Examples D in Lines

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This set of examples is for students as college students or high school students taking courses in calculus.

0. Suppose $\boldsymbol{y = f(x) = 2x + 1}$ when $\boldsymbol{-\frac{1}{2} \le x \le 2}$, and $\boldsymbol{y = f(x) = 9 - 2x}$ when $\boldsymbol{2 < x \le \frac{9}{2}}$. Then, construct the graph of $\boldsymbol{g(x) = (f \bullet f)(x)}$.

1. Assuming in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, three points $\boldsymbol{P_1(x_1, y_1)}$, $\boldsymbol{P_2(x_2, y_2)}$, and $\boldsymbol{P_3(x_3, y_3)}$ are in a line, find the connective equation among the coordinates.

2. Assuming the distance from a point $\boldsymbol{P_1(2, 1)}$ to a line is the same as the one from another point $\boldsymbol{P_2(-2, -3)}$ to the line, find the line.

Suggestions or Solutions
To the Problem in the Example 0

Suppose $y = f(x) = 2x + 1$ when $-\frac{1}{2} \le x \le 2$, and $y = f(x) = 9 - 2x$ when $2 < x \le \frac{9}{2}$. Then, construct the graph of $g(x) = (f \bullet f)(x)$.

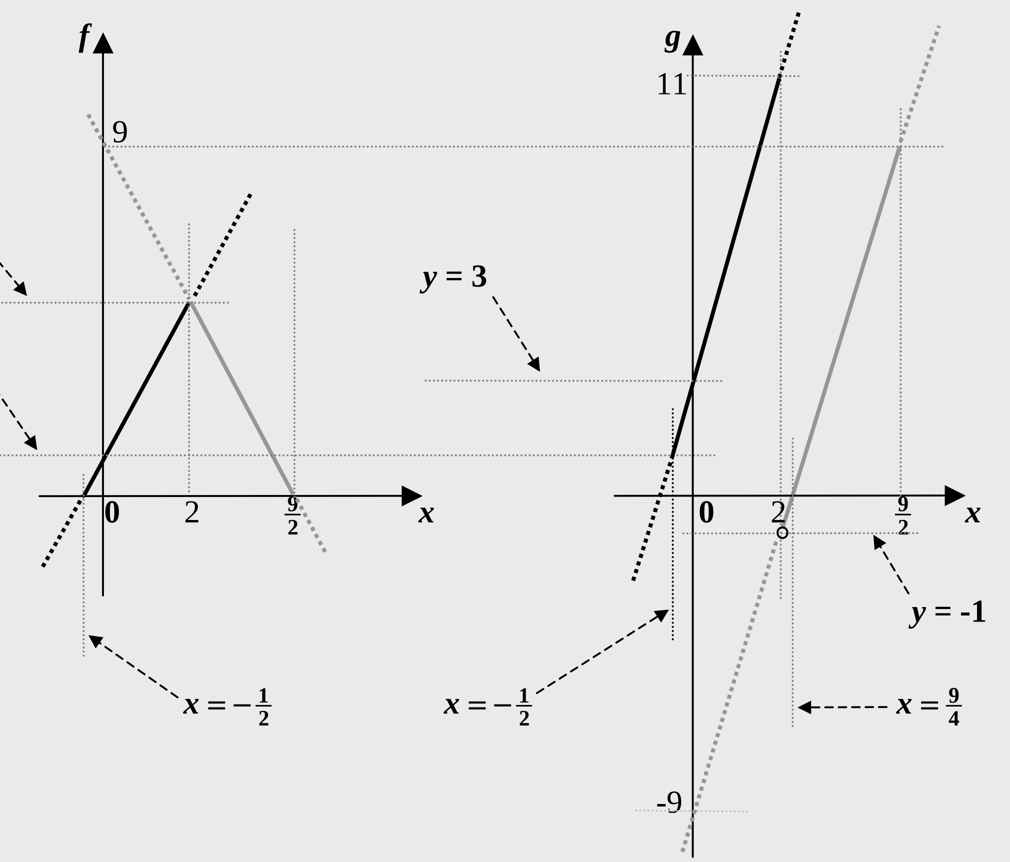
The function g is a composite function from f only. So g has two expressions, and the domain of g is the same as that of f, too. Finding thus, g for each domain, we get

$y = f(x) = 2x + 1$ for $-\frac{1}{2} \le x \le 2$
$\Rightarrow g(x) = (f \bullet f)(x) = f\{f(x)\} = 2(2x + 1) + 1 = 4x + 3$ for $-\frac{1}{2} \le x \le 2$.

