Examples G in Lines

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0. Find a point $\boldsymbol{P(s, t)}$ that is the same distance away from both the two lines as follows.

$$\mathbf{2x - y - 1 = 0} \text{ and } \mathbf{3x + 2y - 2 = 0}.$$

And that's not it, of course.
In this set of examples, we are going to cover how to come up with the formula for the distance from a point to a line, and will cover how to get an area of a triangle using three points given.

Suggestions or Solutions
To the Problem in the Example 0

Find a point $P(s, t)$ that is the same distance away from both the two lines as follows.
$2x - y - 1 = 0$ and $3x + 2y - 2 = 0$.

The distance from a point $P(s, t)$ to a line $ax + by + c = 0$ is $\frac{|as+bt+c|}{\sqrt{a^2+b^2}}$.

Suppose A is the line $2x - y - 1 = 0$, and B is the line $3x + 2y - 2 = 0$.

Then first, the distance d from the point P to the line A is $\frac{|2s-t-1|}{\sqrt{2^2+(-1)^2}} = \frac{|2s-t-1|}{\sqrt{5}}$.

And next, the distance d from the point P to the line B is $\frac{|3s+2t-2|}{\sqrt{3^2+2^2}} = \frac{|3s+2t-2|}{\sqrt{13}}$.

So we get $\frac{|2s-t-1|}{\sqrt{5}} = \frac{|3s+2t-2|}{\sqrt{13}} \Rightarrow \frac{2s-t-1}{\sqrt{5}} = \pm\frac{3s+2t-2}{\sqrt{13}}$.

The last expression above is the connective expression between coordinates of the point P, and is from the fact that P is the same distance away from the two lines A and B.

All the lines are in the x-y system, so replacing s and t with x and y, we get an equation indicating two lines, where every point is an equal distance away from both of the lines A and B, and thus, can be the point P.

One of the two lines is $\frac{2x-y-1}{\sqrt{5}} + \frac{3x+2y-2}{\sqrt{13}} = 0$, and the other is $\frac{2x-y-1}{\sqrt{5}} - \frac{3x+2y-2}{\sqrt{13}} = 0$.

And simplifying the two equations, we get

$$\frac{2x-y-1}{\sqrt{5}} + \frac{3x+2y-2}{\sqrt{13}} = 0 \Rightarrow \sqrt{13}(2x-y-1) + \sqrt{5}(3x+2y-2) = 0$$

$$\Rightarrow (2\sqrt{13}+3\sqrt{5})x + (2\sqrt{5}-\sqrt{13})y - (2\sqrt{5}+\sqrt{13}) = 0.$$

$$\frac{2x-y-1}{\sqrt{5}} - \frac{3x+2y-2}{\sqrt{13}} = 0 \Rightarrow \sqrt{13}(2x-y-1) - \sqrt{5}(3x+2y-2) = 0$$

$$\Rightarrow (2\sqrt{13}-3\sqrt{5})x - (2\sqrt{5}+\sqrt{13})y + (2\sqrt{5}-\sqrt{13}) = 0.$$

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, what is the distance from a point to a line? What do we mean by the distance from the point ***P*** to both the two lines?

Taking the distance from a point ***P*** to a line ***L***, for instance, we take the length of the shortest line segment connecting the point ***P*** and the line ***L***.

Such a line segment is perpendicular to the line ***L***, and in fact, connects two points. One is the point ***P***, and the other is one of the points in the line ***L***.

So the distance is the length of the line segment that begins with the point ***P***, is perpendicular to the line, and ends at a point in the line ***L***, of course.

Now, suppose ***A*** is the line $\mathbf{2x - y - 1 = 0}$, and ***B*** is the line $\mathbf{3x + 2y - 2 = 0}$.

Then, taking the distance from the point $\mathbf{P(s, t)}$ to the line ***A***, we first, need to find the line containing the point ***P*** and perpendicular to the line ***A***.

The same is true for the distance from the point ***P*** to the line ***B***, also.

Suppose now, ***C*** is the line perpendicular to the line ***A***, ***D*** is the line perpendicular to the line ***B***, and the two lines ***C*** and ***D*** pass through the point ***P***. Then, where do the two lines ***C*** and ***D*** meet each other?

The lines ***C*** and ***D*** meet at the point ***P***, of course, which is the same distance away from each of the two lines ***A*** and ***B***.

