Examples 8 in Lines

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0. Assuming a line includes a point **(2, 5)**, and the slope is 2, find a point **(*s*, *t*)** from which the distance is 5 to the line.

1. Find a point **(*s*, *t*)** that is the same distance away from a point **(-2, 1)** as away from a point **(3, 2)**.

Suggestions or Solutions
To the Problem in the Example 0

Assuming a line includes a point (2, 5), and the slope is 2, find a point (*s*, *t*) from which the distance is 5 to the line.

Suppose that L is the given line, that R_1 and R_2 are two points in the line L, and also, are in a circle centered at **(2, 5)** with a radius of 5, and that N is a line perpendicular to the line L, and passes through R_1, R_2, and **(2, 5)**.

Suppose also, P and Q are two lines parallel to the line L, the line P includes the point R_1, and the line Q passes through the point R_2.

Then, each of the lines P and Q is 5 away from the line L, so every point in the two lines is 5 away from the line L, and thus, can be the point (s, t).

If n is the slope of N, we get $2n = -1$, for the slope of L is 2, and L is perpendicular to N. So we get $n = -\frac{1}{2}$. And the line N is $y - 5 = -\frac{1}{2}(x - 2)$, since N has the point **(2, 5)**.

And the equation of the circle is $(x - 2)^2 + (y - 5)^2 = 5^2$.

Next, we want to find the two points R_1 and R_2, where the line N meets the circle.
To begin with, eliminating y to find the x-coordinates at the points, we get

$$y - 5 = -\tfrac{1}{2}(x - 2) = \tfrac{2 - x}{2} \Rightarrow 5^2 = (x-2)^2 + (y-5)^2 = (x-2)^2 + (\tfrac{x-2}{2})^2 = \tfrac{5(x-2)^2}{4}$$

$\Rightarrow 5 = \frac{(x-2)^2}{4} \Rightarrow (x-2)^2 = 20 \Rightarrow x = 2 \pm 2\sqrt{5}$. So next, getting the y-coordinates, we get

$$x = 2 + 2\sqrt{5} \Rightarrow y - 5 = -\tfrac{x-2}{2} = -\tfrac{2+2\sqrt{5}-2}{2} = -\sqrt{5} \Rightarrow y = 5 - \sqrt{5}.$$

$$x = 2 - 2\sqrt{5} \Rightarrow y - 5 = -\tfrac{x-2}{2} = -\tfrac{2-2\sqrt{5}-2}{2} = \sqrt{5} \Rightarrow y = 5 + \sqrt{5}.$$

Now, setting $R_1 = (s_1, t_1) = (2 + 2\sqrt{5}, 5 - \sqrt{5})$, and $R_2 = (s_2, t_2) = (2 - 2\sqrt{5}, 5 + \sqrt{5})$, and assuming the point $R_1(s_1, t_1)$ is in the line P, and the point $R_2(s_2, t_2)$ is in the line Q, we can say that the line P is $y - t_1 = 2(x - s_1)$, the line Q is $y - t_2 = 2(x - s_2)$, and every point in the two lines P and Q is 5 away from the line L.

If not quite sure of the idea behind the processes above, follow the steps below.

It's not true that there is only one point that is a particular distance away from a line. There can be infinitely many. All such points constitute particular lines, and each of the points can be the point ***(s, t)***. What particular lines, then?

Suppose now, that ***L*** is the line given in this problem.
Then, the particular lines are parallel to the line ***L***, and are 5 away from ***L***.
That is, the distance from the line ***L*** to each of the particular lines is 5.
Thus, we can say that each of all the points in the particular lines is 5 away from ***L***.
How many particular lines are there, then?

There are two, each of which is on each side of the line ***L***, because ***L*** is in the ***x-y*** *plane*.
What if the line ***L*** is in a **3-D** *space* as the ***x-y-z*** space?

Then, all such points will form a circular cylinder where the radius is 5, and the line ***L*** is parallel to the cylinder, and in fact, passes through the center of the cylinder.
So each of all such points can be the point ***(s, t)***.

Suppose now, that ***P*** and ***Q*** are those two lines parallel to the line ***L*** in the ***x-y*** plane. Then, the line ***P*** has the same slope as the line ***L*** has, and the same is true for the line ***Q***, too. The slope of the line ***L*** is given, and is 2. So the slope in the two lines ***P*** and ***Q*** are 2, also. What else then, do we need finding the line ***P***?

We need a point in the line ***P***, of course. How can we get the point, though?

Suppose ***T*** is the point we need. Then, the point ***T*** is 5 away from the line ***L***. We know one point in the line ***L***, and the point is **(2, 5)**. So the point ***T*** is in a circle of radius 5 centered at **(2, 5)**. Where in the circle is the point ***T***, then?

