Examples E in Lines

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Examples E in Lines

As in the case of the Examples C, this set of examples is for students as college students or high school students taking courses in calculus.

0. Suppose that

$\boldsymbol{T}$**:** $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (3\boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y})$.

$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{ax}$ where $\boldsymbol{a}$ is constant, and $\boldsymbol{T} \bullet \boldsymbol{f} = \boldsymbol{f}$.

Then, find $\boldsymbol{a}$.

1. Suppose that

$\boldsymbol{T}$**:** $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (3\boldsymbol{x} - 2\boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y})$.

$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{ax}$ where $\boldsymbol{a}$ is constant, and $\boldsymbol{T} \bullet \boldsymbol{f} = \boldsymbol{f}$.

Then, find $\boldsymbol{a}$.

2. Suppose that

$\boldsymbol{T}$**:** $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (2\boldsymbol{x} + \boldsymbol{y} + 3, 3\boldsymbol{x} + 2\boldsymbol{y} - 2)$.

$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{ax} + \boldsymbol{b}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant, and $\boldsymbol{T} \bullet \boldsymbol{f} = \boldsymbol{f}$.

Then, find $\boldsymbol{a}$ and $\boldsymbol{b}$.

Suggestions or Solutions
To the Problem in the Example 0

Suppose that
$\boldsymbol{T}$: $(x, y) \longrightarrow (3x - y, x - y)$.
$y = f(x) = ax$ **where** a **is constant, and** $T \bullet f = f$.

Then, find a**.**

Suppose $g = T \bullet f$, and $t = g(s)$. Then, we get
$\boldsymbol{T}$: $(x, y) \longrightarrow (s, t) = (3x - y, x - y)$.

So we get $s = 3x - y$, and $t = x - y$. Solving the system above for x and y, we get
$s - t = 3x - y - (x - y) = 2x \Rightarrow s - t = 2x \Rightarrow x = \frac{s-t}{2}$.
$s = 3x - y \Rightarrow y = 3 \cdot \frac{s-t}{2} - s = \frac{3s-3t-2s}{2} = \frac{s-3t}{2} \Rightarrow y = \frac{s-3t}{2}$.
Next, applying x and y above to the function f to get the new function g, we get

$y = f(x) = ax \Rightarrow \frac{s-3t}{2} = f(\frac{s-t}{2}) = a \cdot \frac{s-t}{2} \Rightarrow s - 3t = a(s - t) = as - at$

$\Rightarrow s - 3t = as - at \Rightarrow (a-3)t = (a-1)s \Rightarrow t = \frac{a-1}{a-3}s$. So we get $t = g(s) = \frac{a-1}{a-3}s$.

Now, putting g into the x-y system, we get $y = g(x) = \frac{a-1}{a-3}x$. We have $f(x) = ax$, too.

So we get $\frac{a-1}{a-3} = a$ where $a \neq 3$. Next, solving for a, we get

$\frac{a-1}{a-3} = a \Rightarrow a - 1 = a(a-3) = a^2 - 3a \Rightarrow a^2 - 4a + 1 = a^2 - 4a + 4 - 4 + 1 = (a-2)^2 - 3 = 0$
$\Rightarrow (a-2)^2 = 3 \Rightarrow a - 2 = \pm\sqrt{3}$. Therefore, we get $a = 2 \pm \sqrt{3}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Transforming a function, we get a new function.
So the function transformed can be called the original function.

What we want to do in this problems is to determine the value of a so that the function f having the curve of an equation $y = ax$ does not get changed even if f gets transformed by the transformation T.

So in this case, the new function is the same as the function original.
If not sure of transformations, refer to **GRAPH OPERATIONS**.

Suppose now, that $g = T \bullet f$.
Then, f is the original function, and g is the new function.
And since $T \bullet f = f$, and $y = f(x) = ax$, we want to find a that makes $g = f$.

First, assuming $t = g(s)$, we get
$T: (x, y) \longrightarrow (s, t) = (3x - y, x - y)$.

So we get $s = 3x - y$, and $t = x - y$.

