Examples F in Lines

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0. Put in one equation, two lines, $\boldsymbol{y = x}$ and $\boldsymbol{y = -x}$.

1. Put in one equation two lines $\boldsymbol{y = -x + 1}$ and $\boldsymbol{y = x}$.

2. An equation $\mathbf{2}\boldsymbol{x}^{\mathbf{2}} - \mathbf{2}\boldsymbol{y}^{\mathbf{2}} - \mathbf{3}\boldsymbol{xy} - \boldsymbol{x} + \mathbf{2}\boldsymbol{y} = \mathbf{0}$ indicates two lines. Find the two lines.

3. Find $\boldsymbol{t}$ for which an equation $\mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{4}\boldsymbol{xy} - \boldsymbol{ty}^{\mathbf{2}} + \mathbf{4}\boldsymbol{x} + \mathbf{2}\boldsymbol{y} - \mathbf{2} = \mathbf{0}$ where $\boldsymbol{t}$ is a constant represents two lines.

Suggestions or Solutions
To the Problem in the Example 0

Put in one equation two lines $y = x$ and $y = -x$.

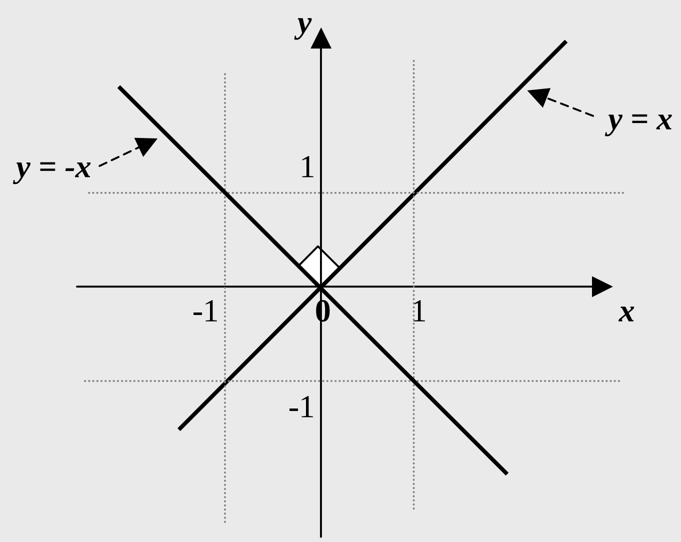
($y = -x$ or $y = x$) $\Rightarrow$ ($x + y = 0$ or $x - y = 0$) $\Rightarrow$ $(x + y)(x - y) = 0$, which is the equation we want, that is, the solution to this problem, and can indicate the two lines.

Let's see how the reverse works.

$(x + y)(x - y) = 0 \Rightarrow x^2 - xy + xy - y^2 = 0 \Rightarrow x^2 - y^2 = 0 \Rightarrow y^2 = x^2 \Rightarrow y = x$ or $y = -x$.

Thus, $y = -x$ or $y = x \Rightarrow (x + y)(x - y) = 0$, and also, $(x + y)(x - y) = 0 \Rightarrow y = -x$ or $y = x$.
In sum, we can put both at once this way: ($y = -x$ or $y = x$) $\Leftrightarrow$ $(x + y)(x - y) = 0$.

Fig. 0



Note that factorizations of (factoring) polynomials are often indispensable for doing algebra and solving equations.
If not quite familiar with it, refer to the book, **POLYNOMIAL FACTORIZATIONS**.

In short:

($y = -x$ or $y = x$) $\Rightarrow$ ($x + y = 0$ or $x - y = 0$) $\Rightarrow$ $(x + y)(x - y) = 0$.

Suggestions or Solutions
To the Problem in the Example 1

Put in one equation two lines $y = -x + 1$ and $y = x$.

$(y = -x + 1$ or $y = x) \Rightarrow (x + y - 1 = 0$ or $x - y = 0) \Rightarrow (x + y - 1)(x - y) = 0$, which is the equation we want, and can indicate the two lines at once.

Let's have a look at though, how the reverse works.