The point ***T*** is also, in a line passing through the point **(2, 5)** and perpendicular to the line ***L***. In fact, there are two points that can be the point ***T***. One is in the line ***P***, and the other is in the line ***Q***. How come?

The perpendicular line includes the diameter of the circle, which is 10, and thus, each of the two lines ***P*** and ***Q*** passes through each of the two endpoints of the diameter.

Suppose now, that ***N*** is the line perpendicular to the line ***L***.
Then, the circle meets the line ***N*** at two points, each of which can be the point ***T***.

So we want to solve a system of two equations, one is the equation of the circle, and the other indicates the line N. What do we mean by an equation of a circle, though?

An equation of a circle can give us the coordinates of every point in a circle. A circle is a set of every point that is a particular distance away from the center of the circle, and the particular distance is the radius of the circle. The details are covered in **CONICS 3**.

A point (x, y) can indicate an arbitrary point in a circle in the x-y plane.
So the equation of the circle in this problem is the math expression stating that 5 is the distance from the center $(2, 5)$ to the arbitrary point (x, y).
So simply using the distance formula, we can get the equation of the circle, and the equation of the circle is $(x - 2)^2 + (y - 5)^2 = 5^2$.
How then, do we find the two points where the circle meets the line N?

We need to solve a system of two equations indicating the circle and N respectively.
Then, the solution says the coordinates of the two points. One is in the line P, and the other is in the line Q. Besides, P and Q are parallel to the line L where the slope is 2.
So we can readily find the two lines P and Q using the form, $y - y_1 = a(x - x_1)$ where a is the slope and (x_1, y_1) is a point in the line.
Let's now, get started with the line N, which is perpendicular to the line L.

To begin with, if n is the slope of N, we get $2n = -1 \Rightarrow n = -\frac{1}{2}$ because the slope of L is 2, and also, the product of the slopes of two lines perpendicular to each other is -1.
And the line N includes the center $(2, 5)$, so the line N is $y - 5 = -\frac{1}{2}(x - 2)$.

Next, finding the point where N meets the circle, we need to solve a system where $5^2 = (x - 2)^2 + (y - 5)^2$, and $y - 5 = -\frac{1}{2}(x - 2)$.

Let's first, eliminate y to find the x-coordinate at the point where N meets the circle.
To begin with, since we have $y - 5 = -\frac{1}{2}(x - 2)$, we can get

$$5^2 = (x-2)^2 + (y-5)^2 = (x-2)^2 + \left(\tfrac{x-2}{2}\right)^2 = \tfrac{5(x-2)^2}{4} \Rightarrow 5^2 = \tfrac{5(x-2)^2}{4} \Rightarrow 5 = \tfrac{(x-2)^2}{4}$$
$$\Rightarrow (x-2)^2 = 20 \Rightarrow x - 2 = \pm\sqrt{20} = \pm 2\sqrt{5} \Rightarrow x = 2 \pm 2\sqrt{5}.$$

So we get two values for the x-coordinate, which means the line N meets the circle at two points. Next, getting the y-coordinates, we get

$x = 2 + 2\sqrt{5} \Rightarrow y - 5 = -\frac{x-2}{2} = -\frac{2+2\sqrt{5}-2}{2} = -\sqrt{5} \Rightarrow y = 5 - \sqrt{5}.$

$x = 2 - 2\sqrt{5} \Rightarrow y - 5 = -\frac{x-2}{2} = -\frac{2-2\sqrt{5}-2}{2} = \sqrt{5} \Rightarrow y = 5 + \sqrt{5}.$

Thus, the circle meets the line N at $(2+2\sqrt{5}, 5-\sqrt{5})$ and $(2-2\sqrt{5}, 5+\sqrt{5})$.

Suppose now, that $s_1 = 2+2\sqrt{5}$, $t_1 = 5-\sqrt{5}$, $s_2 = 2-2\sqrt{5}$, and $t_2 = 5+\sqrt{5}$.