$y = f(x) = 9 - 2x$ for $2 < x \le \frac{9}{2}$
$\Rightarrow g(x) = (f \bullet f)(x) = f\{f(x)\} = 9 - 2(9 - 2x) = 9 - 18 + 4x = 4x - 9$ for $2 < x \le \frac{9}{2}$.

So putting each of f and g in its graph, we can each graph the way below.

Fig. 0



Suggestions or Solutions
To the Problem in the Example 1

Assuming in the x-y plane, three points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ are in a line, find the connective equation among the coordinates of the three points.

Using two points of course, we can get a slope, then a line using one of the two points. So we can get the line using two of the three points given.
Then, putting the other point into the line we got, we can get the connective equation. So let's get the slope first.

A slope of a line in the x-y plane can be defined by a rate $\frac{\Delta y}{\Delta x}$ where Δx is the change in the x-coordinate, and Δy is the change in the y-coordinate.

So assuming a is the slope, and using the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, we get

$$a = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Next, using the point $P_2(x_2, y_2)$, along with the slope a, we get the line as follows.

$$y - y_2 = a(x - x_2) = \tfrac{y_2 - y_1}{x_2 - x_1}(x - x_2) \Rightarrow y - y_2 = \tfrac{y_2 - y_1}{x_2 - x_1}(x - x_2).$$

Now, the three points P_1, P_2, and P_3 are in the line. So either of the three points satisfies the equation above. Thus, putting P_3 into the line, we get

$$y_3 - y_2 = \tfrac{y_2 - y_1}{x_2 - x_1}(x_3 - x_2) \Rightarrow (y_3 - y_2)(x_2 - x_1) = (y_2 - y_1)(x_3 - x_2)$$

$$\Rightarrow y_3x_2 - y_3x_1 - y_2x_2 + y_2x_1 = y_2x_3 - y_2x_2 - y_1x_3 + y_1x_2$$

$$\Rightarrow y_3x_2 - y_3x_1 + y_2x_1 = y_2x_3 - y_1x_3 + y_1x_2$$

$$\Rightarrow y_1x_3 + y_2x_1 + y_3x_2 = y_1x_2 + y_2x_3 + y_3x_1$$

$$\Rightarrow y_1(x_3 - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_1) = 0.$$

So the final expression above can be the connective equation we want in this problem, and also, can be taken as a condition under which three points can be in a line.

And of course, we can put the same the way below, too.

Using two of the three points, we can get the slope, then the line using the other point.

So we can get the line using the three points given.

Then putting into the line either of the three points, we can get the connective equation.

So using $\boldsymbol{P_3(x_3, y_3)}$, together with the slope $\boldsymbol{a}$ found above, we can get the line as follows.

$$y - y_3 = a(x - x_3) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_3) \Rightarrow y - y_3 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_3).$$

Now, the line has three points $\boldsymbol{P_1}$, $\boldsymbol{P_2}$, and $\boldsymbol{P_3}$. So either of the three satisfies the equation above. So for instance, putting $\boldsymbol{P_1}$ into the line, we get

$$y_1 - y_3 = \frac{y_2 - y_1}{x_2 - x_1}(x_1 - x_3) \Rightarrow (y_1 - y_3)(x_2 - x_1) = (y_2 - y_1)(x_1 - x_3)$$

$$\Rightarrow y_1x_2 - y_1x_1 - y_3x_2 + y_3x_1 = y_2x_1 - y_2x_3 - y_1x_1 + y_1x_3$$

$$\Rightarrow y_1x_2 - y_3x_2 + y_3x_1 = y_2x_1 - y_2x_3 + y_1x_3$$

$$\Rightarrow y_1x_2 + y_2x_3 + y_3x_1 = y_1x_3 + y_2x_1 + y_3x_2$$

$$\Rightarrow y_1(x_3 - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_1) = 0.$$

Why do we get the same result, though?

