Examples 7 in Lines

Click this to start.

Examples 7 in Lines

0. Assuming a line includes two points **(1, 2)** and **(3, 5)**, find the distance from the line to a point **(-5, 3)**.

1. Assuming a line of slope 2 includes a point **(2, 5)**, and one of two points symmetric about the line is **(3, 2)**, find the other point.

Suggestions or Solutions
To the Problem in the Example 0

Assuming a line includes two points (1, 2) and (3, 5), find the distance from the line to a point (-5, 3).

Suppose that

L is the line that includes the two points **(1, 2)** and **(5, 3)**, and that ***P*** is the point **(-5, 3)**.
N is the line that includes the point ***P***, and is perpendicular to the line ***L***.
Q is the point where the line ***L*** meets the line ***N***.

Then, first, the slope of the line ***L*** is $\frac{5-2}{3-1}=\frac{3}{2}$.

Next, since ***L*** has **(1, 2)**, ***L*** is $y-2=\frac{3}{2}(x-1)\Rightarrow y=\frac{3}{2}(x-1)+2=\frac{3}{2}x+\frac{1}{2}$.

Next, since ***N*** is perpendicular to ***L***, if ***n*** is the slope of ***N***, we get $\frac{3}{2}n=-1\Rightarrow n=-\frac{2}{3}$.

Next, ***N*** has **(-5, 3)**, so ***N*** is $y-3=-\frac{2}{3}(x+5)\Rightarrow y=-\frac{2}{3}(x+5)+3=-\frac{2}{3}x-\frac{1}{3}$.

Next, getting the point ***Q***, and getting the ***x***-coordinate first, we can get it the way below.

$\frac{3x}{2}+\frac{1}{2}=-\frac{2x}{3}-\frac{1}{3}\Rightarrow(\frac{3}{2}+\frac{2}{3})x=-\frac{1}{3}-\frac{1}{2}\Rightarrow\frac{13x}{6}=-\frac{5}{6}\Rightarrow x=-\frac{5}{13}$.

So next, getting the ***y***-coordinate, we get $y=\frac{3}{2}x+\frac{1}{2}=\frac{3}{2}\cdot\frac{-5}{13}+\frac{1}{2}=\frac{-15}{26}+\frac{1}{2}=\frac{-2}{26}=\frac{-1}{13}$.

Therefore, ***Q*** is at $(-\frac{5}{13},-\frac{1}{13})$.

Next, if ***D*** is the distance between ***P*** and ***Q***, ***D*** is the solution, and we can get ***D*** this way:

$$D^2=\{-\frac{5}{13}-(-5)\}^2+(-\frac{1}{13}-3)^2=(-\frac{5}{13}+5)^2+(\frac{1}{13}+3)^2=(\frac{60}{13})^2+(\frac{40}{13})^2$$

$$=\frac{60^2+40^2}{13^2}=\frac{3600+1600}{13^2}=\frac{5200}{13^2}=\frac{13\cdot4\cdot10^2}{13^2}=\frac{4\cdot10^2}{13}=\frac{20^2}{13}\Rightarrow D^2=\frac{20^2}{13}\Rightarrow D=\pm\frac{20}{\sqrt{13}}=\pm\frac{20\sqrt{13}}{13}.$$

Therefore, the distance is $\frac{20\sqrt{13}}{13}$.

If not quite sure of the idea behind the processes above, follow the steps below.

What do we mean by the distance from a particular point to a line?

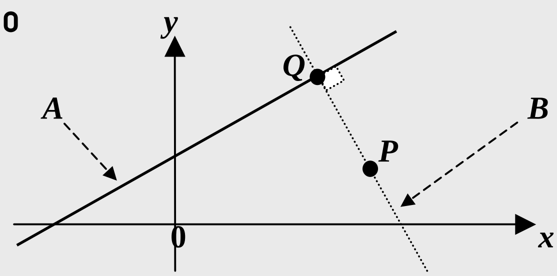
In math, the distance is the shortest distance between the line and the particular point. What then, do we mean by the shortest distance?

It can be called the perpendicular distance from the particular point to the line.
What perpendicular distance?

It's the length of the line segment perpendicular to the line, beginning at a point in the line, and ending at the particular point.

In other words, It's the length of the line segment perpendicular to the line, and in the line segment, one end is the particular point, and the other end belongs to the line.

Fig. 0



Suppose for instance, ***A*** is the line, ***P*** is the particular point, and ***B*** is a line the particular point ***P*** belongs to, and is perpendicular to the line ***A***.

Suppose also, that ***Q*** is the point where the line ***B*** meets the line ***A***.

Then, the shortest distance is the distance between the particular point ***P*** and the point ***Q***. That is, the distance between ***A*** and ***P*** is the length of $\overline{PQ}$.

So we now can see what we want to get first, to get the distance this problem is asking.

We want to get the line that has the two points **(1, 2)** and **(3, 5)**, the line perpendicular to it, and get the point where the two lines perpendicular to each other meet.

So suppose first, ***L*** is the line the two points **(1, 2)** and **(3, 5)** belong to, ***P*** is **(-5, 3)**, ***N*** is the line perpendicular to the line ***L***, and ***Q*** is the point where ***L*** and ***N*** meet each other.

