Examples 9 in Lines

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0. Suppose that

Six points $\boldsymbol{P_1(1, 1)}$, $\boldsymbol{P_2(1, 3)}$, $\boldsymbol{P_3(2, 1)}$, $\boldsymbol{P_4(4, 1)}$, $\boldsymbol{P_5(s, t)}$, and $\boldsymbol{P_6(u, v)}$ are in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane.
The two points $\boldsymbol{P_5}$ and $\boldsymbol{P_6}$ are above $\boldsymbol{P_3}$ and $\boldsymbol{P_4}$, and also, on the right of $\boldsymbol{P_2}$.
The sum of the areas of two triangles $\boldsymbol{P_5P_1P_2}$ and $\boldsymbol{P_5P_3P_4}$ is 3.

0.0. Then, specify the point $\boldsymbol{P_5}$.

0.1. In addition to the supposition above, suppose also, that

$\boldsymbol{D_1}$ is the distance between the two points $\boldsymbol{P_6}$ and $\boldsymbol{P_4}$.
$\boldsymbol{D_2}$ is the distance between the two points $\boldsymbol{P_6}$ and $\boldsymbol{P_2}$.
$\boldsymbol{D_1^2 - D_2^2 = 5}$.

Then, specify the point $\boldsymbol{P_6}$.

1. Suppose that the distance between two points $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane is 5.
Suppose also, that
$\boldsymbol{D_1}$ is the distance from the point $\boldsymbol{P_1}$ to another point $\boldsymbol{P_3}$.
$\boldsymbol{D_2}$ is the distance between the two points $\boldsymbol{P_2}$ and $\boldsymbol{P_3}$.
$\boldsymbol{D_1^2 - D_2^2 = 10}$.

Then, specify the point $\boldsymbol{P_3}$.

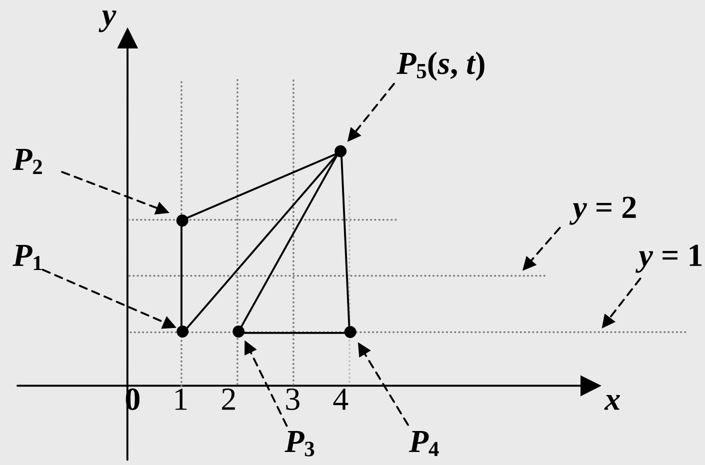
Suggestions or Solutions
To the Problem 0 in the Example 0

Suppose that
Six points $P_1(1, 1)$, $P_2(1, 3)$, $P_3(2, 1)$, $P_4(4, 1)$, $P_5(s, t)$, and $P_6(u, v)$ are in the x-y plane.
The two points P_5 and P_6 are above P_3 and P_4, and also, on the right of P_2.
The sum of the areas of two triangles $P_5P_1P_2$ and $P_5P_3P_4$ is 3.

Then, specify the point P_5.

To begin with, putting the given points in the ***x-y*** plane, we can see better the solution's whereabouts.

Fig. 0



Next, examining the graph above, we can see the problem gets much easier moving all the points so that P_1 can sit at the origin. That is, relocating all the points in the graph so that the point P_1 is positioned at the origin, we can make the problem much smaller.

Provided that all the points can maintain their relative positions to each other's, no matter how much they may make their movements, the problem itself remains the same.

Moving the coordinate axes instead, we can get the same effect, too. In such a case, also, we want to move of course, both axes in the same manner.

That is, applying the same translation to each axis, and making sure that the point P_2 is positioned at the origin, we can get the problem much smaller.

Translating the axes however, we want to make sure the direction is opposite of the one for the points, because the axes will get moved the way opposite of the one the points get moved if we move the points instead. Anyway, the movement of the points is a translation by $\Delta x = -1$ and $\Delta y = -1$.
What in this case though, do we mean by those Δx and Δy?