Fig. 0

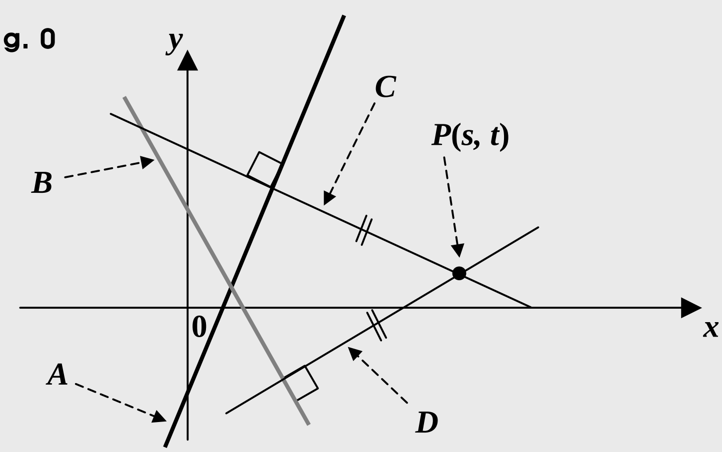
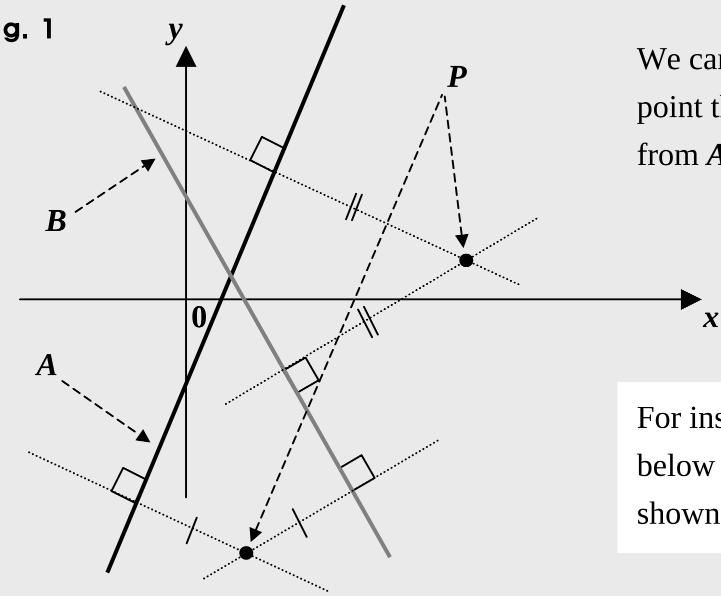


Fig. 1



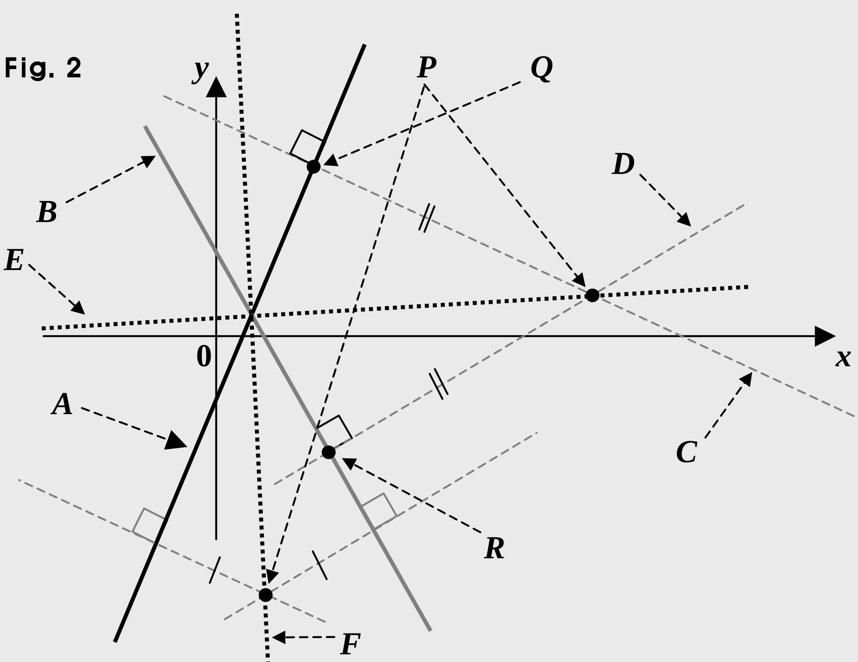
We can see however, another point that is equal distance away from ***A*** and ***B*** both.

For instance, it can be a point below the ***x***-axis, which is shown on the left.

In fact, there can be infinitely many points, each of which is equal distance away from both of two the lines ***A*** and ***B***. So there are infinitely many points each of which can be the point ***P***. What exactly though, are all those points?

All such points constitute two lines. And indicating the two lines by two dotted lines, we can put the two dotted in the graph the way below.

Fig. 2



So we want to find the two dotted lines, that is, the equations of the two dotted lines. Suppose now, ***E*** and ***F*** are the dotted lines.

Then, the point ***P*(*s*, *t*)** is not just a particular point. But the point ***P*** represents all the points in each of the lines ***E*** and ***F***, too.
So every point in ***E*** and ***F*** can be the point ***P*(*s*, *t*)**.

We can approach the two lines ***E*** and ***F*** in such a way as follows.

Suppose first, in the graph above, ***C*** is a line perpendicular to the line ***A***, and ***D*** is a line perpendicular to the line ***B***. Then, we can get the two lines ***E*** and ***F*** if finding two points. What two points though?

At one of the two points, the line ***A*** meets the line ***C***, and at the other, the line ***B*** meets the line ***D***. How then, can we get the two lines ***E*** and ***F***?

We can get the lines ***E*** and ***F*** by means of the fact that the point ***P*** is the same distance away from each of ***A*** and ***B***. We want to find however, the two lines ***C*** and ***D***, first.