All the three points are in the same situation, that is, they are in the same line.

So no matter what two points out of the three we may pick, we get the same slope, and the same line.

So no matter what point we may put into the equation of the line, we get the same expression in terms of the coordinates of the three points.

Let's check though, to see if we get the same putting $\boldsymbol{P_2}$ into the line. Then, we get

$$y_2 - y_3 = \frac{y_2 - y_1}{x_2 - x_1}(x_2 - x_3) \Rightarrow (y_2 - y_3)(x_2 - x_1) = (y_2 - y_1)(x_2 - x_3)$$

$\Rightarrow y_2x_2 - y_2x_1 - y_3x_2 + y_3x_1 = y_2x_2 - y_2x_3 - y_1x_2 + y_1x_3$

$\Rightarrow -y_2x_1 - y_3x_2 + y_3x_1 = -y_2x_3 - y_1x_2 + y_1x_3$

$\Rightarrow y_1x_2 + y_2x_3 + y_3x_1 = y_1x_3 + y_2x_1 + y_3x_2$

$\Rightarrow y_1(x_3 - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_1) = \mathbf{0}$.

For simple instance, three points $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (2, 2)$, and $(x_3, y_3) = (3, 3)$ are in a line. The line is $y = x$, of course. Then, we get

$y_1(x_3 - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_1) = 1(3 - 2) + 2(1 - 3) + 3(2 - 1) = 1 - 4 + 3 = \mathbf{0}$.

Suppose for another instance, a line $y = 2x + 1$ has three points as follows.

$(x_1, y_1) = (1, 3)$, $(x_2, y_2) = (3, 7)$, and $(x_3, y_3) = (5, 11)$. Then, we get

$y_1(x_3 - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_1) = 3(5 - 3) + 7(1 - 5) + 11(3 - 1) = 6 - 28 + 22 = \mathbf{0}$.

Suggestions or Solutions
To the Problem in the Example 2

Assuming the distance from a point $P_1(2, 1)$ to a line is the same as the one from another point $P_2(-2, -3)$ to the line, find the line.

A distance in math is a length of a line segment connecting two points.

We can get the length by the distance formula, often called Pythagorean theorem.

What if we take a distance from a point $\boldsymbol{P}$ to a curve $\boldsymbol{C}$ as a line, parabola, circle, etc.?

The distance is still the length of a line segment connecting two points. One of the two is the point $\boldsymbol{P}$, and the other is a particular point in the curve $\boldsymbol{C}$.

Of all the points in the curve $\boldsymbol{C}$, the particular point is the closest to the point $\boldsymbol{P}$.

How then, do we get the particular point closest to the point $\boldsymbol{P}$?

Suppose first, the point $\boldsymbol{P}$ is given, and is at $\boldsymbol{(s, t)}$ where $\boldsymbol{s}$ and $\boldsymbol{t}$ are constant, and $\boldsymbol{Q}$ is the particular point closest to the point $\boldsymbol{P}$.

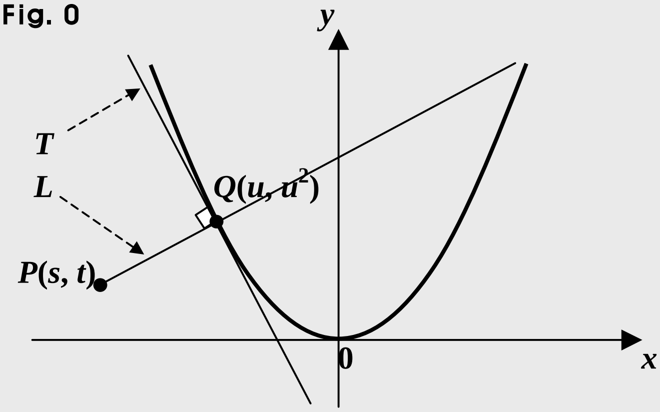
Then, $\boldsymbol{Q}$ is a point in the curve $\boldsymbol{C}$, we can set $\boldsymbol{P = P(s, t)}$, and a line tangent to the curve $\boldsymbol{C}$ at the point $\boldsymbol{Q}$ is perpendicular to a line connecting the point $\boldsymbol{P}$ and the point $\boldsymbol{Q}$.