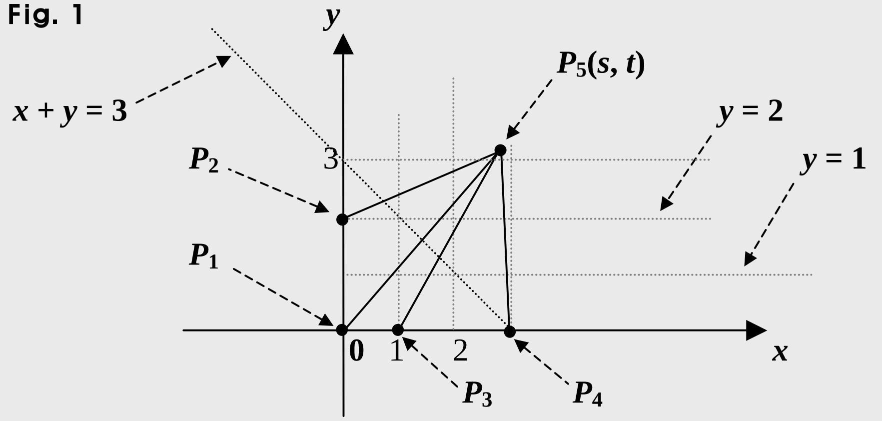
If $\Delta x = a$ where a is constant, we move the point in the amount of a in the direction of the x-axis. In short, if $\Delta x = a$, we move the point by a along the x-axis.
So if $\Delta x = 1$, we move the point by 1 in the direction of the x-axis. That is, we move it by 1 to the right. What if however, $\Delta x = -1$?

Then, we move the point by -1 in the direction of the x-axis. That is, we move it by 1 to the left. By the same token, if $\Delta y = 1$, we move it upward by 1, and if $\Delta y = -1$, we move it downward by 1.

After getting the solution though, we don't want to forget to move the solution to the proper location, because the solution after the translation by $\Delta x = -1$ and $\Delta y = -1$ is the one not to the problem original but to the problem translated by $\Delta x = -1$ and $\Delta y = -1$.

So after getting the solution, we want to translate it back by $\Delta x = 1$ and $\Delta y = 1$.
Now, after the translation of all the points by $\Delta x = -1$ and $\Delta y = -1$, the graph above will get changed to the one shown below.

Fig. 1



Let's put the sum of the areas of the triangles in terms of the coordinates at the point P_5.

The area of the triangle $P_5P_1P_2 = \frac{1}{2}\cdot 2\cdot s = s$, since s is the height of the triangle.

The area of the triangle $P_5P_3P_4 = \frac{1}{2}\cdot 2\cdot t = t$, for t is the height of the triangle.

So since the sum of the two areas is 3, we can put it this way, too: $s + t = 3$.

It's not true though, that there is only one point that can be P_5, which is (s, t). In fact, the choices for s and t are unlimited. More precisely, every point in a line can be the point $P_5(s, t)$, and the line is $x + y = 3$. In other words, all points in the line $x + y = 3$ can be P_5.

So P_5 is has to be in the line $x + y = 3$. However, it's not quite the case either. If P_5 is in the line, but is on the left of P_2, or below P_4, the sum of the areas is larger than 3. So the solution is not the entire line but a line segment in the line.

The problem says in fact, "The point P_5 is above P_3 and P_4, and also, on the right of P_2." Now, setting $y = 0$ in $x + y = 3$, we can readily see that the x-intercept is 3.

So every point in the line segment $x + y = 3$ for $0 < x < 3$ can be P_5.

Now that we have found all the points that can be the point $P_5(s, t)$, we want to translate them back to the proper position. So we want to move the line by $\Delta x = 1$ and $\Delta y = 1$, for the translation was done by $\Delta x = -1$ and $\Delta y = -1$.

The points are now, in a line segment $x + y = 3$ for $0 < x < 3$.
So the solution to this problem is the set of all the points in a line segment as follows.
$(x - 1) + (y - 1) = 3$ for $1 < x < 4$.

That is, translating a line segment $x + y = 3$ for $0 < x < 3$ by 1 along the x-axis, and by 1 along the y-axis, we get a new line segment $x - 1 + y - 1 = 3$ for $1 < x < 4$, and every point in the new line segment is the solution. Such a translation is called a parallel transformation, on which the details are covered in the book **Function Transformations**.
Let's this time, translate the line $x + y = 3$ in a straight manner.
First, pick a point in the line, and then, move the point by $\Delta x = 1$ and $\Delta y = 1$.