Then, finding the point ***Q***, we can get the distance from ***P*** to ***Q*** by the distance formula, and the distance is the solution. So finding the point ***Q***, what do we need to get first?

We want to get the two lines ***L*** and ***N***. Of the two though, we can find the line ***L*** first, since we know two points in ***L***. So we can readily find ***L***, but how do we get the line ***N***?

Once we've got the line ***L***, we can get the slope of ***L***. Then, we can get the slopes of ***N*** from the slope of ***L***, since both lines are perpendicular to each other. The product of the slopes of the two lines is -1.

So to begin with, the line ***L*** includes **(1, 2)** and **(3, 5)**, so its slope is $\frac{\Delta x}{\Delta y} = \frac{5-2}{3-1} = \frac{3}{2}$.

Next, ***L*** includes **(1, 2)**, so ***L*** can be put this way: $y - 2 = \frac{3}{2}(x-1)$, where $\frac{3}{2}$ is the slope.

So the slope of the line ***N*** is $-\frac{2}{3}$, since ***N*** is perpendicular to ***L***, and –1 is the product of their slopes. And the line ***N*** includes **(-5, 3)**, so ***N*** can be put this way: $y - 3 = -\frac{2}{3}(x+5)$.

What then, is the next?

We want to find the point ***Q*** where the line ***L*** meets the line ***N***.

The line ***L*** is $y - 2 = \frac{3}{2}(x-1)$, and the line ***N*** is $y - 3 = -\frac{2}{3}(x+5)$.

Beginning with the ***x***-coordinate at ***Q***, we want to eliminate ***y***.

So preparing first, for the elimination, we get

$$y - 2 = \frac{3}{2}(x-1) \Rightarrow y = \frac{3}{2}(x-1) + 2 = \frac{3x}{2} + \frac{1}{2}.$$

$$y - 3 = -\frac{2}{3}(x+5) \Rightarrow y = -\frac{2}{3}(x+5) + 3 = -\frac{2x}{3} - \frac{1}{3}.$$

Now, we get $\frac{3x}{2} + \frac{1}{2} = -\frac{2x}{3} - \frac{1}{3} \Rightarrow (\frac{3}{2} + \frac{2}{3})x = -\frac{1}{3} - \frac{1}{2} \Rightarrow \frac{13x}{6} = -\frac{5}{6} \Rightarrow x = -\frac{5}{13}$.

Next, using $y = \frac{3x}{2} + \frac{1}{2}$, we get $y = \frac{3}{2}x + \frac{1}{2} = \frac{3}{2} \cdot \frac{-5}{13} + \frac{1}{2} = \frac{-15}{26} + \frac{1}{2} = \frac{-2}{26} = \frac{-1}{13} \Rightarrow y = -\frac{1}{13}$.

Therefore, we can see that the point ***Q*** is at $(-\frac{5}{13}, -\frac{1}{13})$. What then, is the next?

Now, we can get the distance between the two points ***P*** and ***Q***, which is the solution.

We know that the point ***P*** is at **(-5, 3)**, and that the formula is $D^2 = (\Delta x)^2 + (\Delta y)^2$, where ***D*** is the distance, Δx is the difference between the ***x***-coordinates at the two points, and Δy is the difference between the ***y***-coordinates. So we get

$$D^2 = \{-\tfrac{5}{13} - (-5)\}^2 + (-\tfrac{1}{13} - 3)^2 = (-\tfrac{5}{13} + 5)^2 + (\tfrac{1}{13} + 3)^2 = (\tfrac{60}{13})^2 + (\tfrac{40}{13})^2$$

$$= \tfrac{60^2+40^2}{13^2} = \tfrac{3600+1600}{13^2} = \tfrac{5200}{13^2} = \tfrac{13\cdot 4\cdot 10^2}{13^2} = \tfrac{4\cdot 10^2}{13} = \tfrac{20^2}{13} \Rightarrow D^2 = \tfrac{20^2}{13} \Rightarrow D = \pm\tfrac{20}{\sqrt{13}} = \pm\tfrac{20\sqrt{13}}{13}.$$

Therefore, the distance is $\frac{20\sqrt{13}}{13}$, since a distance is $\geq \mathbf{0}$. Why not just $D = \frac{20}{\sqrt{13}}$, though?

Normally, a rational number is preferred in the denominator, since it makes more sense. By the way, we have a formula for the distance between a point and a line. Assuming D is the distance from a point (s, t) to a line $ax + by + c = 0$, we get $D = \dfrac{|as + bt + c|}{\sqrt{a^2 + b^2}}$.

We will get to see how to get it in the next section.

In short:

Suppose that

L is the line that includes **(1, 2)** and **(5, 3)**, and P is the point **(-5, 3)**.
N is the line that includes the point P, and is perpendicular to the line L.
Q is the point where the line L meets the line N.

Then, first, the slope of the line L is $\frac{5-2}{3-1} = \frac{3}{2}$.

Next, L has **(1, 2)**, so L is $y - 2 = \frac{3}{2}(x-1) \Rightarrow y = \frac{3}{2}(x-1) + 2 = \frac{3}{2}x + \frac{1}{2}$.

Next, N is perpendicular to L, so if n is the slope of N, we get $\frac{3}{2}n = -1 \Rightarrow n = -\frac{2}{3}$.