$(x + y - 1)(x - y) = 0 \Rightarrow x^2 - xy + xy - y^2 - x + y = 0 \Rightarrow x^2 - y^2 - x + y = 0$

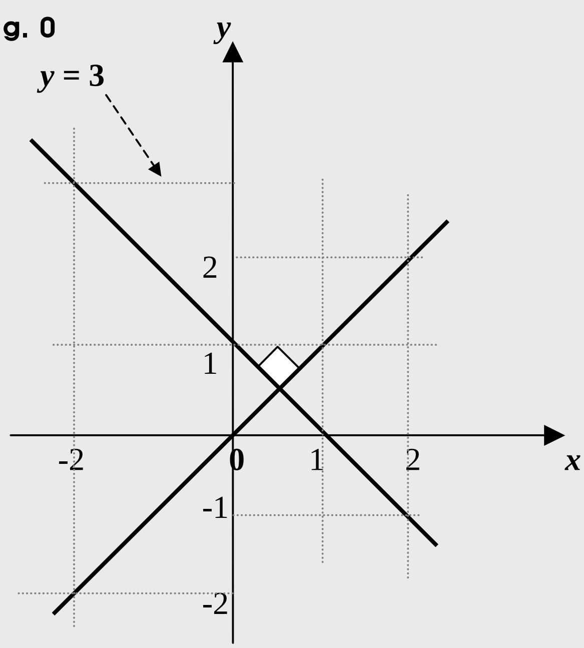
$\Rightarrow x^2 - x = y^2 - y \Rightarrow y^2 - y + \frac{1}{4} - \frac{1}{4} = x^2 - x + \frac{1}{4} - \frac{1}{4} \Rightarrow (y - \frac{1}{2})^2 - \frac{1}{4} = (x - \frac{1}{2})^2 - \frac{1}{4}$

$\Rightarrow (y - \frac{1}{2})^2 = (x - \frac{1}{2})^2 \Rightarrow y - \frac{1}{2} = \pm(x - \frac{1}{2}) \Rightarrow y = \frac{1}{2} \pm (x - \frac{1}{2})$

$\Rightarrow y = \frac{1}{2} + x - \frac{1}{2} = x$ or $y = \frac{1}{2} - x + \frac{1}{2} = -x + 1$

$\Rightarrow y = x$ or $y = -x + 1$. So we get $(y = -x + 1$ or $y = x) \Leftrightarrow (x + y - 1)(x - y) = 0$.

Fig. 0



In short:

$(y = -x + 1$ or $y = x) \Rightarrow (x + y - 1 = 0$ or $x - y = 0) \Rightarrow (x + y - 1)(x - y) = 0$.

Suggestions or Solutions
To the Problem in the Example 2

An equation $2x^2 - 2y^2 - 3xy - x + 2y = 0$ indicates two lines. Find the two lines.

$$2x^2 - 2y^2 - 3xy - x + 2y = 2x^2 - 3xy - x - 2y^2 + 2y = x(2x - 3y - 1) - 2y(y - 1)$$

$$= x(2x - 4y + y - 1) - 2y(y - 1) = x(2x - 4y) + x(y - 1) - 2y(y - 1)$$

$$= 2x(x - 2y) + (x - 2y)(y - 1) = (x - 2y)(2x + y - 1) \Rightarrow 2x + y - 1 = 0 \text{ and } x - 2y = 0.$$

If not quite sure of the idea behind the processes above, follow the steps below.

The solution is a matter of factorization.
In fact, solutions to most equations are matters of factorization.
The factorization in this case is much easier than factorization in ordinary circumstances where we solve algebraic equations.

That's because we've already known the form of the factors. Each of the factors is an equation of a line.

Therefore, we can assume the factors to be $ax + by + c$ and $dx + ey + f$, for instance.

So factorizing the equation given, we get

$$2x^2 - 2y^2 - 3xy - x + 2y = 2x^2 - 3xy - x - 2y^2 + 2y = x(2x - 3y - 1) - 2y(y - 1)$$

$$= x(2x - 4y + y - 1) - 2y(y - 1) = x(2x - 4y) + x(y - 1) - 2y(y - 1)$$

$$= 2x(x - 2y) + (x - 2y)(y - 1) = (x - 2y)(2x + y - 1).$$

Therefore, the two lines are $x - 2y = 0$, and $2x + y - 1 = 0$.

What if however, we can't factorize it the way above?

Then, we can try the complete square method.

So let's this time, try the complete square method.

$$2x^2-2y^2-3xy-x+2y=2x^2-(1+3y)x-2y^2+2y=2(x^2-\tfrac{(1+3y)x}{2})-2y^2+2y$$

$$=2(x^2-\tfrac{(1+3y)x}{2}+\tfrac{(1+3y)^2}{16}-\tfrac{(1+3y)^2}{16})-2y^2+2y=2(x-\tfrac{1+3y}{4})^2-\tfrac{(1+3y)^2}{8}-2y^2+2y.$$