A point **(1, 2)** is in the line, so it will be moved by $\Delta x = 1$ and $\Delta y = 1$, and then, become **(2, 3)**. Thus, the line we get after the translation has to have the point **(2, 3)**.

Also, translations do not change the slope of a line, so the same is true for a line segment, too. Therefore, the slope of the line segment after the translation is **-1**, too.

We know $y - c = a(x - b)$ is the line of slope a passing through a point (b, c).

Now, the line we get after the translation has a point **(2, 3)**, and the slope is **-1**.
So the equation of the line after the translation is $y - 3 = (-1)(x - 2)$.

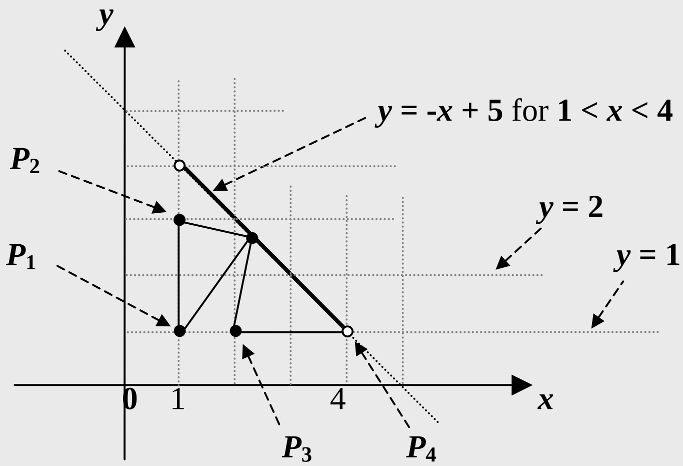
Let's see though, if the line above is the very line $(x - 1) + (y - 1) = 3$ indeed.

$$y - 3 = (-1)(x - 2) \Rightarrow x - 2 + y - 3 = 0 \Rightarrow x - 1 + y - 1 - 1 - 2 = 0 \Rightarrow x - 1 + y - 1 = 3.$$

The equation of the line segment is $(x - 1) + (y - 1) = 3$ for $1 < x < 4$.

Putting it in the standard form, we get $y = -x + 5$ for $1 < x < 4$.

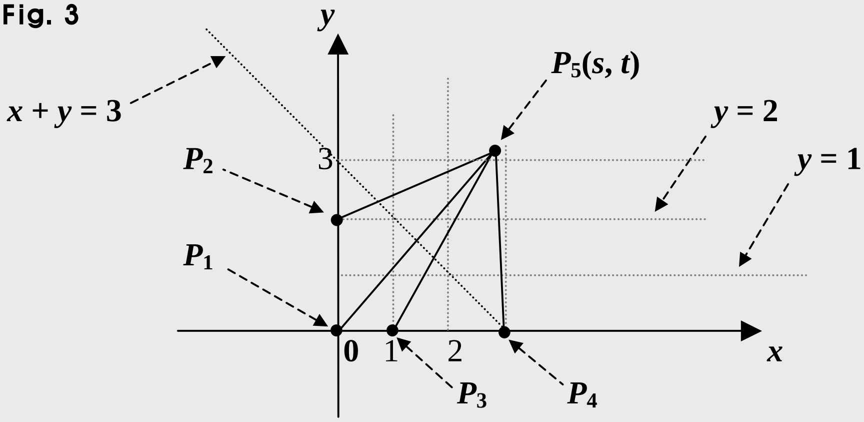
Fig. 2



In short:

Putting all the given points in a graph, and translating them by $\Delta x = -1$ and $\Delta y = -1$, we get a graph as follows.

Fig. 3



The area of the triangle $P_5P_1P_2 = \frac{1}{2}\cdot 2\cdot s = s$, and that of the triangle $P_5P_3P_4 = \frac{1}{2}\cdot 2\cdot t = t$.

Thus, since the sum of the areas is 3, we get $s + t = 3$.

So each of the points in the line $x + y = 3$ can be $P_5(s, t)$, yet P_5 is above P_3 and P_4, and also, on the right of P_2.