Next, N has **(-5, 3)**, so N is $y - 3 = -\frac{2}{3}(x+5) \Rightarrow y = -\frac{2}{3}(x+5) + 3 = -\frac{2}{3}x - \frac{1}{3}$.

Next, getting the point Q, and getting the x-coordinate first, we can get it the way below.

$\frac{3x}{2} + \frac{1}{2} = -\frac{2x}{3} - \frac{1}{3} \Rightarrow (\frac{3}{2} + \frac{2}{3})x = -\frac{1}{3} - \frac{1}{2} \Rightarrow \frac{13x}{6} = -\frac{5}{6} \Rightarrow x = -\frac{5}{13}$. So next, getting the y-coordinate, we get $y = \frac{3}{2}x + \frac{1}{2} = \frac{3}{2}\cdot\frac{-5}{13} + \frac{1}{2} = \frac{-15}{26} + \frac{1}{2} = \frac{-2}{26} = \frac{-1}{13}$. So Q is at $(-\frac{5}{13}, -\frac{1}{13})$.

Next, if D is the distance between P and Q, D is the solution, and we can get D this way:

$$D^2 = \{-\tfrac{5}{13} - (-5)\}^2 + (-\tfrac{1}{13} - 3)^2 = (-\tfrac{5}{13} + 5)^2 + (\tfrac{1}{13} + 3)^2 = (\tfrac{60}{13})^2 + (\tfrac{40}{13})^2$$

$$= \tfrac{60^2+40^2}{13^2} = \tfrac{3600+1600}{13^2} = \tfrac{5200}{13^2} = \tfrac{13\cdot 4\cdot 10^2}{13^2} = \tfrac{4\cdot 10^2}{13} = \tfrac{20^2}{13} \Rightarrow D^2 = \tfrac{20^2}{13} \Rightarrow D = \pm\tfrac{20}{\sqrt{13}} = \pm\tfrac{20\sqrt{13}}{13}.$$

Therefore, the distance is $\frac{20\sqrt{13}}{13}$.

Suggestions or Solutions
To the Problem in the Example 1

Assuming a line of slope 2 includes a point (2, 5), and one of two points symmetric about the line is (3, 2), find the other point.

Suppose first, ***P*** is **(3, 2)**, ***L*** is the line of slope 2 passing through ***P***, ***Q*(*s*, *t*)** is the point symmetric to ***P*** about the line ***L***. ***N*** is the line perpendicular to the line ***L***, and has two points ***P*** and ***Q***, and ***R*(*u*, *v*)** is the midpoint between the two points ***P*** and ***Q***.

Then, the line ***L*** is the axis of symmetry, the midpoint ***R*** is the point where the two lines ***L*** and ***N*** meet each other as well as the midpoint between the two points ***P*** and ***Q***.

So first, since the line ***L*** has **(2, 5)** and the slope is 2, the line ***L*** is $y - 5 = 2(x - 2)$.

Next, in the line ***L***, the slope is 2.
So if ***n*** is the slope of the line ***N***, we get $2n = -1 \Rightarrow n = -\frac{1}{2}$.
Thus, the line ***N*** is $y - 2 = -\frac{1}{2}(x - 3)$ since the point ***P*(3, 2)** is in the line ***N***.

Next, ***R*(*u*, *v*)** is the midpoint between the two points ***P*(3, 2)** and ***Q*(*s*, *t*)**.
So we get $u = \frac{3+s}{2}$, and $v = \frac{2+t}{2}$.

Next, the line ***L*** is $y - 3 = 2(x - 1)$, so putting the midpoint ***R*** into the line ***L***, we get
$y - 5 = 2(x - 2) \Rightarrow v - 5 = 2(u - 2) \Rightarrow \frac{2+t}{2} - 5 = 2(\frac{3+s}{2} - 2)$
$\Rightarrow 2 + t - 10 = 4(\frac{3+s}{2} - 2) = 2s - 2 \Rightarrow t - 8 = 2s - 2 \Rightarrow 2s - t = -6$.

Next, the line ***N*** is $y - 2 = -\frac{1}{2}(x - 3)$, so putting the midpoint ***R*** into the line ***N***, we get
$y - 2 = -\frac{1}{2}(x - 3) \Rightarrow v - 2 = -\frac{1}{2}(u - 3) \Rightarrow \frac{t+2}{2} - 2 = -\frac{1}{2}(\frac{3+s}{2} - 3) \Rightarrow t + 2 - 4 = 3 - \frac{3+s}{2}$
$\Rightarrow 2t - 4 = 6 - (3 + s) = 3 - s \Rightarrow s + 2t = 7$.

Now, we need to solve the system of equations, $2s - t = -6$ and $s + 2t = 7$.
To begin with, we can get $2s - t = -6 \Rightarrow 4s - 2t = -12$. Thus, we get
$(4s - 2t) + (s + 2t) = -12 + 7 \Rightarrow 5s = -5 \Rightarrow s = -1 \Rightarrow s + 2t = -1 + 2t = 7 \Rightarrow 2t = 8 \Rightarrow t = 4$.

