Examples A in Lines

Click this to start.

Examples A in Lines

0. Suppose $\boldsymbol{T}$ is a triangle connecting three points $\boldsymbol{P_1(-2, 3)}$, $\boldsymbol{P_2(-2, -1)}$, and $\boldsymbol{P_3(2, 2)}$.

Suppose also, $\boldsymbol{A}$ is a line connecting $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$, and another point $\boldsymbol{P_5}$ is in the line $\boldsymbol{A}$.

Then, assuming another line passing through $\boldsymbol{P_5}$ and perpendicular to the line $\boldsymbol{A}$ divides the triangle $\boldsymbol{T}$ into two equal parts in area, find the point $\boldsymbol{P_5}$.

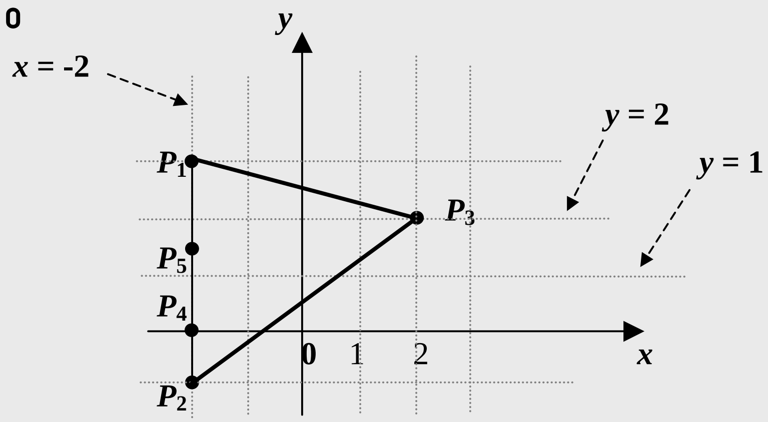
1. Assuming again, $\boldsymbol{T}$ is a triangle connecting three points $\boldsymbol{P_1(-2, 3)}$, $\boldsymbol{P_2(-2, -1)}$, and $\boldsymbol{P_3(2, 2)}$, find another line $\boldsymbol{B}$ that includes a point $\boldsymbol{P_4(-2, 0)}$, and divides the triangle $\boldsymbol{T}$ into two equal parts in area.

Suggestions or Solutions
To the Problem in the Example 0

Suppose in the *x*-*y* plane, *T* is a triangle connecting three points $P_1(-2, 3)$, $P_2(-2, -1)$, and $P_3(2, 2)$. Suppose also, *A* is a line connecting P_1 and P_2, and another point P_5 is in the line *A*. Then, assuming another line passing through P_5 and perpendicular to the line *A* divides the triangle *T* into two equal parts in area, find the point P_5.

To begin with, let's put the given points in a graph, and construct the triangle ***T***.
Then, we can see better whereabouts the solution is.

Fig. 0



Now, examining the graph above, we can notice that a simple translation can make the problem much easier. Translating all the points in the graph in the same manner so that the point $\boldsymbol{P_2}$ is positioned at the origin, we can make the problem much smaller. That is, as long as each of all the points maintains its relative position to those of all the others, no matter how much they may get moved, the problem itself doesn't get changed.

Also, moving the coordinate axes instead, we can get the same effect. In such a case, too, of course, we want to move both axes in the same manner. That is, we want to apply the same translation to each axis. Translating the axes in this problem, we want to make sure that the point $\boldsymbol{P_2}$ is positioned at the origin, of course. However, the translation will be the opposite of the one for the points, because the axes will get moved the way opposite of the way the points get moved if we move the points instead.

Anyway, the movement of the points is a translation by $\boldsymbol{\Delta x = 2}$ and $\boldsymbol{\Delta y = 1}$.

What do we mean by **Δ*x* = 2** and **Δ*y* = 1**, though?

If **Δ*x* = *a*** where ***a*** is constant, we move the point in the amount of ***a*** in the direction of the ***x***-axis. In short, if **Δ*x* = *a***, we move the point by ***a*** along the ***x***-axis.

So if **Δ*x* = 2**, we move the point by 2 in the direction of the ***x***-axis. That is, we move it by 2 to the right. What if **Δ*x* = -2**, though?