Next, we take the equations above as a system of equations for x and y.
Then, solving the system, we can put each of x and y in terms of s and t.

So getting x first, we get
$s - t = 3x - y - (x - y) = 2x \Rightarrow s - t = 2x \Rightarrow x = \frac{s-t}{2}$.

Next, we have $s = 3x - y$, so we get

$s = 3x - y \Rightarrow y = 3 \cdot \frac{s-t}{2} - s = \frac{3s-3t-2s}{2} = \frac{s-3t}{2} \Rightarrow y = \frac{s-3t}{2}$.

Now, we can apply x and y above to the original function f, we get the new function g.
How?

Applying those, we put them respectively into x and y in the original function f.
The original function f is $y = f(x) = ax$. So we get

$y = f(x) = ax \Rightarrow \frac{s-3t}{2} = f(\frac{s-t}{2}) = a \cdot \frac{s-t}{2} \Rightarrow s - 3t = a(s-t) = as - at$

$\Rightarrow s - 3t = as - at \Rightarrow (a-3)t = (a-1)s \Rightarrow t = \frac{a-1}{a-3}s$.

Thus, we get $t = g(s) = \frac{a-1}{a-3}s$.

Now, putting g into the x-y system, we simply change the letters s to x, and t to y.

Then, we get $y = g(x) = \frac{a-1}{a-3}x$.

And we have $\boldsymbol{f(x) = ax}$, too. So we get $\frac{a-1}{a-3} = a$ where $a \neq 3$. Next, solving for a, we get

$$\tfrac{a-1}{a-3} = a \Rightarrow a - 1 = a(a-3) = a^2 - 3a \Rightarrow a^2 - 4a + 1 = a^2 - 4a + 4 - 4 + 1 = (a-2)^2 - 3 = 0$$

$$\Rightarrow (a-2)^2 = 3 \Rightarrow a - 2 = \pm\sqrt{3}.$$ Therefore, we get $a = 2 \pm \sqrt{3}$.

In short:

Suppose $g = T \bullet f$, and $t = g(s)$. Then, we get

$T: (x, y) \longrightarrow (s, t) = (3x - y, x - y)$.

So we get $s = 3x - y$, and $t = x - y$. Solving the system above for x and y, we get

$$s - t = 3x - y - (x - y) = 2x \Rightarrow s - t = 2x \Rightarrow x = \tfrac{s-t}{2}.$$

$$s = 3x - y \Rightarrow y = 3 \cdot \tfrac{s-t}{2} - s = \tfrac{3s-3t-2s}{2} = \tfrac{s-3t}{2} \Rightarrow y = \tfrac{s-3t}{2}.$$

Next, applying x and y above to the function f to get the new function g, we get

$$y = f(x) = ax \Rightarrow \tfrac{s-3t}{2} = f\left(\tfrac{s-t}{2}\right) = a \cdot \tfrac{s-t}{2} \Rightarrow s - 3t = a(s-t) = as - at$$

$$\Rightarrow s - 3t = as - at \Rightarrow (a-3)t = (a-1)s \Rightarrow t = \tfrac{a-1}{a-3}s.$$ So we get $t = g(s) = \frac{a-1}{a-3}s$.

Now, putting g into the x-y system, we get $y = g(x) = \frac{a-1}{a-3}x$. We have $f(x) = ax$, too.

So we get $\frac{a-1}{a-3} = a$ where $a \neq 3$. Next, solving for a, we get

$$\tfrac{a-1}{a-3} = a \Rightarrow a - 1 = a(a-3) = a^2 - 3a \Rightarrow a^2 - 4a + 1 = a^2 - 4a + 4 - 4 + 1 = (a-2)^2 - 3 = 0$$

$$\Rightarrow (a-2)^2 = 3 \Rightarrow a - 2 = \pm\sqrt{3}.$$ Therefore, we get $a = 2 \pm \sqrt{3}$.

Suggestions or Solutions
To the Problem in the Example 1

Suppose that

T: $(x, y) \longrightarrow (3x - 2y, x + y)$.