Now, finding ***C*** and ***D***, we can begin with two facts below.

- The line ***C*** is perpendicular to the line ***A***, and the line ***D*** is perpendicular to the line ***B***.
- If two lines are perpendicular to each other, the product of their slopes is -1.

So let's first, get the slopes of the lines ***A*** and ***B***. Then, we get

The line ***A***: $\boldsymbol{2x - y - 1 = 0} \Rightarrow \boldsymbol{y = 2x - 1} \Rightarrow$ the slope of the line ***A*** is 2.

The line ***B***: $\boldsymbol{3x + 2y - 2 = 0} \Rightarrow \boldsymbol{y = -\frac{3}{2}x + 1} \Rightarrow$ the slope of the line ***B*** is $-\frac{3}{2}$.

Thus, the slope of the line ***C*** is $-\frac{1}{2}$, since it is perpendicular to the line ***A***.
So the line ***C*** is $\boldsymbol{y - t = -\frac{1}{2}(x - s)}$, since it includes the point ***P*(*s*, *t*)**
And putting the line ***C*** in the point-slope form, we get $\boldsymbol{y = -\frac{1}{2}x + t + \frac{s}{2}}$.

Next, the slope of the line ***D*** is $\frac{2}{3}$ since it is perpendicular to the line ***B***.
So the line ***D*** is $y - t = \frac{2}{3}(x - s)$ since the point ***P*(*s*, *t*)** is in the line ***D***, too.
And putting the line ***D*** in the point-slope form, we get $y = \frac{2}{3}x + t - \frac{2s}{3}$.

Next, assuming ***Q*** is the point where the line ***A*** meets the line ***C***, find the point ***Q***.
The line ***A*** is $y = 2x - 1$, and the line ***C*** is $y = -\frac{1}{2}x + t + \frac{s}{2}$.
So beginning with the ***x***-coordinate of ***Q***, we get
$2x - 1 = -\frac{1}{2}x + t + \frac{s}{2} \Rightarrow \frac{5}{2}x = t + \frac{s}{2} + 1 = \frac{2t+s+2}{2} \Rightarrow x = \frac{2t+s+2}{5}$.
So next, the ***y***-coordinate is $y = 2x - 1 = \frac{2(2t+s+2)}{5} - 1 = \frac{4t+2s+4-5}{5} = \frac{4t+2s-1}{5}$.
Therefore, the point ***Q*** is at $(\frac{2t+s+2}{5}, \frac{4t+2s-1}{5})$, which is in the line ***A***.

Next, assuming that ***R*** is the point where the line ***B*** meets the line ***D***, find the point ***R***.
The line ***B*** is $y = -\frac{3}{2}x + 1$, and the line ***D*** is $y = \frac{2}{3}x + t - \frac{2s}{3}$.
Thus, beginning with the ***x***-coordinate, we get
$-\frac{3}{2}x + 1 = \frac{2}{3}x + t - \frac{2s}{3} \Rightarrow (\frac{2}{3} + \frac{3}{2})x = 1 - t + \frac{2s}{3} \Rightarrow \frac{13}{6}x = \frac{2s-3t+3}{3} = \frac{4s-6t+6}{6} \Rightarrow x = \frac{4s-6t+6}{13}$.
So next, the ***y***-coordinate is $y = -\frac{3}{2}x + 1 = -\frac{3}{2} \cdot \frac{4s-6t+6}{13} + 1 = \frac{-3(2s-3t+3)}{13} + 1 = \frac{-6s+9t-9+13}{13}$.
Therefore, the point ***R*** is at $(\frac{4s-6t+6}{13}, \frac{-6s+9t+4}{13})$, which is in the line ***B***.

Now, we can get the two lines ***E*** and ***F*** by means of the same distance from ***P*** to each of the two lines ***A*** and ***B***. And getting them, we are going to use two facts below, too.

- The distance between ***P*** and ***Q*** is the same as the one between ***P*** and ***R***.
- The formula for such a distance is $d^2 = (\Delta x)^2 + (\Delta y)^2$, where ***d*** is the distance.

So assuming d_1 is the distance from ***P*** to ***Q***, and d_2 is the one from ***P*** to ***R***, and beginning with d_1, we get $\Delta x = s - \frac{2t+s+2}{5} = \frac{5s-2t-s-2}{5} = \frac{4s-2t-2}{5}$, $\Delta y = t - \frac{4t+2s-1}{5} = \frac{5t-4t-2s+1}{5} = \frac{t-2s+1}{5}$.
So we get $d_1{}^2 = (\frac{4s-2t-2}{5})^2 + (\frac{t-2s+1}{5})^2 = \frac{1}{25}\{(4s-2t-2)^2 + (t-2s+1)^2\}$. Meanwhile,
$(t - 2s + 1)^2 = \{-(2s - t - 1)\}^2 = (2s - t - 1)^2$, and $(4s - 2t - 2)^2 = 2^2(2s - t - 1)^2$.
So we get $d_1{}^2 = \frac{1}{25}\{(4s-2t-2)^2 + (2s-t-1)^2\} = \frac{5(2s-t-1)^2}{25} = \frac{(2s-t-1)^2}{5}$.

