendicular to the line ***L***, the product of the slopes is **-1**.
So first, we can get the slope of ***N*** from the slope of ***L***. So let's get the slope of ***L***, now.

Using the two points **(2, 1)** and **(3, 5)** the line ***L*** has, we can readily get the slope of ***L***.
The slope of ***L*** = $\frac{\Delta y}{\Delta x} = \frac{5-1}{3-2} = \frac{4}{1} = 4$. So the slope of ***N*** $= -\frac{1}{4}$, since the product is -1.

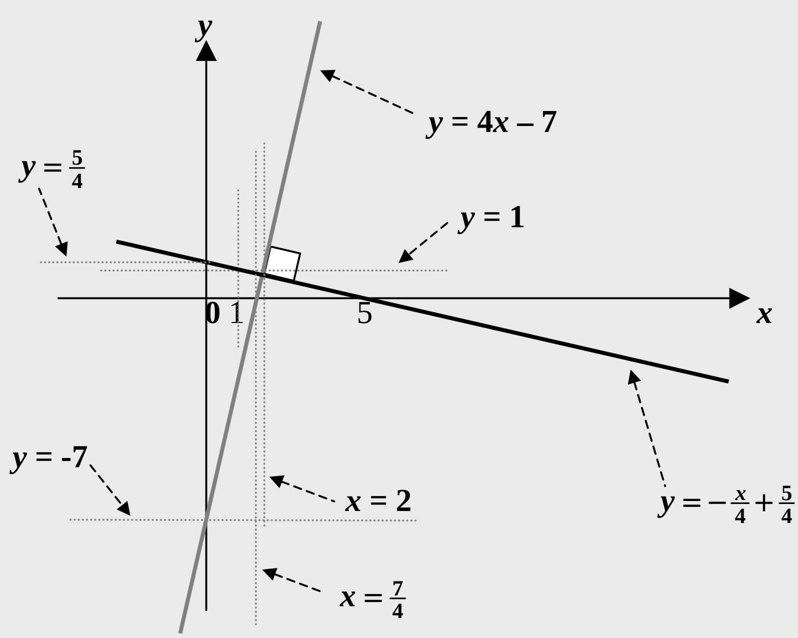
Next, the line ***N*** has a point **(1, 1)**, so the equation of ***N*** is $y - 1 = -\frac{1}{4}(x-1)$.

Exposing the ***y***-intercept, that is, putting it in the slope-point form, we can put it this way: $y = -\frac{1}{4}x + \frac{1}{4} + 1 = -\frac{1}{4}x + \frac{5}{4} \Rightarrow y = -\frac{1}{4}x + \frac{5}{4}$, where $\frac{5}{4}$ is the ***y***-intercept.

And for reference, the equation of the line ***L*** is $y - 1 = 4(x - 2)$, which equals $y = 4x - 7$.

And putting the two lines in one graph, we get

Fig. 4



Suggestions or Solutions
To the Problem in the Example 1

Suppose a and b are nonzero constants, A is a line $3x + \frac{y}{a} - 2 = 0$,
B is a line $5x - by + 3 = 0$, and C is a line $2x + \frac{y}{1-2b} - 5 = 0$.
Suppose also, the line A is perpendicular to the line B, but is parallel to the line C.

Find a and b.

The line A: $3x + \frac{y}{a} - 2 = 0 \Rightarrow y = -3ax + 2a$.
The line B: $5x - by + 3 = 0 \Rightarrow y = \frac{5x}{b} - \frac{3}{b}$.
The line C: $2x + \frac{y}{1-2b} - 5 = 0 \Rightarrow y = 2(2b-1)x + 5(2b-1)$.

The line A is perpendicular to the line B, and is parallel to the line C.
So the line B is perpendicular to both of the lines A and C. Thus, we get

$$\frac{5}{b} \cdot 2(2b-1) = -1 \Rightarrow 10(2b-1) = -b \Rightarrow 20b - 10 + b = 0 \Rightarrow 21b = 10 \Rightarrow b = \frac{10}{21}.$$

$$(-3a)\frac{5}{b} = -1 \Rightarrow 15a = b = \frac{10}{21} \Rightarrow a = \frac{10}{21} \cdot \frac{1}{15} = \frac{2}{63}.$$

If not quite sure of the idea behind the processes above, follow the steps below.

What matters in a line is the slope, and if two lines are perpendicular to each other, the product of the slopes is -1. If lines are parallel to each other, the slopes are the same.

So let's examine the slopes.

Putting the equations in the point-slope form, we can readily see the slopes.
So putting each of the lines in such a form, we get

The line A: $3x + \frac{y}{a} - 2 = 0 \Rightarrow y = -3ax + 2a$.

The line $\boldsymbol{B}$: $\mathbf{5}\boldsymbol{x} - \boldsymbol{by} + \mathbf{3} = \mathbf{0} \Rightarrow \boldsymbol{y} = \frac{5x}{b} - \frac{3}{b}$.

The line $\boldsymbol{C}$: $2x + \frac{y}{1-2b} - 5 = 0 \Rightarrow y = 2(2b - 1)x + 5(2b - 1)$.

Thus, the slope of the line $\boldsymbol{A}$ is **-3a**, and the slope of the line $\boldsymbol{B}$ is $\frac{5}{b}$.

Since the line $\boldsymbol{A}$ is perpendicular to the line $\boldsymbol{B}$, we get $(-3a)\frac{5}{b} = -1 \Rightarrow 15a = b$.

Besides, since the line $\boldsymbol{A}$ is parallel to the line $\boldsymbol{C}$, the line $\boldsymbol{B}$ is perpendicular to the line $\boldsymbol{C}$, too, and the slope of the line $\boldsymbol{C}$ is $2(2b - 1)$. So we get $\frac{5}{b} \cdot 2(2b - 1) = -1$. Thus, we get

$$\frac{5}{b} \cdot 2(2b - 1) = -1 \Rightarrow 10(2b - 1) = -b \Rightarrow 20b - 10 + b = 0 \Rightarrow 21b = 10 \Rightarrow b = \frac{10}{21}.$$

We have $15a = b$, too. Thus, we get $a = \frac{b}{15} = \frac{\frac{10}{21}}{15} = \frac{10}{21 \cdot 15} = \frac{2}{21 \cdot 3} = \frac{2}{63}$.

In short:

The line $\boldsymbol{A}$: $3x + \frac{y}{a} - 2 = 0 \Rightarrow y = -3ax + 2a$.

The line $\boldsymbol{B}$: $5x - by + 3 = 0 \Rightarrow y = \frac{5x}{b} - \frac{3}{b}$.

The line $\boldsymbol{C}$: $2x + \frac{y}{1-2b} - 5 = 0 \Rightarrow y = 2(2b - 1)x + 5(2b - 1)$.

The line $\boldsymbol{A}$ is perpendicular to the line $\boldsymbol{B}$, and is parallel to the line $\boldsymbol{C}$.
So the line $\boldsymbol{B}$ is perpendicular to both of the lines $\boldsymbol{A}$ and $\boldsymbol{C}$. Thus, we get

$$\frac{5}{b} \cdot 2(2b - 1) = -1 \Rightarrow 10(2b - 1) = -b \Rightarrow 20b - 10 + b = 0 \Rightarrow 21b = 10 \Rightarrow b = \frac{10}{21}.$$

$$(-3a)\frac{5}{b} = -1 \Rightarrow 15a = b = \frac{10}{21} \Rightarrow a = \frac{10}{21} \cdot \frac{1}{15} = \frac{2}{63}.$$