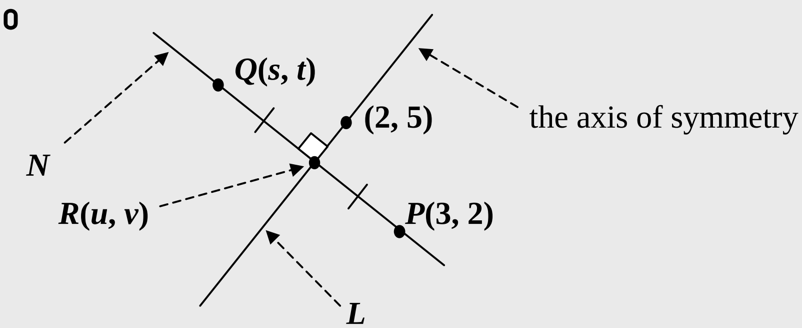
Therefore, ***Q*** is at **(-1, 4)**.

If not quite sure of the idea behind the processes above, follow the steps below.

If two points are symmetric about a line, the distance from one of the two points to the line is the same as the distance from the other to the line. Then, the line is called the axis of symmetry, and the two points are in a line perpendicular to the axis of symmetry.

So to begin with, putting in a graph the points and the line given, along with some points deemed to be necessary or relevant, we can see better the solution's whereabouts.

Fig. 0



Suppose now, that

P is the point **(3, 2)**.
L is the line of slope 2 passing through the point ***P***.
Q*(*s*, *t*)** is the point symmetric to the point ***P about the line ***L***.
N is the line perpendicular to the line ***L***, and has two points ***P*** and ***Q***.
R*(*u*, *v*)** is the midpoint between the two points ***P and ***Q***.

Then, the line ***L*** is the axis of symmetry, and the midpoint ***R*** is the point where the two lines ***L*** and ***N*** meet each other as well as the midpoint between the two points ***P*** and ***Q***.

So the problem solving procedure can go as follows.
To begin with, find the axis of symmetry ***L***, then the line ***N***.
Next, find the midpoint ***R***, then the point ***Q*** symmetric to the point ***P***.

Now, let's get started with finding the line ***L***, which is the axis of symmetry.
Since the line ***L*** has **(2, 5)** and the slope is 2, the line ***L*** is $\boldsymbol{y - 5 = 2(x - 2)}$.

Next, let's move on to the line ***N***.
To begin with, -1 is the product of the slopes of two lines perpendicular to each other.

So since the line $\boldsymbol{N}$ is perpendicular to the line $\boldsymbol{L}$, the slope of the line $\boldsymbol{N}$ can be found from the slope of the line $\boldsymbol{L}$. In the line $\boldsymbol{L}$, the slope is 2.

So if $\boldsymbol{n}$ is the slope of the line N, we get $\mathbf{2}\boldsymbol{n} = \mathbf{-1} \Rightarrow \boldsymbol{n} = -\frac{1}{2}$.

Thus, the line $\boldsymbol{N}$ is $\boldsymbol{y} - \mathbf{2} = -\frac{1}{2}(\boldsymbol{x} - \mathbf{3})$ since the point $\boldsymbol{P}(\mathbf{3}, \mathbf{2})$ is in the line $\boldsymbol{N}$.

We are now ready to get the midpoint $\boldsymbol{R}$.

• The coordinates of the midpoint between two points are the respective averages of the $\boldsymbol{x}$-coordinates and $\boldsymbol{y}$-coordinates of the two points.

So the $\boldsymbol{x}$-coordinate of $\boldsymbol{R}$ is the average of the $\boldsymbol{x}$-coordinates of the two points $\boldsymbol{P}$ and $\boldsymbol{Q}$, and the $\boldsymbol{y}$-coordinate of $\boldsymbol{R}$ is the average of the $\boldsymbol{y}$-coordinates of the two points.

The coordinates of the point $\boldsymbol{R}(\boldsymbol{u}, \boldsymbol{v})$ are $\boldsymbol{u}$ and $\boldsymbol{v}$, each of which can be put in terms of $\boldsymbol{s}$ and $\boldsymbol{t}$, since $\boldsymbol{R}$ is the midpoint between the two points $\boldsymbol{P}(\mathbf{3}, \mathbf{2})$ and $\boldsymbol{Q}(\boldsymbol{s}, \boldsymbol{t})$.

Also, the midpoint $\boldsymbol{R}$ is the point where the line $\boldsymbol{L}$ meets the line $\boldsymbol{N}$.
So the point $\boldsymbol{R}$ belongs to not only the line $\boldsymbol{L}$ but the line $\boldsymbol{N}$, too.

Thus, putting $\boldsymbol{R}$ into the equations of the lines $\boldsymbol{L}$ and $\boldsymbol{N}$ respectively, we will eventually get a system of two equations for $\boldsymbol{s}$ and $\boldsymbol{t}$. So solving the system, we can get the point $\boldsymbol{Q}$.

So to begin with, getting the midpoint $\boldsymbol{R}$ between $\boldsymbol{P}$ and $\boldsymbol{Q}$, we get $\boldsymbol{u} = \frac{3+s}{2}$, and $\boldsymbol{v} = \frac{2+t}{2}$.