Now, setting $y = 0$ in $x + y = 3$, we get the x-intercept, which is 3.
Thus, every point in the line segment $x + y = 3$ for $0 < x < 3$ can be P_5.

We now, need to move the line segment back to the proper place by $\Delta x = 1$ and $\Delta y = 1$.

To begin with, a point $(1, 2)$ is in the line $x + y = 3$, and getting translated by $\Delta x = 1$ and $\Delta y = 1$, it becomes $(2, 3)$.

Next, the slope of the line is -1, and even after the translation, the slope remains the same, so the slope of the line we get after the translation is -1.

Thus, the line we get after the translation is $y - 3 = (-1)(x - 2) \Rightarrow y = 5 - x$.

Besides, the interval $0 < x < 3$ will be $1 < x < 4$ after the translation by $\Delta x = 1$.
So the proper line segment is $y = 5 - x$ for $1 < x < 4$.

Thus, each of all the points in the line segment above can be $P_5(s, t)$.

Suggestions or Solutions
To the Problem 1 in the Example 0

Suppose that

Six points $P_1(1, 1)$, $P_2(1, 3)$, $P_3(2, 1)$, $P_4(4, 1)$, $P_5(s, t)$, and $P_6(u, v)$ are in the x-y plane.
The two points P_5 and P_6 are above P_3 and P_4, and also, on the right of P_2.
The sum of the areas of two triangles $P_5P_1P_2$ and $P_5P_3P_4$ is 3.

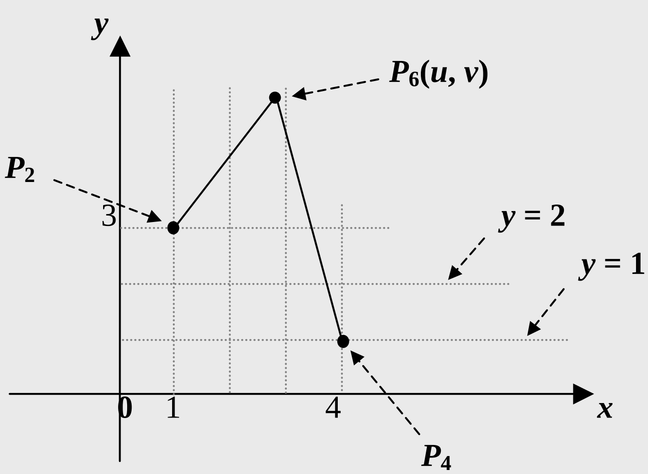
Suppose also, that

D_1 is the distance between the two points P_6 and P_4.
D_2 is the distance between the two points P_6 and P_2. And $D_1^2 - D_2^2 = 5$.

Then, specify the point P_6.

If we put the problem in a graph, it can be easier. That is, putting in a graph the points under consideration, we can see better the solution's whereabouts.

Fig. 0



Since D_1 is the distance from $P_6(u, v)$ to $P_4(4, 1)$, we get $D_1^2 = (u - 4)^2 + (v - 1)^2$, and since D_2 is the distance from $P_6(u, v)$ to $P_2(1, 3)$, we get $D_2^2 = (u - 1)^2 + (v - 3)^2$.

Thus, we get $D_1^2 - D_2^2 = 5 \Rightarrow (u - 4)^2 + (v - 1)^2 - (u - 1)^2 - (v - 3)^2 = 5$
$\Rightarrow u^2 - 8u + 16 + v^2 - 2v + 1 - u^2 + 2u - 1 - v^2 + 6v - 9 = 5 \Rightarrow -6u + 4v + 7 = 5$
$\Rightarrow v = \frac{3}{2}u - \frac{1}{2}$.