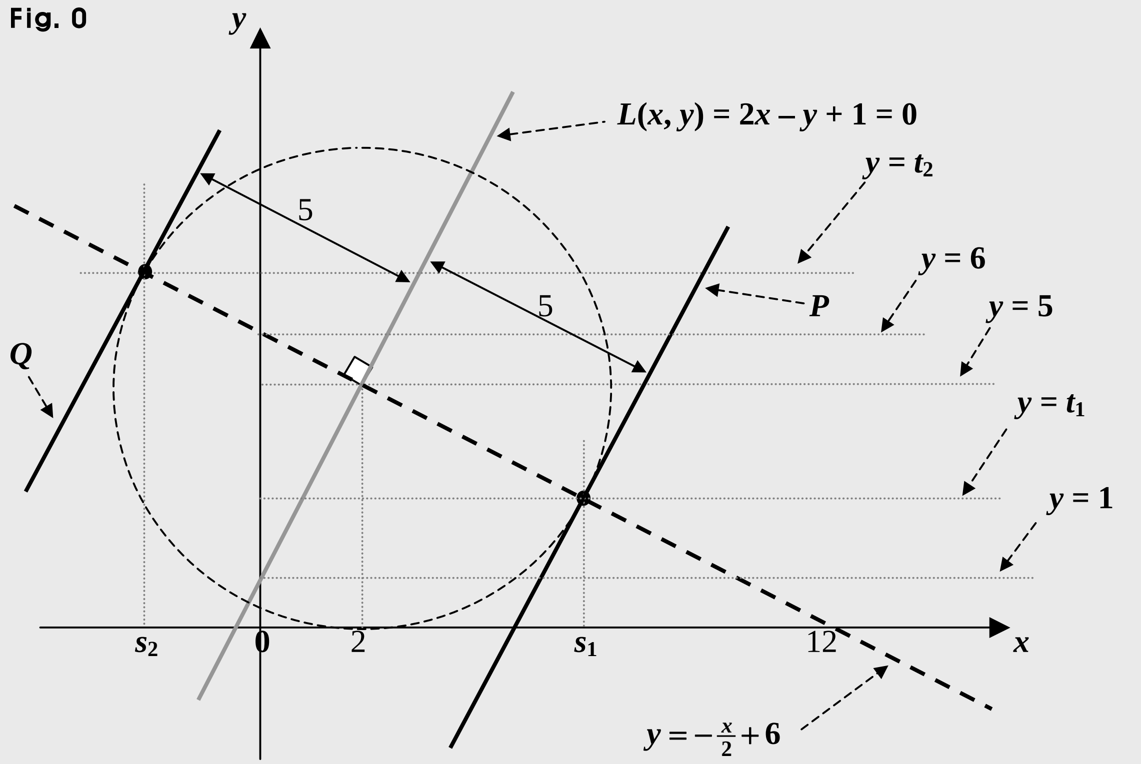
Suppose also, that the point (s_1, t_1) is in the line P, and the point (s_2, t_2) is in the line Q. Then, the line P is $y - t_1 = 2(x - s_1)$, and the line Q is $y - t_2 = 2(x - s_2)$. And every point in the two lines P and Q is 5 away from the line L. How come though, the slope is 2?

That's because the two lines P and Q are parallel to the line L where the slope is 2.

Let's now, put in a graph all those points and lines.

$s_1 = 2+2\sqrt{5} \approx 6.47$, $t_1 = 5-\sqrt{5} \approx 2.76$, $s_2 = 2-2\sqrt{5} \approx -2.47$, and $t_2 = 5+\sqrt{5} \approx 7.24$.

Fig. 0



The two lines P and Q are in black, and are 5 away from the line L, which is in gray.

Suggestions or Solutions
To the Problem in the Example 1

Find a point (*s*, *t*) that is the same distance away from a point (-2, 1) as away from a point (3, 2).

Suppose ***M*** is the midpoint between the two points **(-2, 1)** and **(3, 2)**, ***L*** is the line passing through the two points, and ***N*** is the line passing through ***M*** and perpendicular to ***L***.

Then, each of all the points in the ***N*** can be the point **(*s*, *t*)**.

To begin with, the midpoint ***M*** is at $\left(\frac{-2+3}{2}, \frac{1+2}{2}\right) = \left(\frac{1}{2}, \frac{3}{2}\right)$.

Next, the slope of the line ***L*** is $\frac{\Delta y}{\Delta x} = \frac{2-1}{\{3-(-2)\}} = \frac{1}{5}$, so the slope of the line ***N*** is –5, because ***N*** is perpendicular to the line ***L***.

Thus, ***N*** is $y - \frac{3}{2} = -5\left(x - \frac{1}{2}\right)$, and each and every point in the line ***N*** can be the point **(*s*, *t*)**.

If not quite sure of the idea behind the processes above, follow the steps below.

Of course, we can say that the point **(*s*, *t*)** is the midpoint between the two given points. Then, is the midpoint only the only point that can be the solution?

No, it is not the case. Infinitely many points can be the point **(*s*, *t*)**. Using a particular point, along with the two given points, we can construct an isosceles triangle where the base is the line segment connecting the two points given. Thus, the particular point is the same distance away from the two points given, so it can be the point **(*s*, *t*)**, too. Then, what does the solution have to be?   How many such triangles as above can we make?

Infinitely many, of course! Every point in a line perpendicular to the line connecting the two points given and passing through the midpoint between the two given points can be the point **(*s*, *t*)**, and thus, the line perpendicular is the solution if you will.

How then, can we find the perpendicular line?