$y = f(x) = ax$ where a is constant, and $T \bullet f = f$.

Then, find a.

Suppose $g = T \bullet f$, and $t = g(s)$. Then, we get

T: $(x, y) \longrightarrow (s, t) = (3x - 2y, x + y)$.

So we get $s = 3x - 2y$, and $t = x + y$.
Solving the system above for x and y, we get

$$s + 2t = 3x - 2y + 2(x + y) = 5x \Rightarrow s + 2t = 5x \Rightarrow x = \frac{s+2t}{5}.$$

$$t = x + y \Rightarrow y = t - \frac{s+2t}{5} = \frac{5t-s-2t}{5} = \frac{3t-s}{5} \Rightarrow y = \frac{3t-s}{5}.$$

Applying the expressions above to f to get g, we get

$$y = f(x) = ax \Rightarrow \frac{3t-s}{5} = f\left(\frac{s+2t}{5}\right) = a \cdot \frac{s+2t}{5} \Rightarrow 3t - s = a(s + 2t) = as + 2at$$

$$\Rightarrow 3t - s = as + 2at \Rightarrow (3 - 2a)t = (a + 1)s \Rightarrow t = \frac{a+1}{3-2a}s.$$

So we get $t = g(s) = \frac{a+1}{3-2a}s$.

And putting g into the x-y system, we get $y = g(x) = \frac{a+1}{3-2a}x$.

And we have $f(x) = ax$, too. So we get $\frac{a+1}{3-2a} = a$ where $a \neq \frac{3}{2}$.

Next, solving for a, we get

$$\frac{a+1}{3-2a} = a \Rightarrow a + 1 = a(3 - 2a) = 3a - 2a^2 \Rightarrow 2a^2 - 2a + 1 = 0 \Rightarrow a^2 - 2a + \frac{1}{2} = 0$$

$\Rightarrow a^2 - a + \frac{1}{2} = a^2 - a + \frac{1}{4} - \frac{1}{4} + \frac{1}{2} = (a - \frac{1}{2})^2 + \frac{1}{2}$, which however, cannot be 0.

Therefore, there is no value for a that can satisfy $T \bullet f = f$.

If not quite sure of the idea behind the processes above, follow the steps below.

This time too, we need to determine the value of a so that the function f having the curve of an equation $y = ax$ remains the same even if transformed by the transformation T.

Thus, assuming that $g = T \bullet f$, and that $t = g(s)$, we want to find a that makes $g = f$.

First, since $t = g(s)$, we get

T**:** $(x, y) \longrightarrow (s, t) = (3x - 2y, x + y)$.

So we get $s = 3x - 2y$, and $t = x + y$.

Next, taking the equations above for a system of equations for x and y, then solving the system, we can put each of x and y in terms of s and t. So getting x first, we get

$$s + 2t = 3x - 2y + 2(x + y) = 5x \Rightarrow s + 2t = 5x \Rightarrow x = \tfrac{s+2t}{5}.$$

Next, we have $t = x + y$, too, so we get

$$t = x + y \Rightarrow y = t - \tfrac{s+2t}{5} = \tfrac{5t-s-2t}{5} = \tfrac{3t-s}{5} \Rightarrow y = \tfrac{3t-s}{5}.$$

Then, apply the expressions above to the original function f to get the new function g. The original function f is $y = f(x) = ax$.

So we get $y = f(x) = ax \Rightarrow \tfrac{3t-s}{5} = f(\tfrac{s+2t}{5}) = a \cdot \tfrac{s+2t}{5} \Rightarrow 3t - s = a(s + 2t) = as + 2at$

$\Rightarrow 3t - s = as + 2at \Rightarrow (3 - 2a)t = (a + 1)s \Rightarrow t = \tfrac{a+1}{3-2a}s.$ Thus, we get $t = g(s) = \tfrac{a+1}{3-2a}s$.

Now, putting g into the x-y system, we simply change the letters s to x, and t to y.

Then, we get $y = g(x) = \tfrac{a+1}{3-2a}x$. And we have $f(x) = ax$, too.