Moving next, on to the other distance $\boldsymbol{d_2}$, we get

$\Delta\boldsymbol{x} = \boldsymbol{s} - \frac{4s-6t+6}{13} = \frac{13s-4s+6t-6}{13} = \frac{9s+6t-6}{13}$, and $\Delta\boldsymbol{y} = \boldsymbol{t} - \frac{-6s+9t+4}{13} = \frac{13t+6s-9t-4}{13} = \frac{6s+4t-4}{13}$.

Meanwhile, $\frac{9s+6t-6}{13} = \frac{3(3s+2t-2)}{13}$, and $\frac{6s+4t-4}{13} = \frac{2(3s+2t-2)}{13}$.

So we get $\boldsymbol{d_2}^2 = (\frac{3(3s+2t-2)}{13})^2 + (\frac{2(3s+2t-2)}{13})^2 = \frac{13(3s+2t-2)^2}{13^2} = \frac{(3s+2t-2)^2}{13}$.

And thus now, since we have $\boldsymbol{d_1}^2 = \boldsymbol{d_2}^2$, we get $\frac{(2s-t-1)^2}{5} = \frac{(3s+2t-2)^2}{13}$.

Now, the final equation above is the connective equation between the coordinates of the point ***P*(*s, t*)**, and is from the fact that ***P*** is the same distance away from ***A*** and ***B***.

We know the point ***P*** represents all the points in the two lines ***E*** and ***F***, where each of all the points is an equal distance away from both the lines ***A*** and ***B***.

So the final equation above is the equation that indicates the two lines ***E*** and ***F***.
Both lines ***E*** and ***F*** are in the ***x-y*** coordinate system, so replacing ***s*** and ***t*** with ***x*** and ***y*** in the equation above, we can put the equation in the ***x-y*** system. That is to say that we get

$\frac{(2x-y-1)^2}{5} = \frac{(3x+2y-2)^2}{13}$.

Then, factorizing the equation above, we can extract the two lines ***E*** and ***F***.   How?

We can readily factorize the equation into the form as follows,
$\boldsymbol{(ax + by + c)(dx + ey + f) = 0}$ where ***a***, ***b***, ***c***, ***d***, ***e***, and ***f*** are constant.

And we have a factorization identity, $\boldsymbol{A^2 - B^2 = (A + B)(A - B)}$.   So we get

$\frac{(2x-y-1)^2}{5} = \frac{(3x+2y-2)^2}{13} \Rightarrow \frac{(2x-y-1)^2}{5} - \frac{(3x+2y-2)^2}{13} = (\frac{2x-y-1}{\sqrt{5}} + \frac{3x+2y-2}{\sqrt{13}})(\frac{2x-y-1}{\sqrt{5}} - \frac{3x+2y-2}{\sqrt{13}}) = \mathbf{0}$.

Now, the equation above indicates the two lines ***E*** and ***F***.
How then, do we know which one is for the line ***E*** or ***F***?

Looking at the graph above, we can see that the line ***E*** has a positive slope, and the line ***F*** has a negative slope.

So the line ***E*** is $\frac{2x-y-1}{\sqrt{5}} - \frac{3x+2y-2}{\sqrt{13}} = \mathbf{0}$, and the line ***F*** is $\frac{2x-y-1}{\sqrt{5}} + \frac{3x+2y-2}{\sqrt{13}} = \mathbf{0}$.

We will simplify the equations later.

• Next, let's this time, work out this example by means of another distance formula.

The distance formula we are going to use is not for a distance between two points but for a distance from a point to a line.

We often need to find a distance from a point to a line, so we have a formula for it.
We are going to however, find out first, how the formula gets constructed. That is in fact, another purpose of this example.

The construction procedure is quite straightforward though, and is as follows,

• Suppose $\boldsymbol{L}$ is a line $\boldsymbol{ax + by + c = 0}$ where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are constant, and $\boldsymbol{D}$ is the distance from a point $\boldsymbol{P(s, t)}$ to the line $\boldsymbol{L}$.

• Then first, get the line passing through the point $\boldsymbol{P}$ and perpendicular to the line $\boldsymbol{L}$.

• Next, get the point where the line $\boldsymbol{L}$ meets the perpendicular line.

• Then, using the formula for a distance between two points, get the distance from $\boldsymbol{P}$ to the point. Then, the distance is $\boldsymbol{D}$, the very distance from the point $\boldsymbol{P}$ to the line $\boldsymbol{L}$.

Suppose now, $\boldsymbol{N}$ is the line passing through $\boldsymbol{P(s, t)}$ and perpendicular to the line $\boldsymbol{L}$.
Then first, let's get the slope of the line $\boldsymbol{L}$ to get the slope of the line $\boldsymbol{N}$.

$\boldsymbol{ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}} \Rightarrow$ the slope of $\boldsymbol{L}$ is $-\frac{a}{b}$, so the slope of the line $\boldsymbol{N}$ is $\frac{b}{a}$.