Meanwhile, $-\tfrac{(1+3y)^2}{8}-2y^2+2y=-\tfrac{1++6y+9y^2}{8}-2y^2+2y=-\tfrac{1}{8}-\tfrac{3y}{4}-\tfrac{9y^2}{8}-2y^2+2y$

$$=-\tfrac{25y^2}{8}+\tfrac{5y}{4}-\tfrac{1}{8}=-\tfrac{1}{8}(25y^2-10y+1)=-\tfrac{1}{8}(5y-1)^2.$$

Thus, we get

$$2x^2-2y^2-3xy-x+2y=2(x-\tfrac{1+3y}{4})^2-\tfrac{1}{8}(5y-1)^2=\tfrac{1}{8}\{16(x-\tfrac{1+3y}{4})^2-(5y-1)^2\}.$$

And we have a factorization identity, $\boldsymbol{A^2-B^2=(A+B)(A-B)}$.

So we get $16(x-\tfrac{1+3y}{4})^2-(5y-1)^2=\{4(x-\tfrac{1+3y}{4})+5y-1\}\{4(x-\tfrac{1+3y}{4})-5y+1\}$

$$=(4x-1-3y+5y-1)(4x-1-3y-5y+1)=(4x+2y-2)(4x-8y)$$

$$=2(2x+y-1)4(x-2y)=8(2x+y-1)(x-2y).$$

Thus, we get

$$\tfrac{1}{8}\{16(x-\tfrac{1+3y}{4})^2-(5y-1)^2\}=\tfrac{1}{8}\cdot 8(2x+y-1)(x-2y)=(2x+y-1)(x-2y).$$

Therefore, we get $\boldsymbol{2x^2-2y^2-3xy-x+2y=(2x+y-1)(x-2y)}$.

In short:

$$2x^2-2y^2-3xy-x+2y=2x^2-3xy-x-2y^2+2y=x(2x-3y-1)-2y(y-1)$$

$$=x(2x-4y+y-1)-2y(y-1)=x(2x-4y)+x(y-1)-2y(y-1)$$

$$=2x(x-2y)+(x-2y)(y-1)=(x-2y)(2x+y-1)\Rightarrow 2x+y-1=0 \text{ and } x-2y=0.$$

Suggestions or Solutions
To the Problem in the Example 3

Find t for which an equation $2x^2 + 4xy - ty^2 + 4x + 2y - 2 = 0$ where t is a constant represents two lines.

Assuming first, $\boldsymbol{P = 2x^2 + 4xy - ty^2 + 4x + 2y - 2}$, and putting $\boldsymbol{P}$ in terms of $\boldsymbol{x}$, we get
$\boldsymbol{P = 2x^2 + (4y + 4)x - ty^2 + 2y - 2}$.

Meanwhile, $\boldsymbol{2x^2 + (4y + 4)x = 2\{x^2 + 2(y + 1)x\} = 2\{x^2 + 2(y + 1)x + (y + 1)^2 - (y + 1)^2\}}$
$\boldsymbol{= 2\{x + (y + 1)\}^2 - 2(y + 1)^2}$.

So we get $\boldsymbol{P = 2(x + y + 1)^2 - \{2(y + 1)^2 + ty^2 - 2y + 2\}}$.

Assuming next, $\boldsymbol{Q = 2(y + 1)^2 + ty^2 - 2y + 2}$, and putting $\boldsymbol{Q}$ in terms of $\boldsymbol{y}$, we get
$\boldsymbol{Q = 2(y + 1)^2 + ty^2 - 2y + 2 = 2(y^2 + 2y + 1) + ty^2 - 2y + 2 = (t + 2)y^2 + 2y + 4}$.

Thus, we get $\boldsymbol{P = 2(x + y + 1)^2 - Q}$.

If $\boldsymbol{Q}$ is a complete square, the polynomial $\boldsymbol{P}$ can get factorized to a product of two factors, each of which is a polynomial of first degree in each of $\boldsymbol{x}$ and $\boldsymbol{y}$, and thus, can be used in an equation of a line.

If $\boldsymbol{Q}$ can be put in a complete square, its discriminant has to be 0.
The discriminant is $\boldsymbol{1 - (t + 2)4 = -4t - 7}$, which has to be 0. Therefore, $\boldsymbol{t = -\frac{7}{4}}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Assuming first, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are constant, we can set up an equation as follows.
$\boldsymbol{(ax + by + c)(ux + vy + w) = 0}$, which can indicate at the same time, two lines as follows.
one is $\boldsymbol{ax + by + c = 0}$, and the other is $\boldsymbol{ux + vy + w = 0}$.

Assuming now, $\boldsymbol{P}$ is the polynomial $\boldsymbol{2x^2 + 4xy - ty^2 + 4x + 2y - 2}$ in the equation above, and putting somehow $\boldsymbol{P}$ in the form of $\boldsymbol{(ax + by + c)(ux + vy + w)}$, we can extract the two lines the equation $\boldsymbol{P = 0}$ indicates.