So we get $\tfrac{a+1}{3-2a} = a$ where $a \neq \tfrac{3}{2}$. Next, solving for a, we get

$$\tfrac{a+1}{3-2a} = a \Rightarrow a + 1 = a(3 - 2a) = 3a - 2a^2 \Rightarrow 2a^2 - 2a + 1 = 0 \Rightarrow a^2 - 2a + \tfrac{1}{2} = 0.$$

Thus, we get $a^2 - a + \tfrac{1}{2} = a^2 - a + \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{2} = (a - \tfrac{1}{2})^2 + \tfrac{1}{2}$, which however, cannot be 0.

Therefore, there is no value for a that can satisfy this: $T \bullet f = f$.

Suggestions or Solutions
To the Problem in the Example 2

Suppose that

T: $(x, y) \longrightarrow (2x + y + 3, 3x + 2y - 2)$.

$y = f(x) = ax + b$ where a and b are constant, and $T \bullet f = f$.

Then, find a and b.

Suppose $g = T \bullet f$, and $t = g(s)$. Then, we get

T: $(x, y) \longrightarrow (s, t) = (2x + y + 3, 3x + 2y - 2)$.

So we get $s = 2x + y + 3$, and $t = 3x + 2y - 2$.

Next, solving the system above for x and y, we get

$2s - t = 4x + 2y + 6 - 3x - 2y + 2 = x + 8 \Rightarrow x = 2s - t - 8$.

$s = 2x + y + 3 \Rightarrow y = s - 2x - 3 = s - 2(2s - t - 8) - 3 = s - 4s + 2t + 16 - 3$

$= 2t - 3s + 13 \Rightarrow y = 2t - 3s + 13$.

Next, applying x and y above to the original function f, we get

$y = f(x) = ax + b \Rightarrow 2t - 3s + 13 = f(2s - t - 8) = a(2s - t - 8) + b$

$\Rightarrow 2t - 3s + 13 = a(2s - t - 8) + b \Rightarrow (2 + a)t = (2a + 3)s - 8a + b - 13$

$\Rightarrow t = \frac{2a+3}{a+2} s + \frac{b-8a-13}{a+2}$. So $t = g(s) = \frac{2a+3}{a+2} s + \frac{b-8a-13}{a+2}$.

And putting g into the x-y system, we get $y = g(x) = \frac{2a+3}{a+2} x + \frac{b-8a-13}{a+2}$.

And we have $f(x) = ax + b$, too. So we get $\frac{2a+3}{a+2} = a$, and $\frac{b-8a-13}{a+2} = b$ where $a \neq -2$.

Next, solving for a, first, we get

$\frac{2a+3}{a+2} = a \Rightarrow 2a + 3 = a(a + 2) = a^2 + 2a \Rightarrow a^2 = 3 \Rightarrow a = \pm\sqrt{3}$.

Next, solving for b, we get

$\frac{b-8a-13}{a+2} = b \Rightarrow b - 8a - 13 = b(2 + a) \Rightarrow b(1 - 2 - a) = 8a + 13 \Rightarrow b = \frac{8a+13}{-a-1}$.

Thus, we get

$$a = \sqrt{3} \Rightarrow b = \frac{8a+13}{-a-1} = \frac{8\sqrt{3}+13}{-\sqrt{3}-1} = \frac{(8\sqrt{3}+13)(\sqrt{3}-1)}{-(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{24+5\sqrt{3}-13}{-2} = -\frac{5\sqrt{3}+11}{2}.$$

$$a = -\sqrt{3} \Rightarrow b = \frac{8a+13}{-a-1} = \frac{-8\sqrt{3}+13}{\sqrt{3}-1} = \frac{(-8\sqrt{3}+13)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{-24+5\sqrt{3}+13}{2} = \frac{5\sqrt{3}-11}{2}.$$

If not quite sure of the idea behind the processes above, follow the steps below.

This example is about the same as the previous two. We just have one more unknown to find. So we want to determine a and b so that the function f having the curve of an equation $y = ax + b$ remains the same even if transformed by the transformation T.