Thus, the line $\boldsymbol{N}$ is $\boldsymbol{y - t = \frac{b}{a}(x - s)}$, since the point $\boldsymbol{P(s, t)}$ is in the line $\boldsymbol{N}$.

Next, let's get the point where the line $\boldsymbol{N}$ meets the line $\boldsymbol{L}$.

We have $\boldsymbol{y - t = \frac{b}{a}(x - s)}$, and $\boldsymbol{y = -\frac{a}{b}x - \frac{c}{b}}$.

So getting the $\boldsymbol{x}$-coordinate first, we get $\boldsymbol{y - t = (-\frac{a}{b}x - \frac{c}{b}) - t = \frac{b}{a}(x - s)}$

$\boldsymbol{\Rightarrow -\frac{a}{b}x - \frac{c}{b} - t = \frac{b}{a}(x - s) \Rightarrow -a^2x - ac - abt = b^2(x - s)}$

$\boldsymbol{\Rightarrow x(a^2 + b^2) = sb^2 - ac - abt \Rightarrow x = \frac{sb^2 - ac - abt}{a^2 + b^2}}$.

Next, moving on to the $\boldsymbol{y}$-coordinate, we get

$$y-t=\frac{b}{a}(x-s)\Rightarrow y-t=\frac{b}{a}\left(\frac{sb^2-ac-abt}{a^2+b^2}-s\right)=\frac{b}{a}\cdot\frac{sb^2-ac-abt-sa^2-sb^2}{a^2+b^2}=\frac{b}{a}\cdot\frac{-sa^2-ac-abt}{a^2+b^2}$$

$$=\frac{-sab-bc-b^2t}{a^2+b^2}\Rightarrow y=\frac{-sab-bc-b^2t}{a^2+b^2}+t=\frac{-sab-bc-b^2t+ta^2+tb^2}{a^2+b^2}=\frac{-sab-bc+ta^2}{a^2+b^2}.$$

So the line $\boldsymbol{L}$ meets the line $\boldsymbol{N}$ at $\left(\frac{sb^2-ac-abt}{a^2+b^2},\frac{ta^2-bc-abs}{a^2+b^2}\right)$.

Suppose now, $\boldsymbol{Q(u, v)}$ is the point where the line $\boldsymbol{L}$ meets the line $\boldsymbol{N}$.

Then, we get $u=\frac{sb^2-ac-abt}{a^2+b^2}$, and $v=\frac{ta^2-bc-abs}{a^2+b^2}$.

Suppose next, $\boldsymbol{d}$ is the distance from the point $\boldsymbol{P(s, t)}$ to the point $\boldsymbol{Q(u, v)}$.
Then, we get $d^2=(u-s)^2+(v-t)^2$. So let's get $(u-s)$ and $(v-t)$. Then,

$$u-s=\frac{sb^2-ac-abt}{a^2+b^2}-s=\frac{sb^2-ac-abt-sa^2-sb^2}{a^2+b^2}=\frac{-sa^2-ac-abt}{a^2+b^2}=\frac{-a(sa+c+bt)}{a^2+b^2}.$$

$$v-t=\frac{ta^2-bc-abs}{a^2+b^2}-t=\frac{ta^2-bc-abs-ta^2-tb^2}{a^2+b^2}=\frac{-tb^2-bc-abs}{a^2+b^2}=\frac{-b(tb+c+as)}{a^2+b^2}.$$

So we get $d^2=(u-s)^2+(v-t)^2=\frac{a^2(as+bt+c)^2}{(a^2+b^2)^2}+\frac{b^2(as+bt+c)^2}{(a^2+b^2)^2}=\frac{(a^2+b^2)(as+bt+c)^2}{(a^2+b^2)^2}=\frac{(as+bt+c)^2}{a^2+b^2}$.

Thus, we get $d=\frac{\pm(as+bt+c)}{\sqrt{a^2+b^2}}$. However, $\boldsymbol{d}$ is a distance, so we get $d\geq 0$.

Thus, $d=\dfrac{|as+bt+c|}{\sqrt{a^2+b^2}}$, which is $\boldsymbol{D}$, the very distance from the point $\boldsymbol{P(s, t)}$ to the line $\boldsymbol{L}$.

So it is the formula for a distance from a point $\boldsymbol{P(s, t)}$ to a line $ax+by+c=0$.

Now, with the formula above, let's get back to the problem in this example.

What we are after in this problem is the point $\boldsymbol{P(s, t)}$, which is an equal distance away from both of the line $\boldsymbol{A}$, $2x-y-1=0$ and the line $\boldsymbol{B}$, $3x+2y-2=0$.

Then, by means of the formula above, we get this:

The distance from the point $\boldsymbol{P}$ to the line $\boldsymbol{A}$ is $\frac{|2s-t-1|}{\sqrt{2^2+(-1)^2}}=\frac{|2s-t-1|}{\sqrt{5}}$.

The distance from the point $\boldsymbol{P}$ to the line $\boldsymbol{B}$ is $\frac{|3s+2t-2|}{\sqrt{3^2+2^2}} = \frac{|3s+2t-2|}{\sqrt{13}}$.