Next, the line $\boldsymbol{L}$ is $\boldsymbol{y} - \mathbf{5} = \mathbf{2}(\boldsymbol{x} - \mathbf{2})$, so putting the point $\boldsymbol{R}(\boldsymbol{u}, \boldsymbol{v})$ into the line $\boldsymbol{L}$, we get

$\boldsymbol{y} - 5 = 2(\boldsymbol{x} - 2) \Rightarrow \boldsymbol{v} - 5 = 2(\boldsymbol{u} - 2) \Rightarrow \frac{2+t}{2} - 5 = 2(\frac{3+s}{2} - 2)$ because $\boldsymbol{u} = \frac{3+s}{2}$, and $\boldsymbol{v} = \frac{2+t}{2}$.

So we get $2 + \boldsymbol{t} - 10 = 4(\frac{3+s}{2} - 2) = 2\boldsymbol{s} - 2 \Rightarrow \boldsymbol{t} - 8 = 2\boldsymbol{s} - 2 \Rightarrow 2\boldsymbol{s} - \boldsymbol{t} = -6$.

Next, the line $\boldsymbol{N}$ is $\boldsymbol{y} - 2 = -\frac{1}{2}(\boldsymbol{x} - 3)$, so putting the point $\boldsymbol{R}$ into the line $\boldsymbol{N}$, we get

$\boldsymbol{y} - 2 = -\frac{1}{2}(\boldsymbol{x} - 3) \Rightarrow \boldsymbol{v} - 2 = -\frac{1}{2}(\boldsymbol{u} - 3) \Rightarrow \frac{t+2}{2} - 2 = -\frac{1}{2}(\frac{3+s}{2} - 3) \Rightarrow \boldsymbol{t} + 2 - 4 = 3 - \frac{3+s}{2}$

$\Rightarrow 2\boldsymbol{t} - 4 = 6 - (3 + \boldsymbol{s}) = 3 - \boldsymbol{s} \Rightarrow \boldsymbol{s} + 2\boldsymbol{t} = 7$.

Now, we need to solve the system of equations, $\mathbf{2}\boldsymbol{s} - \boldsymbol{t} = \mathbf{-6}$ and $\boldsymbol{s} + \mathbf{2}\boldsymbol{t} = \mathbf{7}$.

To begin with, we can get $\mathbf{2}\boldsymbol{s} - \boldsymbol{t} = \mathbf{-6} \Rightarrow \mathbf{4}\boldsymbol{s} - \mathbf{2}\boldsymbol{t} = \mathbf{-12}$. And we have $\boldsymbol{s} + \mathbf{2}\boldsymbol{t} = \mathbf{7}$, too.

So next, $(\mathbf{4}\boldsymbol{s} - \mathbf{2}\boldsymbol{t}) + (\boldsymbol{s} + \mathbf{2}\boldsymbol{t}) = \mathbf{-12} + \mathbf{7} \Rightarrow \mathbf{5}\boldsymbol{s} = \mathbf{-5} \Rightarrow \boldsymbol{s} = \mathbf{-1}$.

Next, $\boldsymbol{s} + \mathbf{2}\boldsymbol{t} = \mathbf{7} \Rightarrow \mathbf{-1} + \mathbf{2}\boldsymbol{t} = \mathbf{7} \Rightarrow \mathbf{2}\boldsymbol{t} = \mathbf{8} \Rightarrow \boldsymbol{t} = \mathbf{4}$. Therefore, ***Q*** is at **(-1, 4)**.

Let's now, try another way. It is about the same method as above, though.

We have

P is the point **(3, 2)**.
L is the line of slope 2 passing through the point ***P***.
Q*(*s*, *t*)** is the point symmetric to the point ***P about the line ***L***.
N is the line perpendicular to the line ***L***, and has two points ***P*** and ***Q***.
R is the midpoint between the two points ***P*** and ***Q***.

Thus,

the distance from the point ***P*** to the line ***L*** is the same as the distance from the point ***Q*** to the line ***L***, so the line ***L*** is the axis of symmetry.
The two points ***P*** and ***Q*** are in the line ***N*** perpendicular to the axis of symmetry ***L***, and the midpoint ***R*** is in the line ***N***, too.

Then, the problem solving procedure can go as follows.

First, find the line ***L***, then the line ***N***.

Next, find the midpoint ***R***, where the line ***L*** meets the line ***N***.

Next, find the distance from the point ***P*** to the midpoint ***R***. Call the distance ***D***.

Next, setting the distance from ***P*** to ***R*** equal to the distance from ***R*** to ***Q***, we can get an equation, which is for ***s*** and ***t***.

Next, putting ***Q*** into the line ***N***, we get another equation, which is for ***s*** and ***t***, too.

Finally, solve the system of the two equations for ***s*** and ***t***.

So to begin with, the line ***L*** is $\boldsymbol{y} - \mathbf{5} = \mathbf{2}(\boldsymbol{x} - \mathbf{2})$, and the line ***N*** is $\boldsymbol{y} - \mathbf{2} = -\frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{x} - \mathbf{3})$.

Let's next, find the midpoint ***R*** where the line ***L*** meets the line ***N***.

We can put the two equations above the way below, too.

$\boldsymbol{L}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{2}(\boldsymbol{x} - \mathbf{2}) - (\boldsymbol{y} - \mathbf{5}) = \mathbf{0}$, and $\boldsymbol{N}(\boldsymbol{x}, \boldsymbol{y}) = -\frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{x} - \mathbf{3}) - (\boldsymbol{y} - \mathbf{2}) = \mathbf{0}$.