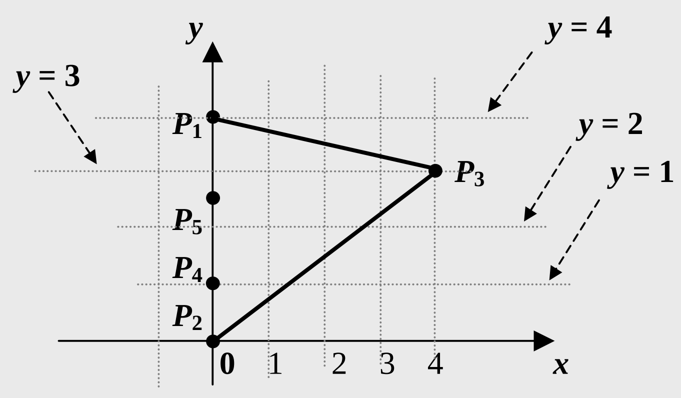
Then, we move the point by -2 in the direction of the ***x***-axis.
That is, we move it by 2 to the left.
By the same token, if **Δ*y* = 2**, we move it upward by 2, and if **Δ*y* = -2**, we move it downward by 2.

After getting the solution though, we need to move the solution back to the original location.
That is because the solution after the translation by **Δ*x* = 2** and **Δ*y* = 1** is not the one to the problem original but the one to the problem translated by **Δ*x* = 2** and **Δ*y* = 1**.

So after getting the solution, we want to translate it back by **Δ*x* = -2** and **Δ*y* = -1**.

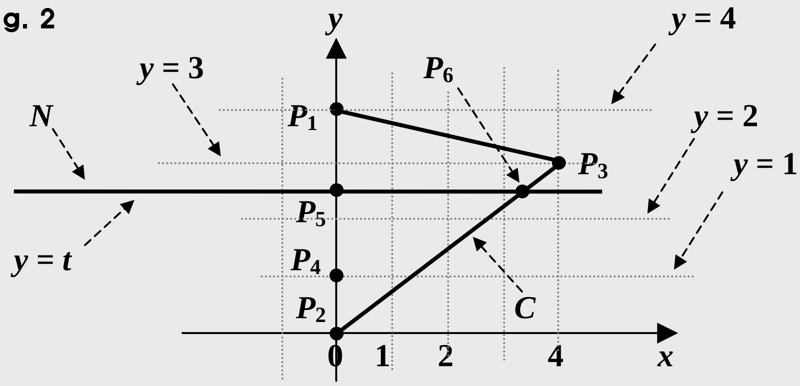
Now, after the translation of all the points by **Δ*x* = 2** and **Δ*y* = 1**, the graph above will get changed to the one as follows.

Fig. 1



Next, putting in the graph above a line including P_5 and perpendicular to the ***y***-axis, we can get the newer graph as shown below, and see that the line meets the line segment connecting P_3 and P_2. Call the perpendicular line, ***N***.

Fig. 2



Suppose now, C is a line that includes the line segment, and P_6 is the point where the line C meets the line N.

Suppose also, L is the smaller triangle below the line N, and the point P_5 is at $(0, t)$.

Then, the equation of the line N is $y = t$. Thus, finding the point P_6, we can express in terms of t the area of the smaller triangle below the line N.

How do we get P_6, though?

Finding the line C, then finding the point where C meets the line N, we can get P_6.
So we may want to get the line C, first.

The line C includes the origin, and the slope is $\frac{\Delta y}{\Delta x} = \frac{3-0}{4-0} = \frac{3}{4}$, so C is $y = \frac{3}{4}x$.

Next, solving the system where $y = t$ and $y = \frac{3}{4}x$, we get $y = \frac{3}{4}x \Rightarrow t = \frac{3}{4}x \Rightarrow x = \frac{4}{3}t$.

Thus, the x-coordinate at P_6 is $\frac{4}{3}t$, and the y-coordinate is t. Therefore, P_6 is at $(\frac{4}{3}t, t)$.

Now, we are ready to work with the areas of the triangles.

Suppose first, that t is the base of the smaller triangle L.

Then, the height is $\frac{4}{3}t$.

Thus, the area of the smaller triangle $\boldsymbol{L}$ is $\frac{1}{2} \cdot t \cdot \frac{4}{3} \cdot t = \frac{2}{3} t^2$.

Next, we want the area of $\boldsymbol{L}$ to be half the area of the triangle $\boldsymbol{T}$.

The area of the triangle $\boldsymbol{T}$ is $\frac{1}{2} \cdot 4 \cdot 4 = 8$.

Thus, we get $\frac{2}{3} t^2 = \frac{8}{2} = 4 \Rightarrow t^2 = 6 \Rightarrow t = \pm\sqrt{6}$.