So we get $\frac{|2s-t-1|}{\sqrt{5}} = \frac{|3s+2t-2|}{\sqrt{13}} \Rightarrow \frac{2s-t-1}{\sqrt{5}} = \pm\frac{3s+2t-2}{\sqrt{13}}$. Why $\pm$?

That's because what's inside the absolute sign can be positive or negative.
Practically speaking though, that's because we get two lines. What two lines?

One line is $\frac{2x-y-1}{\sqrt{5}} + \frac{3x+2y-2}{\sqrt{13}} = \mathbf{0}$, and the other is $\frac{2x-y-1}{\sqrt{5}} - \frac{3x+2y-2}{\sqrt{13}} = \mathbf{0}$.

They are the two lines we have found earlier.
Now, simplifying the two equations, we get

$$\frac{2x-y-1}{\sqrt{5}} + \frac{3x+2y-2}{\sqrt{13}} = 0 \Rightarrow \sqrt{13}(2x-y-1) + \sqrt{5}(3x+2y-2) = 0$$

$$\Rightarrow (2\sqrt{13}+3\sqrt{5})x + (2\sqrt{5}-\sqrt{13})y - (2\sqrt{5}+\sqrt{13}) = 0.$$

$$\frac{2x-y-1}{\sqrt{5}} - \frac{3x+2y-2}{\sqrt{13}} = 0 \Rightarrow \sqrt{13}(2x-y-1) - \sqrt{5}(3x+2y-2) = 0$$

$$\Rightarrow (2\sqrt{13}-3\sqrt{5})x - (2\sqrt{5}+\sqrt{13})y + (2\sqrt{5}-\sqrt{13}) = 0.$$

In short:

One of the two methods:

The distance from a point $\boldsymbol{P(s, t)}$ to a line $\boldsymbol{ax + by + c = 0}$ is $\frac{|as+bt+c|}{\sqrt{a^2+b^2}}$.

Suppose $\boldsymbol{A}$ is the line $\mathbf{2x - y - 1 = 0}$, and $\boldsymbol{B}$ is the line $\mathbf{3x + 2y - 2 = 0}$.

Then first, the distance $\boldsymbol{d}$ from the point $\boldsymbol{P}$ to the line $\boldsymbol{A}$ is $\frac{|2s-t-1|}{\sqrt{2^2+(-1)^2}} = \frac{|2s-t-1|}{\sqrt{5}}$.

And next, the distance $\boldsymbol{d}$ from the point $\boldsymbol{P}$ to the line $\boldsymbol{B}$ is $\frac{|3s+2t-2|}{\sqrt{3^2+2^2}} = \frac{|3s+2t-2|}{\sqrt{13}}$.

So we get $\frac{|2s-t-1|}{\sqrt{5}} = \frac{|3s+2t-2|}{\sqrt{13}} \Rightarrow \frac{2s-t-1}{\sqrt{5}} = \pm\frac{3s+2t-2}{\sqrt{13}}$.

The last expression above is the connective expression between coordinates of the point $\boldsymbol{P}$, and is from the fact that $\boldsymbol{P}$ is the same distance away from the two lines $\boldsymbol{A}$ and $\boldsymbol{B}$.

All the lines are in the ***x-y*** system, so replacing ***s*** and ***t*** with ***x*** and ***y***, we get an equation indicating two lines, where every point is an equal distance away from both of the lines ***A*** and ***B***, and thus, can be the point ***P***.

One of the two lines is $\frac{2x-y-1}{\sqrt{5}}+\frac{3x+2y-2}{\sqrt{13}}=0$, and the other is $\frac{2x-y-1}{\sqrt{5}}-\frac{3x+2y-2}{\sqrt{13}}=0$.

And simplifying the two equations, we get

$$\frac{2x-y-1}{\sqrt{5}}+\frac{3x+2y-2}{\sqrt{13}}=0 \Rightarrow \sqrt{13}(2x-y-1)+\sqrt{5}(3x+2y-2)=0$$

$$\Rightarrow (2\sqrt{13}+3\sqrt{5})x+(2\sqrt{5}-\sqrt{13})y-(2\sqrt{5}+\sqrt{13})=0.$$

$$\frac{2x-y-1}{\sqrt{5}}-\frac{3x+2y-2}{\sqrt{13}}=0 \Rightarrow \sqrt{13}(2x-y-1)-\sqrt{5}(3x+2y-2)=0$$

$$\Rightarrow (2\sqrt{13}-3\sqrt{5})x-(2\sqrt{5}+\sqrt{13})y+(2\sqrt{5}-\sqrt{13})=0.$$

The other method:

Suppose first, ***A*** is the line $2x-y-1=0$, and ***B*** is the line $3x+2y-2=0$. Then,

$2x-y-1=0 \Rightarrow y=2x-1 \Rightarrow$ the slope of the line ***A*** is 2.

$3x+2y-2=0 \Rightarrow y=-\frac{3}{2}x+1 \Rightarrow$ the slope of the line ***B*** is $-\frac{3}{2}$.