Then, we need to solve the system of equations as follows. $\boldsymbol{L}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{0}$, and $\boldsymbol{N}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{0}$.

$\boldsymbol{L(x, y) = 0 \Rightarrow 2(x - 2) - (y - 5) = 2x - 4 - y + 5 = 2x - y + 1 = 0 \Rightarrow y = 2x + 1}$.

$\boldsymbol{N(x, y) = 0 \Rightarrow -\frac{1}{2}(x-3)-(y-2) = -\frac{x}{2}+\frac{3}{2}-y+2 = -\frac{x}{2}+\frac{7}{2}-y = 0 \Rightarrow y = -\frac{x}{2}+\frac{7}{2}}$.

So we get $\boldsymbol{2x+1 = -\frac{x}{2}+\frac{7}{2} \Rightarrow 4x+2 = -x+7 \Rightarrow 5x = 5 \Rightarrow x = 1 \Rightarrow y = 2x+1 = 3}$.

So the midpoint $\boldsymbol{R}$ is at **(1, 3)**. So the next is the distance $\boldsymbol{D}$ between $\boldsymbol{P}$**(3, 2)** and $\boldsymbol{R}$**(1, 3)**.

Finding it, we get $\boldsymbol{D^2 = (3 - 1)^2 + (2 - 3)^2 = 4 + 1 = 5 \Rightarrow D^2 = 5}$. Just leave it at that.

Next, setting up an equation where $\boldsymbol{D}$ is the same as the distance between $\boldsymbol{R}$**(1, 3)** and the symmetric point $\boldsymbol{Q(s, t)}$, we get one of the two equations for $\boldsymbol{s}$ and $\boldsymbol{t}$.

That is, setting $\boldsymbol{(s - 1)^2 + (t - 3)^2 = D^2 = 5}$, we get $\boldsymbol{5 = (s - 1)^2 + (t - 3)^2}$.

Next, putting the point $\boldsymbol{Q(s, t)}$ into the line $\boldsymbol{N}$, we get the other equation.

The line $\boldsymbol{N}$ is $\boldsymbol{y = -\frac{x}{2}+\frac{7}{2}}$, so we get $\boldsymbol{t = -\frac{s}{2}+\frac{7}{2}}$.

Now, we want to solve the system as follows. $\boldsymbol{5 = (s - 1)^2 + (t - 3)^2}$ and $\boldsymbol{t = -\frac{s}{2}+\frac{7}{2}}$.

Eliminating $\boldsymbol{t}$, we get $\boldsymbol{5 = (s-1)^2 + (\frac{-s}{2}+\frac{7}{2}-3)^2 = (s-1)^2 + (\frac{-s+7-6}{2})^2 = (s-1)^2 + (\frac{-s+1}{2})^2}$

$\boldsymbol{= (s-1)^2 + (\frac{s-1}{2})^2 = (s-1)^2 + \frac{(s-1)^2}{4} = \frac{5(s-1)^2}{4} \Rightarrow 5 = \frac{5(s-1)^2}{4} \Rightarrow (s-1)^2 = 4}$.

How come we get a quadratic equation having two different roots, though?

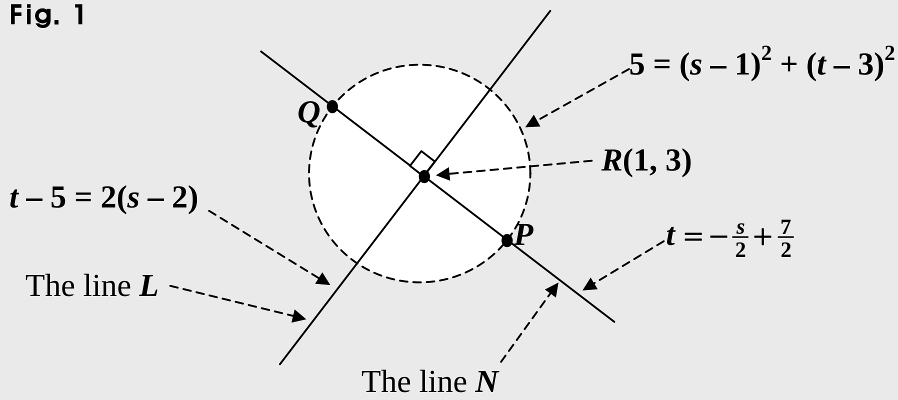
One of the two values of $\boldsymbol{s}$ is the $\boldsymbol{x}$-coordinate of the point $\boldsymbol{P}$, and the other is that of the point $\boldsymbol{Q}$. How come?

The system of the two equations is composed of $\boldsymbol{5 = (s - 1)^2 + (t - 3)^2}$ and $\boldsymbol{t = -\frac{s}{2}+\frac{7}{2}}$.

The first of the two is an equation of a circle of radius $\boldsymbol{\sqrt{5}}$ centered at **(1, 3)**, which is the midpoint $\boldsymbol{R}$.
The second is an equation of a line of slope $\boldsymbol{-\frac{1}{2}}$ passing through $\boldsymbol{P}$**(2, 3)**, and the line is in fact, the line $\boldsymbol{N}$.

Thus, the solution to the system is composed of coordinates of the two points $\boldsymbol{P}$ and $\boldsymbol{Q}$ symmetric to each other about the line $\boldsymbol{L}$. Refer to the figure below.