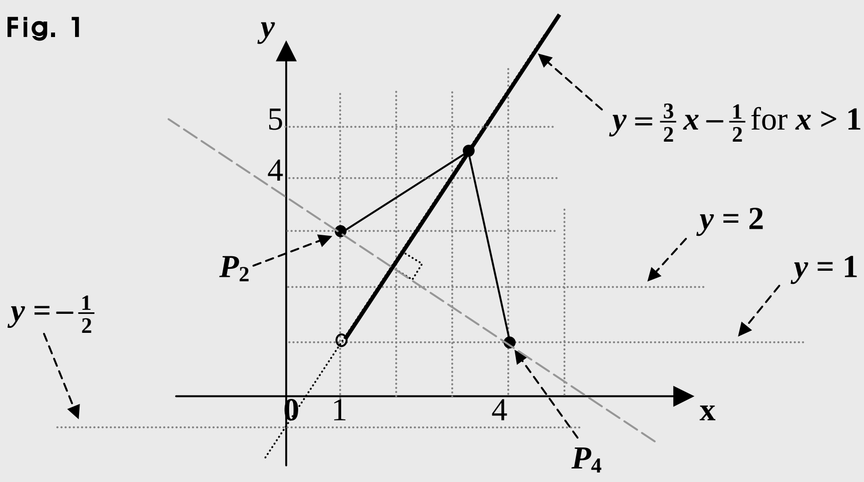
We can see now, that there can be infinitely many points that can satisfy the conditions of the problem, that is, infinitely many points can be the point $\boldsymbol{P_6}$. So choices for $\boldsymbol{u}$ and $\boldsymbol{v}$ are unlimited. In other words, every point in the line $\boldsymbol{y = \frac{3}{2}x - \frac{1}{2}}$ can be the point $\boldsymbol{P_6(u, v)}$. It is not quite the case, though. Why?

That's because $\boldsymbol{P_6}$ is supposed to be above $\boldsymbol{P_4}$, and also, on the right of $\boldsymbol{P_2}$.
So the solution is not every point in the line but every point in a ray. What ray?

The ray we need is $\boldsymbol{y = \frac{3}{2}x - \frac{1}{2}}$ for $\boldsymbol{x > 1}$.
So each and every point in the ray can be the point $\boldsymbol{P_6(u, v)}$.

A ray is called a half-line, too, so it is a part of a line, and expands itself either to the left or to the right only, for instance, but a line proceeds in both directions at the same time.

Fig. 1



Now, the ray is perpendicular to a line that includes the two points $\boldsymbol{P_2}$ and $\boldsymbol{P_4}$.

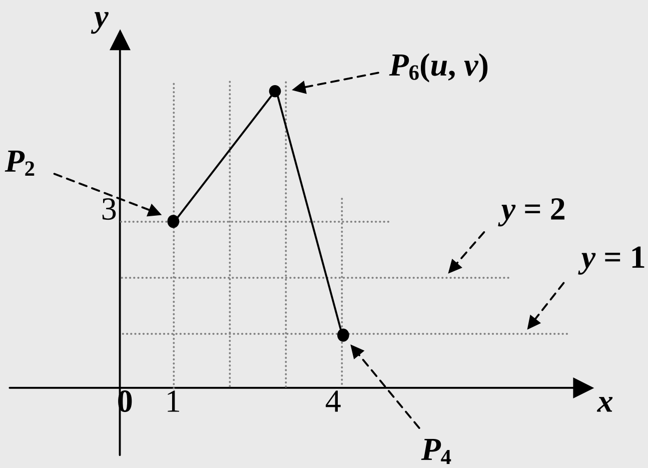
Let's check to see if the product of the slopes of the ray and the perpendicular line is -1.

The slope of the perpendicular line is $\frac{1-3}{4-1} = -\frac{2}{3}$, and product of the slopes is $-\frac{2}{3} \cdot \frac{3}{2} = \boldsymbol{-1}$.

In short:

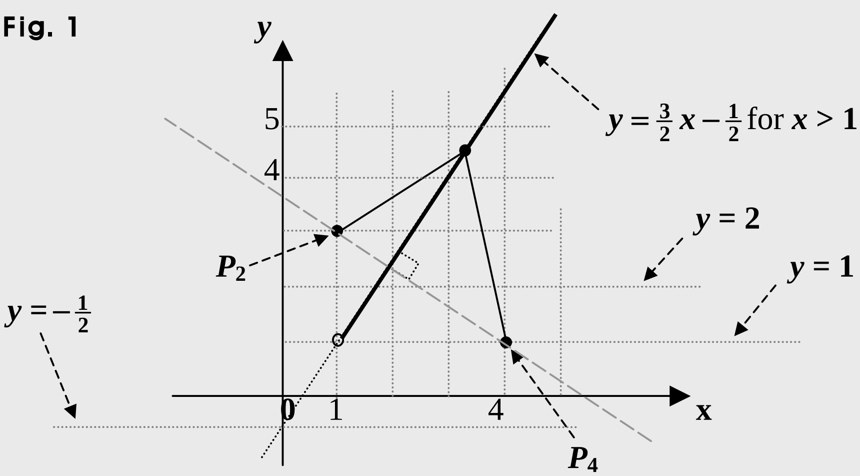
The point $P_6(u, v)$ is supposed to be above P_4, and also, on the right of P_2.