That is, we can try factorizing the polynomial $\boldsymbol{P}$ to an expression, which is in a form of a product of two simple polynomials, each of which is of degree 1 in each of $\boldsymbol{x}$ and $\boldsymbol{y}$.

The two simple polynomials are the factors, of course, and each of the two can be used in an equation of a line. What about $\boldsymbol{t}$, though?

During the factorization, we get to determine the $\boldsymbol{t}$-value. That is, we want to determine the $\boldsymbol{t}$-value so that $\boldsymbol{P}$ can be factorized that way. How then, do we do the factorization?

The polynomial $\boldsymbol{P}$ is a polynomial of degree 2 in each of $\boldsymbol{x}$ and $\boldsymbol{y}$. So what?

We can try the complete square method so that we can convert $\boldsymbol{P}$ into a polynomial in a form of $\boldsymbol{(ax + by + c)^2 - (ux + vy + w)^2}$, which can be readily factorized to a polynomial as follows. $\boldsymbol{(ax + by + c + ux + vy + w)(ax + by + c - ux - vy - w)}$, which is the form we want.

So to begin with, putting $\boldsymbol{P}$ in terms of $\boldsymbol{x}$, we get $\boldsymbol{P = 2x^2 + (4y + 4)x - ty^2 + 2y - 2}$.

Meanwhile, $\boldsymbol{2x^2 + (4y + 4)x = 2\{x^2 + 2(y + 1)x\} = 2\{x^2 + 2(y + 1)x + (y + 1)^2 - (y + 1)^2\}}$
$\boldsymbol{= 2\{x + (y + 1)\}^2 - 2(y + 1)^2 = 2(x + y + 1)^2 - 2(y + 1)^2}$.

So we get

$\boldsymbol{P = 2(x + y + 1)^2 - 2(y + 1)^2 - ty^2 + 2y - 2 = 2(x + y + 1)^2 - \{2(y + 1)^2 + ty^2 - 2y + 2\}}$.

Suppose now, $\boldsymbol{Q = 2(y + 1)^2 + ty^2 - 2y + 2}$. Then, we get $\boldsymbol{P = 2(x + y + 1)^2 - Q}$.

So if we can convert the polynomial $\boldsymbol{Q}$ into a complete square, the polynomial $\boldsymbol{P}$ can be put in a form of $\boldsymbol{A^2 - B^2}$, which gets factorized to $\boldsymbol{(A + B)(A - B)}$.
The conversion depends on the $\boldsymbol{t}$-value, though.

So we are now, going to determine the $\boldsymbol{t}$-value so that the polynomial $\boldsymbol{Q}$ can be put in a complete square. To begin with, putting the polynomial $\boldsymbol{Q}$ in terms of $\boldsymbol{y}$, we get

$\boldsymbol{Q = 2(y + 1)^2 + ty^2 - 2y + 2 = 2(y^2 + 2y + 1) + ty^2 - 2y + 2 = (t + 2)y^2 + 2y + 4}$.

Now, if the polynomial $\boldsymbol{Q = (t + 2)y^2 + 2y + 4}$ gets converted into a complete square, its discriminant has to be 0.

The discriminant is what's inside the radical sign in the quadratic formula.
Suppose for instance, $ax^2 + bx + c = 0$. Then, the quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So the discriminant is $b^2 - 4ac$, which is often called D. If in particular, b above is even, we usually take $\frac{D}{4}$ instead of just D. Then, we get $\frac{D}{4} = \frac{b^2-4ac}{4} = \frac{b^2}{4} - ac$.

If the discriminant is 0, the solution is $-\frac{b}{2a}$ only, so the polynomial $ax^2 + bx + c$ can be in a complete square. And in fact, we have $ax^2 + bx + c = (x + \frac{b}{2a})^2$ if $b^2 - 4ac = 0$.

Now, the discriminant of the polynomial Q is $\frac{D}{4} = 1 - (t + 2)4 = 1 - 4t - 8 = -4t - 7$, which has to be 0, so we get $-4t - 7 = 0 \Rightarrow t = -\frac{7}{4}$.

So if $t = -\frac{7}{4}$, the polynomial $2x^2 + 4xy - ty^2 + 4x + 2y - 2$ can be factorized to a product of two polynomials, each of which can be used in an equation of a line.

That is, $2x^2 + 4xy + \frac{7}{4}y^2 + 4x + 2y - 2 = 0$ is an equation of two lines.