Fig. 1



Note that the lines and circle are in the ***s-t*** plane, now. That's simply because we use ***s*** and ***t*** as the variables in the equations. If we use ***x*** and ***y*** as the variables, the lines and circle will be in the ***x-y*** plane, of course. Let's now, get going. We are now at $(s - 1)^2 = 4$.

So we get $(s - 1)^2 = 4 \Rightarrow (s - 1 = 2$ or $-2) \Rightarrow (s = 3$ or $-1)$.

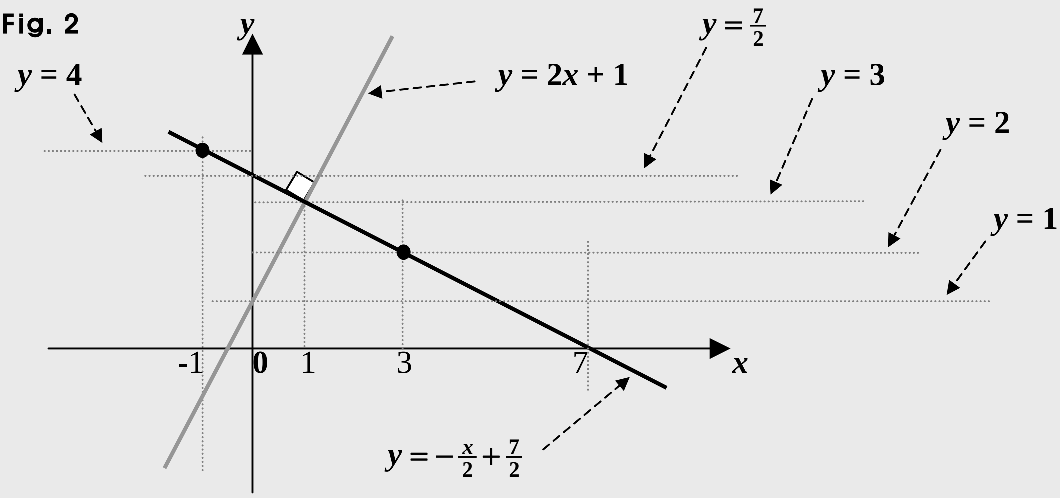
Next, we need to find ***t***. We have $t = -\frac{s}{2} + \frac{7}{2}$. So $s = 3 \Rightarrow t = -\frac{s}{2} + \frac{7}{2} = -\frac{3}{2} + \frac{7}{2} = \frac{4}{2} = 2$.

Thus, we get $(s, t) = (3, 2)$, which is the point ***P*** given already.

Next, when $s = -1$, we get $t = -\frac{s}{2} + \frac{7}{2} = -\frac{-1}{2} + \frac{7}{2} = \frac{8}{2} = 4$. So the point ***Q*** is at **(-1, 4)**.

Now, let's have a look at in the graph below the two points **(3, 2)** and **(-1, 4)** symmetric to each other about the line $2x - y + 1 = 0$, which is equivalent to $y = 2x + 1$, which is the line ***L***.

Fig. 2



In short:

The First Method

Suppose first, $\boldsymbol{P}$ is **(3, 2)**, $\boldsymbol{L}$ is the line of slope 2 passing through $\boldsymbol{P}$, $\boldsymbol{Q(s, t)}$ is the point symmetric to $\boldsymbol{P}$ about the line $\boldsymbol{L}$. $\boldsymbol{N}$ is the line perpendicular to the line $\boldsymbol{L}$, and has two points $\boldsymbol{P}$ and $\boldsymbol{Q}$, and $\boldsymbol{R(u, v)}$ is the midpoint between the two points $\boldsymbol{P}$ and $\boldsymbol{Q}$.

Then, the line $\boldsymbol{L}$ is the axis of symmetry, the midpoint $\boldsymbol{R}$ is the point where the two lines $\boldsymbol{L}$ and $\boldsymbol{N}$ meet each other as well as the midpoint between the two points $\boldsymbol{P}$ and $\boldsymbol{Q}$.

So first, since the line $\boldsymbol{L}$ has **(2, 5)** and the slope is 2, the line $\boldsymbol{L}$ is $\boldsymbol{y - 5 = 2(x - 2)}$.

Next, in the line $\boldsymbol{L}$, the slope is 2.
So if $\boldsymbol{n}$ is the slope of the line $\boldsymbol{N}$, we get $\boldsymbol{2n = -1 \Rightarrow n = -\frac{1}{2}}$.
Thus, the line $\boldsymbol{N}$ is $\boldsymbol{y - 2 = -\frac{1}{2}(x-3)}$ since the point $\boldsymbol{P(3, 2)}$ is in the line $\boldsymbol{N}$.

Next, $\boldsymbol{R(u, v)}$ is the midpoint between the two points $\boldsymbol{P(3, 2)}$ and $\boldsymbol{Q(s, t)}$.
So we get $\boldsymbol{u = \frac{3+s}{2}}$, and $\boldsymbol{v = \frac{2+t}{2}}$.