Thus, assuming $g = T \bullet f$ and $t = g(s)$, we want to find a and b that make $g = f$.

First, since $t = g(s)$, we have

$T: (x, y) \longrightarrow (s, t) = (2x + y + 3, 3x + 2y - 2)$.

So we get $s = 2x + y + 3$, and $t = 3x + 2y - 2$.

Next, solving the system above for x and y, we can put each in terms of s and t.
So getting x first, we get

$2s - t = 4x + 2y + 6 - 3x - 2y + 2 = x + 8 \Rightarrow x = 2s - t - 8$.

And we have $s = 2x + y + 3$, too, so we get

$s = 2x + y + 3 \Rightarrow y = s - 2x - 3 = s - 2(2s - t - 8) - 3 = s - 4s + 2t + 16 - 3$
$= 2t - 3s + 13 \Rightarrow y = 2t - 3s + 13$.

Next, applying x and y above to the original function f, we get the new function g.
The original function f is $y = f(x) = ax + b$. So we get

$$y = f(x) = ax + b \Rightarrow 2t - 3s + 13 = f(2s - t - 8) = a(2s - t - 8) + b$$
$$\Rightarrow 2t - 3s + 13 = a(2s - t - 8) + b \Rightarrow (2 + a)t = (2a + 3)s - 8a + b - 13$$
$$\Rightarrow t = \frac{2a+3}{a+2}s + \frac{b-8a-13}{a+2}.$$

Thus, we get $t = g(s) = \frac{2a+3}{a+2}s + \frac{b-8a-13}{a+2}$.

And putting $\boldsymbol{g}$ into the $\boldsymbol{x}$-$\boldsymbol{y}$ system, we simply change the letters $\boldsymbol{s}$ to $\boldsymbol{x}$, and $\boldsymbol{t}$ to $\boldsymbol{y}$.

Then, we get $\boldsymbol{y = g(x) = \frac{2a+3}{a+2}x + \frac{b-8a-13}{a+2}}$. And we have $\boldsymbol{f(x) = ax + b}$, too.

So we get $\boldsymbol{\frac{2a+3}{a+2} = a}$, and $\boldsymbol{\frac{b-8a-13}{a+2} = b}$ where $\boldsymbol{a \neq -2}$.

Next, solving for $\boldsymbol{a}$, first, we get

$$\boldsymbol{\frac{2a+3}{a+2} = a \Rightarrow 2a+3 = a(a+2) = a^2 + 2a \Rightarrow a^2 = 3 \Rightarrow a = \pm\sqrt{3}}.$$

Next, solving for $\boldsymbol{b}$, we get

$$\boldsymbol{\frac{b-8a-13}{a+2} = b \Rightarrow b - 8a - 13 = b(2+a) \Rightarrow b(1-2-a) = 8a + 13 \Rightarrow b = \frac{8a+13}{-a-1}}.$$

Thus, we get

$$\boldsymbol{a = \sqrt{3} \Rightarrow b = \frac{8a+13}{-a-1} = \frac{8\sqrt{3}+13}{-\sqrt{3}-1} = \frac{(8\sqrt{3}+13)(\sqrt{3}-1)}{-(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{24+5\sqrt{3}-13}{-2} = -\frac{5\sqrt{3}+11}{2}}.$$

$$\boldsymbol{a = -\sqrt{3} \Rightarrow b = \frac{8a+13}{-a-1} = \frac{-8\sqrt{3}+13}{\sqrt{3}-1} = \frac{(-8\sqrt{3}+13)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{-24+5\sqrt{3}+13}{2} = \frac{5\sqrt{3}-11}{2}}.$$

In sum, we can put them this way, too: $\boldsymbol{a = \pm\sqrt{3}}$, and $\boldsymbol{b = \frac{\mp 5\sqrt{3} \pm 11}{2}}$.