Suppose next, the equation of the curve $\boldsymbol{C}$ is $\boldsymbol{y = f(x)}$. For instance, $\boldsymbol{y = x^2}$.

Then, we can set $\boldsymbol{Q = Q(u, u^2)}$, where $\boldsymbol{u}$ is constant, so finding $\boldsymbol{u}$, we get the point $\boldsymbol{Q}$.

Suppose now, $\boldsymbol{T}$ is the line tangent to $\boldsymbol{C}$, $\boldsymbol{L}$ is the line connecting $\boldsymbol{P}$ and $\boldsymbol{Q}$, $\boldsymbol{M}$ is the slope of the tangent line $\boldsymbol{T}$, and $\boldsymbol{N}$ is the slope of the perpendicular line $\boldsymbol{L}$.

Fig. 0



Then first, ***L*** is the line perpendicular to the tangent line ***T***, so we get $\boldsymbol{MN = -1}$ since the product of the slopes of two lines perpendicular to each other is -1.

Next, the line ***L*** has ***P*** and ***Q***, so we get $\boldsymbol{N} = \frac{\Delta y}{\Delta x} = \frac{u^2-t}{u-s}$ since $\boldsymbol{P = P(s, t)}$, and $\boldsymbol{Q = Q(u, u^2)}$.

Next, since $\boldsymbol{MN = -1}$, and $\boldsymbol{N} = \frac{u^2-t}{u-s}$, solving the equation $\boldsymbol{M} \cdot \frac{u^2-t}{u-s} = \boldsymbol{-1}$ for ***u***, we can get the point $\boldsymbol{Q(u, u^2)}$.

Well then, how can we get ***M*** the slope of the tangent line ***T***?

We cannot get it at this stage, because we need to know a tool called the derivative, which is covered in calculus.

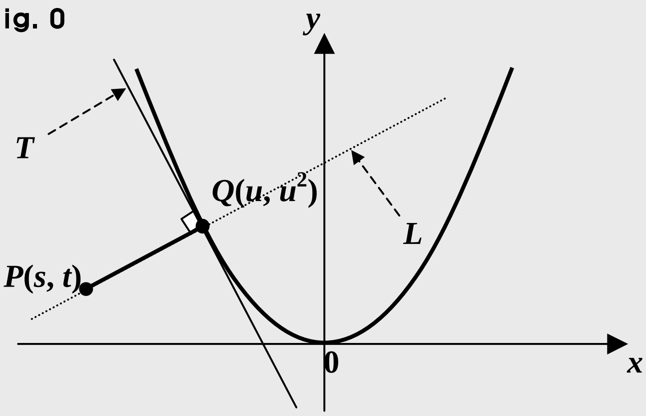
Finding the slope ***M***, we need to take the derivative of $\boldsymbol{x^2}$, which is from the equation of the curve ***C***, and the derivative is $\boldsymbol{2x}$, which is not however, a value indicating an actual slope but an expression representing all slopes of all lines tangent to the curve ***C***.

However, we can get such a derivative in calculus.

The slope ***M*** is in fact, the derivative evaluated at $\boldsymbol{x = u}$, and thus, is $\boldsymbol{2u}$.

So $\boldsymbol{M = 2u}$ is the actual slope of the line tangent to the curve ***C*** at the point $\boldsymbol{Q(u, u^2)}$, and is also called the differential coefficient at $\boldsymbol{x = u}$ in the curve of $\boldsymbol{y = x^2}$, which is ***C***.

Fig. 0



Anyway, the distance is the length of the line segment connecting ***P*** and ***Q***, and the line segment is a part of the perpendicular line ***L*** passing through ***P*** and ***Q***.