What is the perpendicular line perpendicular to?

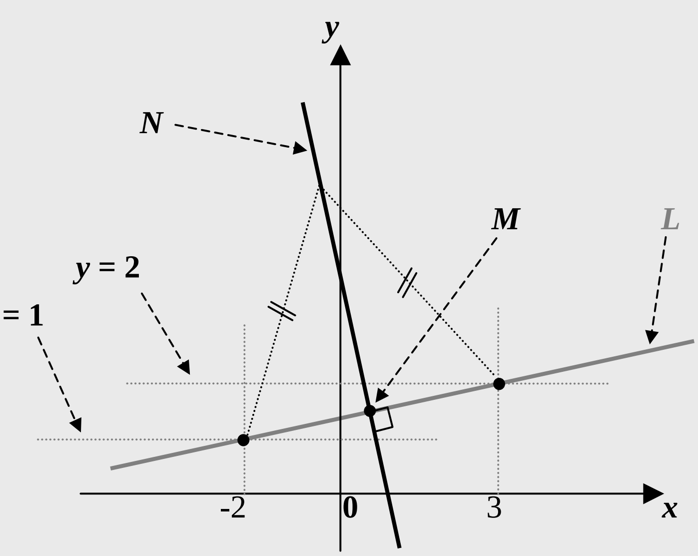
It is the line connecting the two points given, of course. We don't have to however, find the line connecting the two. We only need to find the slope of such a line. We can find it using the fact that -1 is the product of the slopes of two lines perpendicular to each other. Of course, we want to find the midpoint between the two given points, too.

Then, we can get the perpendicular line we want using the quasi standard, which has the form of $\boldsymbol{y - t = a(x - u)}$, where $\boldsymbol{a}$ is the slope, and $\boldsymbol{(s, t)}$ is a point in the line.

Suppose now, $\boldsymbol{M}$ is the midpoint between the two points given, and $\boldsymbol{L}$ is the line passing through the two given points.
Suppose also, $\boldsymbol{N}$ is the line passing through the midpoint $\boldsymbol{M}$ and perpendicular to the line $\boldsymbol{L}$.
Then, each of all the points in the line $\boldsymbol{N}$ is the same distance away from the two points given. Therefore, each of all the points constituting the line $\boldsymbol{N}$ can be the point $\boldsymbol{(s, t)}$.

Let's now, put in a graph the two lines $\boldsymbol{N}$ and $\boldsymbol{L}$, along with the two points given and $\boldsymbol{M}$.

Fig. 0



Let's now, begin with the midpoint $\boldsymbol{M}$, which is the midpoint between **(-2, 1)** and **(3, 2)**.

Then first, we can readily see that $\boldsymbol{M}$ is at $\left(\frac{-2+3}{2}, \frac{1+2}{2}\right) = \left(\frac{1}{2}, \frac{3}{2}\right)$.

Next, the line ***N*** includes the midpoint ***M***, so getting the slope of ***N***, we can readily get the line ***N*** using the form, $\boldsymbol{y - y_1 = a(x - x_1)}$ where ***a*** is the slope and $\boldsymbol{(x_1, y_1)}$ is a point in the line. So let's move on to the slope of the line ***N***.

To begin with, the line ***N*** is perpendicular to the line ***L***, and the product of the slopes of two lines perpendicular to each other is -1.

Thus, getting the slope of the line ***L***, we can readily see the slope of the line ***N***. Next, the line ***L*** has the two points **(-2, 1)** and **(3, 2)**.

Thus, the slope of the line ***L*** is $\frac{\Delta y}{\Delta x} = \frac{2-1}{\{3-(-2)\}} = \frac{1}{5}$, so the slope of the line ***N*** is -5.

Thus, ***N*** is $y - \frac{3}{2} = -5(x - \frac{1}{2})$, and each and every point in the line ***N*** can be the point **(*s*, *t*)**.

In short:

Suppose ***M*** is the midpoint between the two points **(-2, 1)** and **(3, 2)**, ***L*** is the line passing through the two points, and ***N*** is the line passing through ***M*** and perpendicular to ***L***.

Then, each of all the points in the ***N*** can be the point **(*s*, *t*)**.

To begin with, the midpoint ***M*** is at $(\frac{-2+3}{2}, \frac{1+2}{2}) = (\frac{1}{2}, \frac{3}{2})$.

Next, the slope of the line ***L*** is $\frac{\Delta y}{\Delta x} = \frac{2-1}{\{3-(-2)\}} = \frac{1}{5}$, so the slope of the line ***N*** is –5, because ***N*** is perpendicular to the line ***L***.

Thus, ***N*** is $y - \frac{3}{2} = -5(x - \frac{1}{2})$, and each and every point in the line ***N*** can be the point **(*s*, *t*)**.