Fig. 0



We have $D_1^2 = (u - 4)^2 + (v - 1)^2$, and $D_2^2 = (u - 1)^2 + (v - 3)^2$.
Thus, $D_1^2 - D_2^2 = 5 \Rightarrow (u - 4)^2 + (v - 1)^2 - (u - 1)^2 - (v - 3)^2 = 5$
$\Rightarrow u^2 - 8u + 16 + v^2 - 2v + 1 - u^2 + 2u - 1 - v^2 + 6v - 9 = 5 \Rightarrow -6u + 4v + 7 = 5$
$\Rightarrow v = \frac{3}{2}u - \frac{1}{2}$.

Fig. 1



Thus, every point in a ray $y = \frac{3}{2}x - \frac{1}{2}$ for $x > 1$can be the point $P_6(u, v)$.

Suggestions or Solutions
To the Problem in the Example 1

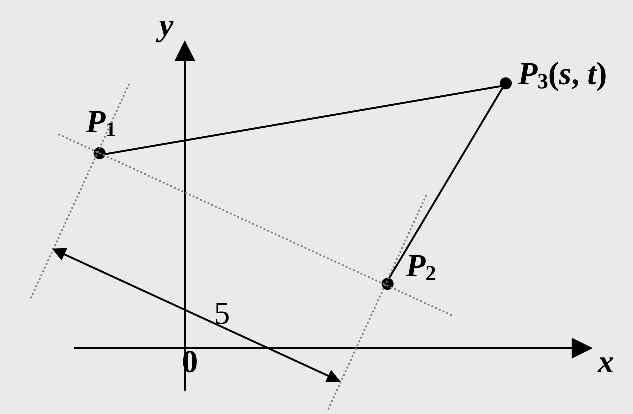
Suppose that the distance between two points P_1 and P_2 in the x-y plane is 5.

Suppose also, that
D_1 is the distance from the point P_1 to another point P_3.
D_2 is the distance between the two points P_2 and P_3.
$D_1^2 - D_2^2 = 10$.

Then, specify the point P_3.

To begin with, let the point P_3 be (s, t), and put all the points on the x-y plane.

Fig. 0



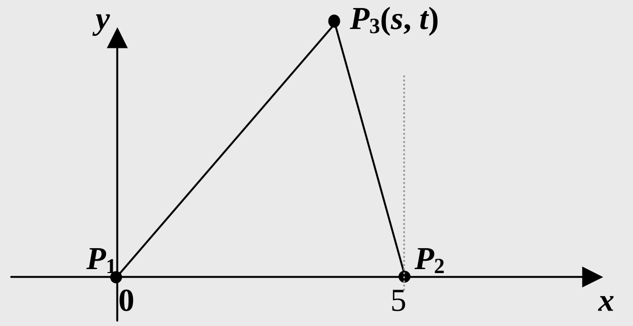
In this example also, we may want to relocate the two points P_1 and P_2 at some proper positions where life gets easier.

This time however, we do not have to consider those Δx and Δy. That's because the constraint is on only the distance between the two points.

In other words, we can put those two points anyway we want as long as the distance is 5.

What we want to find is the position of P_3 relative to the positions of P_1 and P_2.

Fig. 1



Now, the problem looks much smaller, and in the graph above, $\boldsymbol{P_1}$ is at the origin, and $\boldsymbol{P_2}$ is at **(5, 0)**. Let's now, start taking advantage of the suppositions given next.

$\boldsymbol{D_1}$ is the distance from $\boldsymbol{P_1(0, 0)}$ to $\boldsymbol{P_3(s, t)}$. So we get $\boldsymbol{D_1^2 = (s - 0)^2 + (t - 0)^2 = s^2 + t^2}$.