Suppose next, that

C is the line containing the point ***P*(*s*, *t*)** and perpendicular to the line ***A***.

D is the line containing the point ***P*** and perpendicular to the line ***B***.

Q is the point where the line ***A*** meets the line ***C***.

R is the point where the line ***B*** meets the line ***D***.

Then, the slope of the line ***C*** is $-\frac{1}{2}$, and the slope of the line ***D*** is $\frac{2}{3}$. So,

the line ***C*** is $y-t=-\frac{1}{2}(x-s)$ since the point ***P*(*s*, *t*)** is in the line ***C***.

The line ***D*** is $y-t=\frac{2}{3}(x-s)$ since the point ***P*(*s*, *t*)** is in the line ***D***, too.

Thus, finding the point ***Q***, we get

$$2x-1=-\frac{1}{2}x+t+\frac{s}{2} \Rightarrow \frac{5}{2}x=t+\frac{s}{2}+1=\frac{2t+s+2}{2} \Rightarrow x=\frac{2t+s+2}{5}$$

$\Rightarrow y=2x-1=\frac{2(2t+s+2)}{5}-1=\frac{4t+2s+4-5}{5}=\frac{4t+2s-1}{5}$. So the point ***Q*** is at $\left(\frac{2t+s+2}{5},\frac{4t+2s-1}{5}\right)$.

Next, finding the point $\boldsymbol{R}$, we get

$$-\frac{3}{2}x+1=\frac{2}{3}x+t-\frac{2s}{3}\Rightarrow(\frac{2}{3}+\frac{3}{2})x=1-t+\frac{2s}{3}\Rightarrow\frac{13}{6}x=\frac{2s-3t+3}{3}=\frac{4s-6t+6}{6}$$
$$x=\frac{4s-6t+6}{13}\Rightarrow y=-\frac{3}{2}x+1=-\frac{3}{2}\cdot\frac{4s-6t+6}{13}+1=\frac{-3(2s-3t+3)}{13}+1=\frac{-6s+9t-9+13}{13}=\frac{-6s+9t+4}{13}.$$

So the $\boldsymbol{R}$ is at $(\frac{4s-6t+6}{13},\frac{-6s+9t+4}{13})$.

Suppose next, d_1 is the distance from $\boldsymbol{P}$ to $\boldsymbol{Q}$, and d_2 is the one from $\boldsymbol{P}$ to $\boldsymbol{R}$. Then,

first, $\Delta x=s-\frac{2t+s+2}{5}=\frac{5s-2t-s-2}{5}=\frac{4s-2t-2}{5}$, and $\Delta y=t-\frac{4t+2s-1}{5}=\frac{5t-4t-2s+1}{5}=\frac{t-2s+1}{5}$.

So $d_1{}^2=(\frac{4s-2t-2}{5})^2+(\frac{t-2s+1}{5})^2=4(\frac{2s-t-1}{5})^2+(\frac{2s-t-1}{5})^2=\frac{(2s-t-1)^2}{5}$.

Next, $\Delta x=s-\frac{4s-6t+6}{13}=\frac{13s-4s+6t-6}{13}=\frac{9s+6t-6}{13}$, and $\Delta y=t-\frac{-6s+9t+4}{13}=\frac{13t+6s-9t-4}{13}=\frac{6s+4t-4}{13}$.

So $d_2{}^2=(\frac{3(3s+2t-2)}{13})^2+(\frac{2(3s+2t-2)}{13})^2=\frac{13(3s+2t-2)^2}{13^2}=\frac{(3s+2t-2)^2}{13}$.

Then, since $d_1{}^2=d_2{}^2$, we get $\frac{(2s-t-1)^2}{5}=\frac{(3s+2t-2)^2}{13}$, which is the connective equation between coordinates of the point $\boldsymbol{P}$, and is from the fact that $\boldsymbol{P}$ is the same distance away from the two lines $\boldsymbol{A}$ and $\boldsymbol{B}$. So replacing s and t with x and y in the equation above, we get an equation indicating two lines where every point can be the point $\boldsymbol{P}$. And then, factorizing the equation, we get

$$\frac{(2x-y-1)^2}{5}=\frac{(3x+2y-2)^2}{13}\Rightarrow\frac{(2x-y-1)^2}{5}-\frac{(3x+2y-2)^2}{13}=(\frac{2x-y-1}{\sqrt{5}}+\frac{3x+2y-2}{\sqrt{13}})(\frac{2x-y-1}{\sqrt{5}}-\frac{3x+2y-2}{\sqrt{13}})=0.$$

So the equation indicates two lines, and every point in the lines is an equal distance away from both the two lines $\boldsymbol{A}$ and $\boldsymbol{B}$.

One of the two lines is $\frac{2x-y-1}{\sqrt{5}}+\frac{3x+2y-2}{\sqrt{13}}=0$, and the other is $\frac{2x-y-1}{\sqrt{5}}-\frac{3x+2y-2}{\sqrt{13}}=0$.