So a distance in math is often called the perpendicular distance, too. Then, again, what is the distance from a point ***P*** to the curve ***C***?

Finding the distance from a point ***P*** to the curve ***C***, we take the length of a line segment connecting the point ***P*** and the particular point ***Q***, which is the closest to the point ***P*** among all the points in the curve ***C***.

The two points ***P*** and ***Q*** are in a line perpendicular to a line tangent to the curve ***C*** at the point ***Q***. In this example however, the curve is a line.
So the line itself is the very tangent line.

So suppose now, the curve ***C*** is a line.

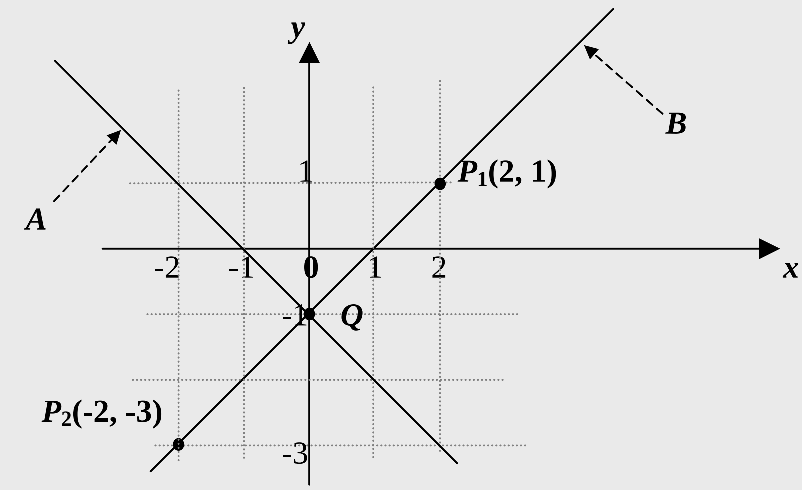
Then, using the slope of the line ***C***, we can readily find a line passing through the point ***P*** and perpendicular to the line ***C***. The product of the slopes of two lines perpendicular to each other is -1.

Then, the point where the two lines meet is the point ***Q***, so solving the system of the two equations of the two lines, we can get the point ***Q***.

Now, in this example, we are after a particular line that is the same distance away from two points $\boldsymbol{P_1(2, 1)}$, and $\boldsymbol{P_2(-2, -3)}$. So assuming that ***A*** is the particular line, we want to find first in the line ***A***, such a point ***Q*** as stated above. How then, do we get the point ***Q***?

The point $\boldsymbol{Q}$ is the same distance away from the two points $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$.
Thus, the point $\boldsymbol{Q}$ is the midpoint between the two points, so just taking the averages of the coordinates of the two points, we get the point $\boldsymbol{Q}$. How then, do we get the line $\boldsymbol{A}$?

Fig. 1



The same distance stated above is from $\boldsymbol{Q}$ to either of $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$.

So $\boldsymbol{Q}$ is in a line connecting $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$, and perpendicular to the line $\boldsymbol{A}$. So assuming $\boldsymbol{B}$ is the line connecting $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$, and using $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$, we can get the slope of the line $\boldsymbol{B}$.

Next, since the line $\boldsymbol{B}$ is perpendicular to the line $\boldsymbol{A}$, we can get the slope of the line $\boldsymbol{A}$, since the product of the two slopes is -1.

Then, using the point $\boldsymbol{Q}$, we can readily get the line $\boldsymbol{A}$ since $\boldsymbol{Q}$ is in the line $\boldsymbol{A}$.

So let's now, begin with the slope of the line $\boldsymbol{B}$ perpendicular to the line $\boldsymbol{A}$.

The line $\boldsymbol{B}$ has $\boldsymbol{P_1(2, 1)}$ and $\boldsymbol{P_2(-2, -3)}$. So the slope of the line $\boldsymbol{B}$ is $\frac{\Delta y}{\Delta x} = \frac{-3-1}{-2-2} = \mathbf{1}$.