We know the point $P_5(0, t)$ is above the origin. So we get $t = \sqrt{6} \approx 2.45$.

Thus, P_5 is at $(0, \sqrt{6})$, which is however, not in the proper position.

That's because P_5, together with all the other points has been translated by $\Delta x = 2$ and $\Delta y = 1$. So we need to translate P_5 back so that it can be in the original position.

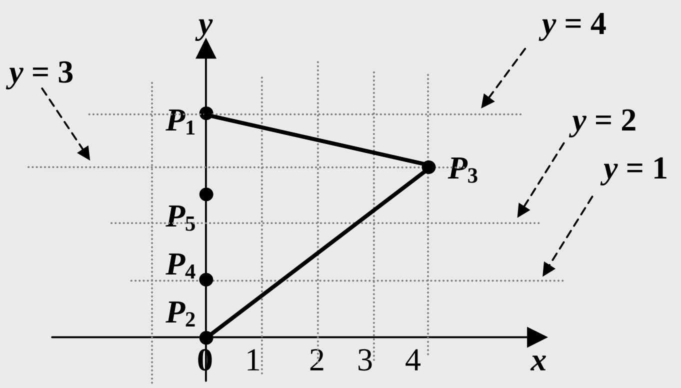
The Δx was 2, and the Δy was 1. Thus, Δx is now, -2, and the Δy is -1, now.

Therefore, the point P_5 is at $(-2, \sqrt{6} - 1)$.

In short:

Moving all the points in the same manner so that P_2 gets moved to the origin, and then putting all the points in a graph, we can see that the graph will be as follows.

Fig. 3



Suppose that

C is a line including the line segment connecting P_3 and P_2.
N is a line including P_5 and perpendicular to the ***y***-axis, and meets the line ***C***.
P_6 is the point where ***C*** meets the line ***N***.
L is the smaller triangle below the line ***N***, and P_5 is at **(0, *t*)**.

Then, ***C*** includes the origin, and the slope is $\frac{3-0}{4-0}=\frac{3}{4}$, so ***C*** is $y=\frac{3}{4}x$.

Next, solving a system where $y = t$, and $y=\frac{3}{4}x$, we can get the ***x***-coordinate of P_6.

Then, we get $y=\frac{3}{4}x \Rightarrow t=\frac{3}{4}x \Rightarrow x=\frac{4}{3}t$.

Since P_6 is in the line ***N***, too, the ***y***-coordinate is ***t***. So P_6 is at $(\frac{4}{3}t, t)$.

Thus, the area of ***L*** is $\frac{1}{2}\cdot t\cdot\frac{4}{3}\cdot t=\frac{2}{3}t^2$, and the area of the triangle ***T*** is $\frac{1}{2}\cdot 4\cdot 4=8$.

The area of ***L*** is half the area of ***T***, so we get $\frac{2}{3}t^2=\frac{8}{2}=4 \Rightarrow t^2=6$.

We know $P_5(0, t)$ is above the origin, so we get $t=\sqrt{6}$.

So P_5 is at $(0, \sqrt{6})$, yet we need to move P_5 to the proper position.

The Δx was 2, and the Δy was 1. So Δx is now, -2, and Δy is -1, now.

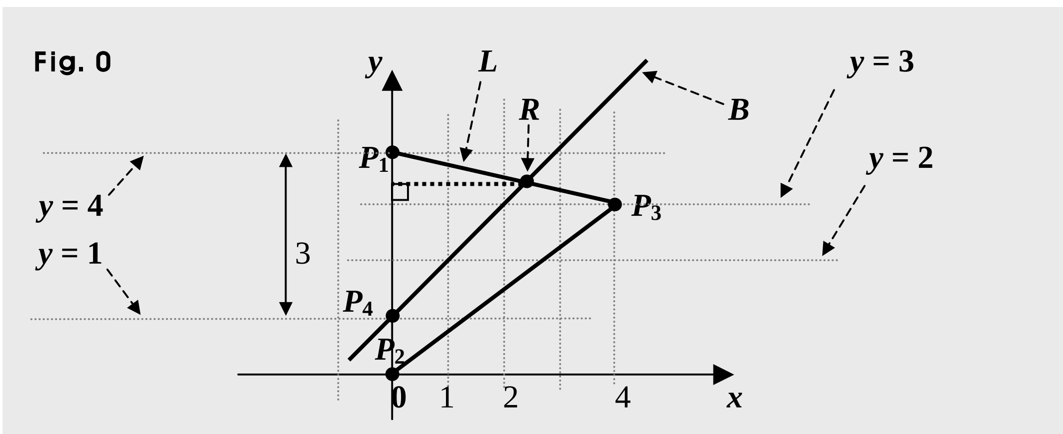
Therefore, P_5 is at $(-2, \sqrt{6}-1)$.