Thus, even if we transform by $\boldsymbol{T}$ a line $\boldsymbol{y = \sqrt{3}x - \frac{5\sqrt{3}+11}{2}}$, we still get the same line, and the same is true for another line $\boldsymbol{y = -\sqrt{3}x + \frac{5\sqrt{3}-11}{2}}$, too.

In short:

Suppose $\boldsymbol{g = T \bullet f}$, and $\boldsymbol{t = g(s)}$, Then, we get
$\boldsymbol{T: (x, y) \longrightarrow (s, t) = (2x + y + 3, 3x + 2y - 2)}$.

So we get $\boldsymbol{s = 2x + y + 3}$, and $\boldsymbol{t = 3x + 2y - 2}$.

Next, solving the system above for $\boldsymbol{x}$ and $\boldsymbol{y}$, we get

$\boldsymbol{2s - t = 4x + 2y + 6 - 3x - 2y + 2 = x + 8 \Rightarrow x = 2s - t - 8}$.
$\boldsymbol{s = 2x + y + 3 \Rightarrow y = s - 2x - 3 = s - 2(2s - t - 8) - 3 = s - 4s + 2t + 16 - 3}$
$\boldsymbol{= 2t - 3s + 13 \Rightarrow y = 2t - 3s + 13}$.

Next, applying $\boldsymbol{x}$ and $\boldsymbol{y}$ above to the original function $\boldsymbol{f}$, we get

$$\boldsymbol{y = f(x) = ax + b \Rightarrow 2t - 3s + 13 = f(2s - t - 8) = a(2s - t - 8) + b}$$
$$\boldsymbol{\Rightarrow 2t - 3s + 13 = a(2s - t - 8) + b \Rightarrow (2 + a)t = (2a + 3)s - 8a + b - 13}$$
$$\boldsymbol{\Rightarrow t = \tfrac{2a+3}{a+2} s + \tfrac{b-8a-13}{a+2}}.\ \text{ So } \boldsymbol{t = g(s) = \tfrac{2a+3}{a+2} s + \tfrac{b-8a-13}{a+2}}.$$

Now, putting $\boldsymbol{g}$ into the $\boldsymbol{x}$-$\boldsymbol{y}$ system, we get $\boldsymbol{y = g(x) = \tfrac{2a+3}{a+2} x + \tfrac{b-8a-13}{a+2}}$.

And we have $\boldsymbol{f(x) = ax + b}$, too. So we get $\boldsymbol{\tfrac{2a+3}{a+2} = a}$, and $\boldsymbol{\tfrac{b-8a-13}{a+2} = b}$ where $\boldsymbol{a \neq -2}$.

Next, solving for $\boldsymbol{a}$, first, we get

$$\boldsymbol{\tfrac{2a+3}{a+2} = a \Rightarrow 2a + 3 = a(a + 2) = a^2 + 2a \Rightarrow a^2 = 3 \Rightarrow a = \pm\sqrt{3}}.$$

Next, solving for $\boldsymbol{b}$, we get

$$\boldsymbol{\tfrac{b-8a-13}{a+2} = b \Rightarrow b - 8a - 13 = b(2 + a) \Rightarrow b(1 - 2 - a) = 8a + 13 \Rightarrow b = \tfrac{8a+13}{-a-1}}.$$

Thus, we get

$$\boldsymbol{a = \sqrt{3} \Rightarrow b = \tfrac{8a+13}{-a-1} = \tfrac{8\sqrt{3}+13}{-\sqrt{3}-1} = \tfrac{(8\sqrt{3}+13)(\sqrt{3}-1)}{-(\sqrt{3}+1)(\sqrt{3}-1)} = \tfrac{24+5\sqrt{3}-13}{-2} = -\tfrac{5\sqrt{3}+11}{2}}.$$

$$\boldsymbol{a = -\sqrt{3} \Rightarrow b = \tfrac{8a+13}{-a-1} = \tfrac{-8\sqrt{3}+13}{\sqrt{3}-1} = \tfrac{(-8\sqrt{3}+13)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \tfrac{-24+5\sqrt{3}+13}{2} = \tfrac{5\sqrt{3}-11}{2}}.$$