And simplifying the two equations above, we get

$$\frac{2x-y-1}{\sqrt{5}}+\frac{3x+2y-2}{\sqrt{13}}=0\Rightarrow\sqrt{13}(2x-y-1)+\sqrt{5}(3x+2y-2)=0$$
$$\Rightarrow(2\sqrt{13}+3\sqrt{5})x+(2\sqrt{5}-\sqrt{13})y-(2\sqrt{5}+\sqrt{13})=0.$$

$$\frac{2x-y-1}{\sqrt{5}}-\frac{3x+2y-2}{\sqrt{13}}=0\Rightarrow\sqrt{13}(2x-y-1)-\sqrt{5}(3x+2y-2)=0$$
$$\Rightarrow(2\sqrt{13}-3\sqrt{5})x-(2\sqrt{5}+\sqrt{13})y+(2\sqrt{5}-\sqrt{13})=0.$$

Let's now, construct another formula.

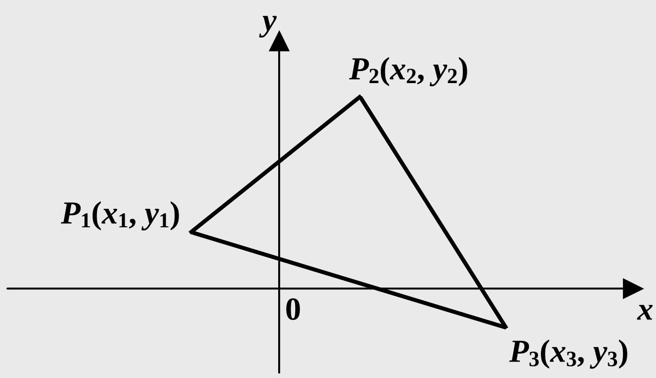
Finding an area of a triangle, we usually use the simple formula saying that the area is half the product of the base and the height of the triangle.

Sometimes though, we get to face trouble finding the height. However, we have several other formulas for an area of a triangle with no specification of its height.

The one we are about to construct is quite useful when we work with a triangle with no specification of its height, but it is defined by three points in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane.

So suppose for instance, $\boldsymbol{P_1(x_1, y_1)}$, $\boldsymbol{P_2(x_2, y_2)}$, and $\boldsymbol{P_3(x_3, y_3)}$ are in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane as below.

Fig. 3



When constructing the formula though, we still need to consult the idea that the area of a triangle is half the product of the base and the height, that is, $\frac{1}{2}(\text{base} \cdot \text{height})$.

Suppose now, that in the graph above, the line segment $\overline{\boldsymbol{P_1P_2}}$ is the base.

Then, the perpendicular distance from the point $\boldsymbol{P_3}$ to the base is the height.
What perpendicular distance though?

Such a perpendicular distance is the length of a line segment that connects the point $\boldsymbol{P_3}$ and a point in the line segment $\overline{\boldsymbol{P_1P_2}}$, and is perpendicular to the segment $\overline{\boldsymbol{P_1P_2}}$.

Let's now, begin with the line passing through the two points $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$.

First, since the line has $\boldsymbol{P_1}$, and the slope is $\frac{y_2-y_1}{x_2-x_1}$, the line is $\boldsymbol{y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1)}$.

Next, putting the equation above in the general form, we get

$$(y - y_1)(x_2 - x_1) = (y_2 - y_1)(x - x_1) \Rightarrow y(x_2 - x_1) - y_1(x_2 - x_1) = x(y_2 - y_1) - x_1(y_2 - y_1)$$

$$\Rightarrow y(x_2 - x_1) - y_1x_2 + y_1x_1 = x(y_2 - y_1) - x_1y_2 + x_1y_1$$

$$\Rightarrow (y_2 - y_1)x - (x_2 - x_1)y + y_1x_2 - x_1y_2 = 0.$$

Now, we can readily find the distance from the point P_3 to the line above by the distance formula between a point and a line. Then, the distance will be the height of the triangle.

The distance from a point $P(s, t)$ to a line $ax + by + c = 0$ is $\frac{|as+bt+c|}{\sqrt{a^2+b^2}}$.

So the height is $\dfrac{|(y_2 - y_1)x_3 - (x_2 - x_1)y_3 + x_2y_1 - x_1y_2|}{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}$.

Next, the base is the length of $\overline{P_1P_2}$, which is $\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$.

And we know that the area of a triangle is $\frac{1}{2}(\text{base} \cdot \text{height})$.

So the area of the triangle $P_1P_2P_3$ is $\frac{1}{2}|(y_2 - y_1)x_3 - (x_2 - x_1)y_3 + x_2y_1 - x_1y_2|$.

Making look a bit nicer what's inside the absolute sign, we get

$$(y_2 - y_1)x_3 - (x_2 - x_1)y_3 + x_2y_1 - x_1y_2 = y_2x_3 - y_1x_3 + (x_1 - x_2)y_3 + x_2y_1 - x_1y_2$$

$$= (x_1 - x_2)y_3 + (x_2 - x_3)y_1 + (x_3 - x_1)y_2.$$

Thus, the area of the triangle $P_1P_2P_3$ is $\frac{1}{2}|(x_1 - x_2)y_3 + (x_2 - x_3)y_1 + (x_3 - x_1)y_2|$.