Suggestions or Solutions
To the Problem in the Example 1

Assuming T is a triangle connecting three points $P_1(-2, 3)$, $P_2(-2, -1)$, and $P_3(2, 2)$, find a line B that includes a point $P_4(-2, 0)$, and divides the triangle T into two equal parts in area.

Moving $\boldsymbol{P_2}$ to the origin, and the other points in the same manner as $\boldsymbol{P_2}$ gets moved, we get a graph as below. The movement is a translation by $\Delta \boldsymbol{x} = \mathbf{2}$, and $\Delta \boldsymbol{y} = \mathbf{1}$.

Fig. 0



What we are after in this problem is the line $\boldsymbol{B}$.

We know a point in the line $\boldsymbol{B}$, which is $\boldsymbol{P_4}(\mathbf{0, 1})$, so what we want to get is the slope of the line $\boldsymbol{B}$. Setting the slope = $\boldsymbol{m}$, for instance, and meeting the conditions given in the problem, we should be able to get the slope, then the line $\boldsymbol{B}$.

Let's see first, of the two line segments $\overline{\boldsymbol{P_1P_3}}$ and $\overline{\boldsymbol{P_2P_3}}$, which one the line $\boldsymbol{B}$ has to meet.

To begin with, the area of the triangle $\boldsymbol{T}$ is $\frac{1}{2}\cdot\mathbf{4}\cdot\mathbf{4}=\mathbf{8}$.

If we take $\overline{\boldsymbol{P_2P_4}}$ for the base, the area of the triangle $\boldsymbol{P_2P_3P_4}$ is $\frac{1}{2}\cdot\mathbf{1}\cdot\mathbf{4}=\mathbf{2}$.

Thus, the line $\boldsymbol{B}$ needs to meet the line segment $\overline{\boldsymbol{P_1P_3}}$.

Suppose next, $\boldsymbol{L}$ is a line that has $\boldsymbol{P_1}$ and $\boldsymbol{P_3}$, and $\boldsymbol{R}$ is a point where $\boldsymbol{L}$ meets the line $\boldsymbol{B}$. Suppose also, $\boldsymbol{H}$ is the triangle $\boldsymbol{P_1P_4R}$.

Then, the area of $\boldsymbol{H}$ is 4 since the area is haft the area of the triangle $\boldsymbol{T}$.
In the graph above, we can see that $\overline{\boldsymbol{P_1P_4}}$ is the base of the triangle $\boldsymbol{H}$, and the length is 3, and that the $\boldsymbol{x}$-coordinate of the point $\boldsymbol{R}$ is the height of the triangle $\boldsymbol{H}$.

Thus, we want to get the $\boldsymbol{x}$-coordinate of the point $\boldsymbol{R}$.
Then, we need the two lines $\boldsymbol{B}$ and $\boldsymbol{L}$. Well… the line $\boldsymbol{B}$ is the one we are after, though.

Let's anyway, begin with the line $\boldsymbol{L}$.

Since the two points $\boldsymbol{P_1(0, 4)}$ and $\boldsymbol{P_3(4, 3)}$ are in the line $\boldsymbol{L}$, the slope is $\frac{\Delta y}{\Delta x} = \frac{3-4}{4-0} = \frac{-1}{4}$.

The line of slope $\boldsymbol{a}$ passing through a point $\boldsymbol{(s, t)}$ is $\boldsymbol{y - t = a(x - s)}$.
So using $\boldsymbol{P_1(0, 4)}$, together with the slope above, we can get the line $\boldsymbol{L}$, which is below.

$$y - 4 = \frac{-1}{4}(x - 0) = \frac{-x}{4} \Rightarrow y = -\frac{x}{4} + 4.$$

Next, the line $\boldsymbol{B}$ has the point $\boldsymbol{P_4(0, 1)}$. So if $\boldsymbol{m}$ is the slope of $\boldsymbol{B}$, we get

$\boldsymbol{y - 1 = m(x - 0) = mx \Rightarrow y = mx + 1}$, which is the line $\boldsymbol{B}$. And we have these, too:

- The line segment $\overline{\boldsymbol{P_1P_4}}$ is the base of the triangle $\boldsymbol{H}$, and the length is 3.
- The $\boldsymbol{x}$-coordinate of the point $\boldsymbol{R}$ is the height of the triangle $\boldsymbol{H}$.