Let's see now, what the two lines look like. Then first, we get

$$t = -\frac{7}{4} \Rightarrow Q = (t+2)y^2 + 2y + 4 = \frac{1}{4}y^2 + 2y + 4 = \frac{1}{4}(y^2 + 8y + 16) = \frac{1}{4}(y+4)^2.$$

So we get $P = 2(x + y + 1)^2 - Q = 2(x + y + 1)^2 - \frac{1}{4}(y+4)^2$.

And we have $A^2 - B^2 = (A + B)(A - B)$, too.

So we get $P = \{\sqrt{2}(x+y+1) + \frac{1}{2}(y+4)\}\{\sqrt{2}(x+y+1) - \frac{1}{2}(y+4)\}$.

Meanwhile,

$$\sqrt{2}(x+y+1) + \frac{1}{2}(y+4) = \sqrt{2}x + (\sqrt{2} + \frac{1}{2})y + \sqrt{2} + 2.$$

$$\sqrt{2}(x+y+1) - \frac{1}{2}(y+4) = \sqrt{2}x + (\sqrt{2} - \frac{1}{2})y + \sqrt{2} - 2.$$

Thus, setting $P = 0$, we get $(\sqrt{2}x + \frac{2\sqrt{2}+1}{2}y + \sqrt{2} + 2)(\sqrt{2}x + \frac{2\sqrt{2}-1}{2}y + \sqrt{2} - 2) = 0$.

So we can get two equations, each of which indicates a line.

One is $\sqrt{2}x + \frac{2\sqrt{2}+1}{2}y + \sqrt{2} + 2 = 0$, and the other is $\sqrt{2}x + \frac{2\sqrt{2}-1}{2}y + \sqrt{2} - 2 = 0$.

Putting them in the point-slope form, we get

First, $\sqrt{2}x + \frac{2\sqrt{2}+1}{2}y + \sqrt{2} + 2 = 0 \Rightarrow y = \frac{-2}{2\sqrt{2}+1}(\sqrt{2}x + \sqrt{2} + 2) = \frac{-2\sqrt{2}}{2\sqrt{2}+1}(x + 1 + \sqrt{2})$.

Meanwhile, $\frac{-2\sqrt{2}}{2\sqrt{2}+1} = \frac{-2\sqrt{2}(2\sqrt{2}-1)}{7} = \frac{-8+2\sqrt{2}}{7}$. Thus, we get $y = \frac{-8+2\sqrt{2}}{7}(x + 1 + \sqrt{2})$.

Next, $\sqrt{2}x + \frac{2\sqrt{2}-1}{2}y + \sqrt{2} - 2 = 0 \Rightarrow y = \frac{-2}{2\sqrt{2}-1}(\sqrt{2}x + \sqrt{2} - 2) = \frac{-2\sqrt{2}}{2\sqrt{2}-1}(x + 1 - \sqrt{2})$.

Meanwhile, $\frac{-2\sqrt{2}}{2\sqrt{2}-1} = \frac{-2\sqrt{2}(2\sqrt{2}+1)}{7} = \frac{-8-2\sqrt{2}}{7}$. Thus, we get $y = \frac{-8-2\sqrt{2}}{7}(x + 1 - \sqrt{2})$.

In short:

Assuming first, $P = 2x^2 + 4xy - ty^2 + 4x + 2y - 2$, and putting P in terms of x, we get $P = 2x^2 + (4y + 4)x - ty^2 + 2y - 2$.

Meanwhile, $2x^2 + (4y + 4)x = 2\{x^2 + 2(y + 1)x\} = 2\{x^2 + 2(y + 1)x + (y + 1)^2 - (y + 1)^2\}$
$= 2\{x + (y + 1)\}^2 - 2(y + 1)^2$.

So we get $P = 2(x + y + 1)^2 - \{2(y + 1)^2 + ty^2 - 2y + 2\}$.

Assuming next, $Q = 2(y + 1)^2 + ty^2 - 2y + 2$, and putting Q in terms of y, we get
$Q = 2(y + 1)^2 + ty^2 - 2y + 2 = 2(y^2 + 2y + 1) + ty^2 - 2y + 2 = (t + 2)y^2 + 2y + 4$.

Thus, we get $P = 2(x + y + 1)^2 - Q$.

If Q is a complete square, the polynomial P can get factorized to a product of two factors, each of which is a polynomial of first degree in each of x and y, and thus, can be used in an equation of a line.

If Q can be put in a complete square, its discriminant has to be 0.

The discriminant is $1 - (t + 2)4 = -4t - 7$, which has to be 0. Therefore, $t = -\frac{7}{4}$.