Thus, the slope of the line $\boldsymbol{A}$ is -1. So next, we want to get the point $\boldsymbol{Q}$.

At the point $\boldsymbol{Q}$, the $\boldsymbol{x}$-coordinate $= \frac{2+(-2)}{2} = \mathbf{0}$, and the $\boldsymbol{y}$-coordinate $= \frac{1+(-3)}{2} = \mathbf{-1}$.

So the point $\boldsymbol{Q}$ is at **(0, -1)**, which belongs to the line $\boldsymbol{A}$, where the slope is -1.

Therefore, the line $\boldsymbol{A}$ is $\boldsymbol{y - (-1) = -1(x - 0)}$.

That is, $\boldsymbol{y = -x - 1}$, which is however, is not the only line.   Why?

Technically, the line itself connecting the two points is also, the same distance away from both of the two points $\boldsymbol{P_1(2, 1)}$ and $\boldsymbol{P_2(-2, -3)}$, because both distances are 0.

There are many more, though.

In fact, there are infinitely many, each of which is the same distance away from both of the two points $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$.

One of the many is a line $\boldsymbol{x = 0}$, which is the $\boldsymbol{y}$-axis.

Another is a line $\boldsymbol{y = -1}$, which is parallel to the $\boldsymbol{x}$-axis.

The others are all the lines parallel to the line connecting the two points.   How?

A distance in math is the shortest distance between two points, and is the length of the line segment connecting the two points.

The distance from a point to a curve is the shortest distance from the point to the curve, and thus, is the length of the shortest of all line segments from the point to all the points in the curve.

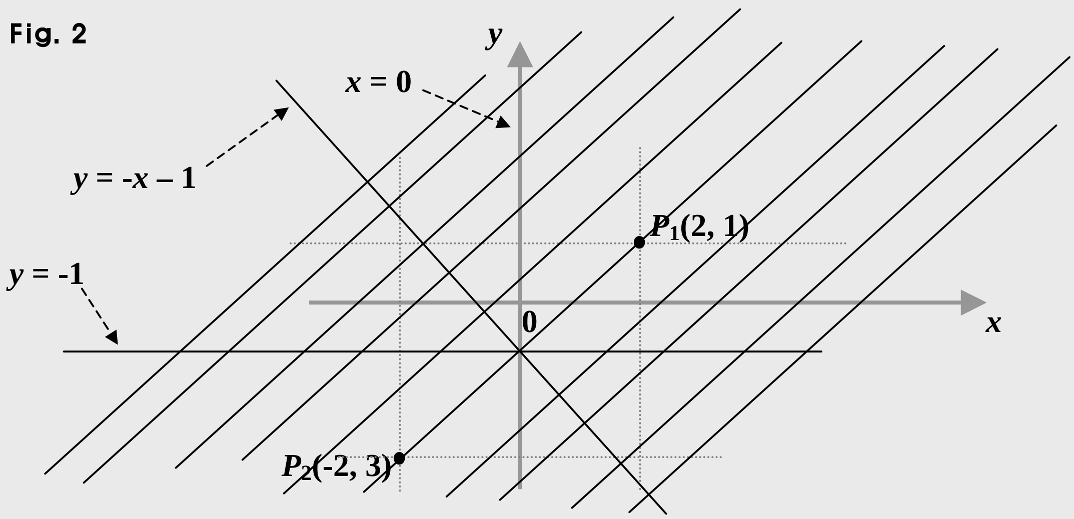
The distance from a point to a curve is the length of a line segment connecting the point to a particular point in the curve, and the line tangent to the curve at the particular point is perpendicular to the line segment.

So the distance is often called the perpendicular distance.

Thus, if the perpendicular distances from particular points to a line are the same, the particular points are the same distance away from the line.

And in the graph below, each dashed line is the same distance away from $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$, also.

Fig. 2



And of course, all the dashed lines are parallel to each other and are perpendicular to the line $y = -x - 1$.

So all the lines that are the same distance away from P_1 and P_2 are the line $y = -x - 1$, the line $y = -1$, the line $x = 0$, that is, the y-axis, and all the dashed lines.