So finding the $\boldsymbol{x}$-coordinate of the point $\boldsymbol{R}$, we can find $\boldsymbol{m}$ using the area of the triangle $\boldsymbol{H}$.
Eliminating $\boldsymbol{y}$ from the system where $y = -\frac{x}{4} + 4$ and $\boldsymbol{y = mx + 1}$, we can find $\boldsymbol{x}$, which is the $\boldsymbol{x}$-coordinate of the point $\boldsymbol{R}$.

$$-\frac{x}{4} + 4 = mx + 1 \Rightarrow -x + 16 = 4mx + 4 \Rightarrow x(4m + 1) = 12 \Rightarrow x = \frac{12}{4m+1}.$$

So the area of the triangle $\boldsymbol{H}$ is $\frac{1}{2} \cdot 3 \cdot \frac{12}{4m+1} = \frac{18}{4m+1}$, which has to be 4.

Thus, we get $\frac{18}{4m+1} = 4 \Rightarrow 18 = 16m + 4 \Rightarrow 16m = 14 \Rightarrow m = \frac{7}{8}$.

So the line $\boldsymbol{B}$ is $y = \frac{7}{8}x + 1$, yet the line $\boldsymbol{B}$ needs to be moved back to the proper position.

The Δx was 2, and the Δy was 1. Thus, the Δx is now -2, and the Δy is -1, now.

So we get $y - \Delta y = \frac{7}{8}(x - \Delta x) + 1 \Rightarrow y + 1 = \frac{7}{8}(x+2) + 1 \Rightarrow y = \frac{7}{8}(x+2)$.

If not sure of what's going on above, refer to the section, **Parallel Transformations**, in the book, **Function Transformations**.

We can get the same result the way below, too.

The line B has the point P_4**(0, 1)**, so if it gets moved by $\Delta x = -2$, and $\Delta y = -1$, so does P_4. That is, the point P_4 will be at **(0 – 2, 1 – 1) = (-2, 0)**.

Translations do not change slopes.
So the slope of the line remains the same even after the translation.

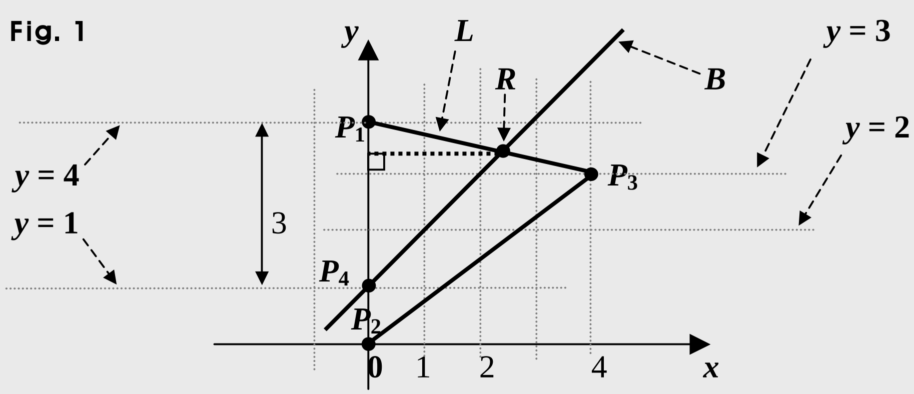
Thus, the slope of the line we get after the translation is $\frac{7}{8}$, too.
Now, the line moved has the point P_4**(-2, 0)** and a slope of $\frac{7}{8}$.
Therefore, we get

$y - 0 = \frac{7}{8}\{x - (-2)\} = \frac{7}{8}(x+2) \Rightarrow y = \frac{7}{8}(x+2)$, which is the line B original.

In short:

Moving P_2 to the origin, and the other points in the same manner as P_2 gets moved, we can get a graph as below. The movement is a translation by $\Delta x = 2$, and $\Delta y = 1$.

Fig. 1



The area of the triangle $\boldsymbol{T}$ is $\frac{1}{2}\cdot\mathbf{4}\cdot\mathbf{4}=\mathbf{8}$.

The area of the triangle $\boldsymbol{P_2P_3P_4}$ is $\frac{1}{2}\cdot\mathbf{1}\cdot\mathbf{4}=\mathbf{2}$, so $\boldsymbol{B}$ needs to meet the line segment $\overline{\boldsymbol{P_1P_3}}$.