Next, the line $\boldsymbol{L}$ is $\boldsymbol{y - 3 = 2(x - 1)}$, so putting the midpoint $\boldsymbol{R}$ into the line $\boldsymbol{L}$, we get
$\boldsymbol{y - 5 = 2(x-2) \Rightarrow v - 5 = 2(u-2) \Rightarrow \frac{2+t}{2} - 5 = 2(\frac{3+s}{2} - 2)}$
$\boldsymbol{\Rightarrow 2 + t - 10 = 4(\frac{3+s}{2} - 2) = 2s - 2 \Rightarrow t - 8 = 2s - 2 \Rightarrow 2s - t = -6}$.

Next, the line $\boldsymbol{N}$ is $\boldsymbol{y - 2 = -\frac{1}{2}(x-3)}$, so putting the midpoint $\boldsymbol{R}$ into the line $\boldsymbol{N}$, we get
$\boldsymbol{y - 2 = -\frac{1}{2}(x-3) \Rightarrow v - 2 = -\frac{1}{2}(u-3) \Rightarrow \frac{t+2}{2} - 2 = -\frac{1}{2}(\frac{3+s}{2} - 3) \Rightarrow t + 2 - 4 = 3 - \frac{3+s}{2}}$
$\boldsymbol{\Rightarrow 2t - 4 = 6 - (3+s) = 3 - s \Rightarrow s + 2t = 7}$.

Now, we need to solve the system of equations, $\boldsymbol{2s - t = -6}$ and $\boldsymbol{s + 2t = 7}$.
To begin with, we can get $\boldsymbol{2s - t = -6 \Rightarrow 4s - 2t = -12.}$ Thus, we get
$\boldsymbol{(4s - 2t) + (s + 2t) = -12 + 7 \Rightarrow 5s = -5 \Rightarrow s = -1 \Rightarrow s + 2t = -1 + 2t = 7 \Rightarrow 2t = 8 \Rightarrow t = 4}$.
Therefore, $\boldsymbol{Q}$ is at **(-1, 4)**.

The Second Method

Suppose that

P is the point **(3, 2)**.
L is the line of slope 2 passing through the point ***P***.
Q*(*s*, *t*)** is the point symmetric to the point ***P about the line ***L***.
N is the line perpendicular to the line ***L***, and has two points ***P*** and ***Q***.
R*(*u*, *v*)** is the midpoint between the two points ***P and ***Q***.

Then,
the line ***L*** is the axis of symmetry.
The midpoint ***R*** is the point where the two lines ***L*** and ***N*** meet each other as well as the midpoint between the two points ***P*** and ***Q***.

So first, the line ***L*** has **(2, 5)**, and the slope is 2, so the line ***L*** is $y - 5 = 2(x - 2)$.
Next, if n is the slope of the line ***N***, $2n = -1 \Rightarrow n = -\frac{1}{2}$ because ***L*** is perpendicular to ***N***.
Thus, the line ***N*** is $y - 2 = -\frac{1}{2}(x - 3)$ since **(3, 2)** is in the line ***N***.

Next, the midpoint ***R*** is the point where the line ***L*** meets the line ***N***.
So finding ***R***, we want to solve the system of equations of two lines ***L*** and ***N***.
Subtracting the equation of ***N*** from the equation of ***L***, we can eliminate ***y***, and can get the ***x***-coordinate at the point ***R***. So to begin with, we get

$$(y-5)-(y-2)=\{2(x-2)\}-\{-\tfrac{1}{2}(x-3)\}=2x-4+\tfrac{x}{2}-\tfrac{3}{2}.$$

Thus, we get $-3 = 2x - 4 + \frac{x}{2} - \frac{3}{2} \Rightarrow 4 - 3 + \frac{3}{2} = 2x + \frac{x}{2} \Rightarrow \frac{5}{2} = \frac{5x}{2} \Rightarrow x = 1$.

Next, finding the ***y***-coordinate, we get $y - 5 = 2(x-2) = 2(1-2) = -2 \Rightarrow y = 3$.
Thus, the midpoint ***R*** is at **(1, 3)**.

Next, the distance from ***P*** to ***R*** equals the distance from ***R*** to ***Q*(*s*, *t*)**.
So we get $(3 - 1)^2 + (2 - 3)^2 = 4 + 1 = 5 = (s - 1)^2 + (t - 3)^2$.
Also, putting ***Q*(*s*, *t*)** into ***N***, we get $t = -\frac{s}{2} + \frac{7}{2} = \frac{7-s}{2} \Rightarrow t = \frac{7-s}{2}$.

Next, we can get $t - 3 = \frac{7-s}{2} - 3 = \frac{1-s}{2}$, so we get $5 = (s-1)^2 + (t-3)^2 = (s-1)^2 + (\frac{1-s}{2})^2$
$= (s-1)^2 + \frac{(s-1)^2}{4} = \frac{5(s-1)^2}{4} \Rightarrow 5 = \frac{5(s-1)^2}{4} \Rightarrow (s-1)^2 = 4 \Rightarrow (s - 1 = 2$ or $-2) \Rightarrow (s = 3$ or $-1)$.
Then first, we get $s = 3 \Rightarrow t = \frac{7-s}{2} = \frac{4}{2} = 2 \Rightarrow (s, t) = (3, 2)$, which is ***P*** given already.
Next, we get $s = -1 \Rightarrow t = \frac{7-s}{2} = \frac{8}{2} = 4$. Thus, the point ***Q*** is at **(-1, 4)**.