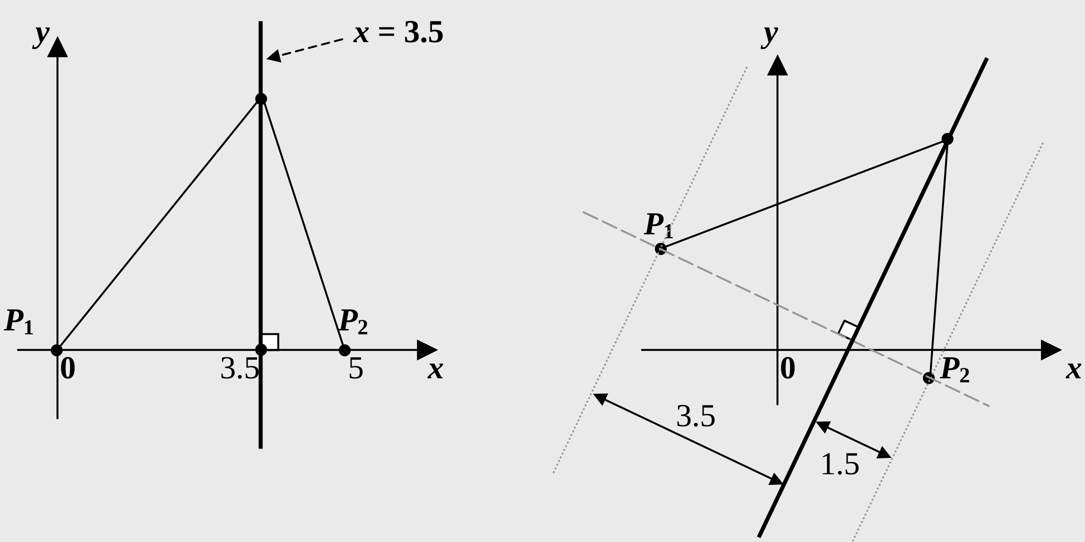
$\boldsymbol{D_2}$ is the distance from $\boldsymbol{P_2(5, 0)}$ to $\boldsymbol{P_3}$. So we get $\boldsymbol{D_2^2 = (s - 5)^2 + (t - 0)^2 = (s - 5)^2 + t^2}$.

In addition, we have $\boldsymbol{D_1^2 - D_2^2 = 10}$. Thus, we get $\boldsymbol{s^2 + t^2 - \{(s - 5)^2 + t^2\} = 10}$

$\boldsymbol{\Rightarrow s^2 + t^2 - (s^2 - 10s + 25 + t^2) = 10 \Rightarrow 10s - 25 = 10 \Rightarrow s = 3.5}$.

Then again, we can see there can be infinitely many points that can satisfy the conditions in the problem. Every point in a line $\boldsymbol{x = 3.5}$ can be $\boldsymbol{P_3}$. More precisely, the solution is a set of all the points in a line, which passes through a point that is **3.5** away from $\boldsymbol{P_1}$, and is **1.5** away from $\boldsymbol{P_2}$. And also, the line is perpendicular to a line passing through the two points $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$.

Fig. 2



In short:

Suppose $\boldsymbol{P_1}$ is at the origin, $\boldsymbol{P_2}$ is at **(5, 0)**, and $\boldsymbol{P_3}$ is at $\mathbf{(\boldsymbol{s}, \boldsymbol{t})}$.

$\boldsymbol{D_1}$ is the distance from $\boldsymbol{P_1}\mathbf{(0, 0)}$ to $\boldsymbol{P_3}(\boldsymbol{s}, \boldsymbol{t})$. So $\boldsymbol{D_1}^2 = (s - 0)^2 + (t - 0)^2 = s^2 + t^2$.

$\boldsymbol{D_2}$ is the distance from $\boldsymbol{P_2}\mathbf{(5, 0)}$ to $\boldsymbol{P_3}$. So $\boldsymbol{D_2}^2 = (s - 5)^2 + (t - 0)^2 = (s - 5)^2 + t^2$.

In addition, we have $\boldsymbol{D_1}^2 - \boldsymbol{D_2}^2 = 10$. So we get $s^2 + t^2 - \{(s - 5)^2 + t^2\} = 10$

$\Rightarrow s^2 + t^2 - (s^2 - 10s + 25 + t^2) = 10 \Rightarrow 10s - 25 = 10 \Rightarrow s = 3.5$.

Thus, the solution is a set of all the points in a line, which passes through a point that is **3.5** away from $\boldsymbol{P_1}$, and is **1.5** away from $\boldsymbol{P_2}$, and also, is perpendicular to a line passing through the two points $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$.