Suppose $\boldsymbol{L}$ is a line that has $\boldsymbol{P_1}$ and $\boldsymbol{P_3}$, and $\boldsymbol{R}$ is a point where $\boldsymbol{L}$ meets the line $\boldsymbol{B}$.

Then, the slope of the line $\boldsymbol{L}$ is $\frac{\Delta y}{\Delta x}=\frac{3-4}{4-0}=\frac{-1}{4}$.

So since the line $\boldsymbol{L}$ has the point $\boldsymbol{P_3}\mathbf{(0,\ 4)}$, L is $\boldsymbol{y-4=\frac{-1}{4}(x-0)=\frac{-x}{4}\Rightarrow y=-\frac{x}{4}+4}$.

Next, the line $\boldsymbol{B}$ has the point $\boldsymbol{P_4}\mathbf{(0,\ 1)}$.
So if $\boldsymbol{m}$ is the slope of the line $\boldsymbol{B}$, $\boldsymbol{B}$ is $\boldsymbol{y-1=m(x-0)=mx\Rightarrow y=mx+1}$.

Now, in the graph above, we can see that $\overline{\boldsymbol{P_1P_4}}$ is the base of the triangle $\boldsymbol{P_1P_4R}$, and has a length of 3, and that the $\boldsymbol{x}$-coordinate of the point $\boldsymbol{R}$ is the height of the triangle.

Then, finding the $\boldsymbol{x}$-coordinate of the point $\boldsymbol{R}$, we get

$\boldsymbol{-\frac{x}{4}+4=mx+1\Rightarrow -x+16=4mx+4\Rightarrow x(4m+1)=12\Rightarrow x=\frac{12}{4m+1}}$.

So the area of the $\boldsymbol{H}$ is $\frac{1}{2}\cdot\mathbf{3}\cdot\frac{12}{4m+1}=\frac{18}{4m+1}$, which has to be 4.

Thus, $\boldsymbol{\frac{18}{4m+1}=4\Rightarrow 18=16m+4\Rightarrow 16m=14\Rightarrow m=\frac{7}{8}}$.

So the line $\boldsymbol{B}$ is $\boldsymbol{y=\frac{7}{8}x+1}$, yet $\boldsymbol{B}$ needs to be moved back to the proper position.

The $\Delta\boldsymbol{x}$ was 2, and the $\Delta\boldsymbol{y}$ was 1. Thus, the $\Delta\boldsymbol{x}$ is now -2, and the $\Delta\boldsymbol{y}$ is -1, now.

Thus, we get $\boldsymbol{y-\Delta y=\frac{7}{8}(x-\Delta x)+1\Rightarrow y+1=\frac{7}{8}(x+2)+1\Rightarrow y=\frac{7}{8}(x+2)}$, which is the line $\boldsymbol{B}$ original, which is the solution.

Summary of the basics in lines:

• Given two points (x_1, y_1) and (x_2, y_2), we get $\frac{y_2 - y_1}{x_2 - x_1}$, which is the slope of the line passing through the two points given.

Thus, we can define the line passing through the two points in such a way as follows.
$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, can be put it this way, too: $(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$.
And also, we can put it this way, too: $(y - y_2)(x_2 - x_1) = (x - x_2)(y_2 - y_1)$.

• Some forms we can use finding a particular line:

Given a slope and a point, we may want to use the form of $y - t = a(x - s)$, where a is the slope, and (s, t) is a point in the line.

Given a slope and a y-intercept or x-intercept, we can use the form of $y = ax + b$, where a is the slope, and b is the y-intercept, or $-\frac{b}{a}$ is the x-intercept.

So if the x-intercept is given and is c, we can find b solving $c = -\frac{b}{a}$ for b since a is given.

Given two points only, we can use either of the two below.
$(y - y_1)(x_2 - x_1) = (y_2 - y_1)(x - x_1)$, and $(y - y_2)(x_2 - x_1) = (y_2 - y_1)(x - x_2)$, in each of which (x_1, y_1) and (x_2, y_2) are two points in the line.

Given the x-intercept and the y-intercept both, we can use the form of $\frac{x}{p} + \frac{y}{q} = 1$, where p is the x-intercept, and q is the y-intercept.

Besides,

• Lines all different from and parallel to each other share one slope only, but they have all different y-intercepts, and thus, have all different x-intercepts, too.

• The product of the slopes of two lines perpendicular to each other is **-1**.