Examples H in Lines

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Suppose $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are constants, $\boldsymbol{abc \neq 0}$, and $\boldsymbol{uvw \neq 0}$.
Suppose also, $\boldsymbol{A}$ is a line $\boldsymbol{ax + by + c = 0}$, and $\boldsymbol{B}$ is a line $\boldsymbol{ux + vy + w = 0}$. Then:

0. Assuming that the two lines are in fact, the same line, find a connective equation between the constants.

1. Assuming $\boldsymbol{A}$ is parallel to $\boldsymbol{B}$, find a connective equation between the constants.

2. Assuming $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular to each other, find a connective equation between the constants.

3. Assuming the two lines meet each other at a point, find a connective equation between the constants.

4. Assuming again, the two lines meet at a point, find another line passing through the point.

Suggestions or Solutions
To the Problem in the Example 1

Suppose *a*, *b*, *c*, *u*, *v*, and *w* are constants, *abc* ≠ 0, and *uvw* ≠ 0. Suppose also, *A* is a line *ax* + *by* + *c* = 0, and *B* is a line *ux* + *vy* + *w* = 0. Then, assuming the two lines are in fact, the same line, find a connective equation between the constants.

To begin with, since $abc \neq 0$, and $uvw \neq 0$, neither of the constants can be 0.
So division by any of the constants can be done. And thus, we can get

$ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$, and $ux + vy + w = 0 \Rightarrow y = -\frac{u}{v}x - \frac{w}{v}$.

If the two equations indicate the same line, we need to get the same slope and the same *y*-intercept from the two equations.

Thus, we get $\frac{a}{b} = \frac{u}{v}$, and $\frac{c}{b} = \frac{w}{v}$.

So we get $\frac{a}{b} = \frac{u}{v} \Rightarrow av = bu \Rightarrow \frac{a}{u} = \frac{b}{v}$, and $\frac{c}{b} = \frac{w}{v} \Rightarrow cv = bw \Rightarrow \frac{b}{v} = \frac{c}{w}$.

Thus, we get $\frac{a}{u} = \frac{b}{v}$, and $\frac{b}{v} = \frac{c}{w}$, so we get $\frac{a}{u} = \frac{b}{v} = \frac{c}{w}$, which is the connective equation.

If not quite sure of the idea behind the processes above, follow the steps below.

Some equations look quite different can in fact, indicate the same line, so they can actually be the same. Then, how can we see if they are actually the same?

We can check to see if it is the case.
Comparing the coefficients or constant terms, we can get a clue.
Constants or numbers in front of variables are often called coefficients, and terms without a variable are called constant terms. Thus, in the equations given, ***a***, ***b***, ***u***, and ***v*** are coefficients, and ***c*** and ***w*** are constant terms, and are constants, of course. And in the equations above, all the letters other than ***x*** and ***y*** are constants.
Examining the relationship between the coefficients or constant terms, we can see if the equations are the same or not.

First, we may want to begin with how the same equations can look different.

For instance, all of the equations, $\mathbf{2x + 3y - 5 = 0}$, $\mathbf{4x + 6y - 10 = 0}$, $\mathbf{-2x - 3y + 5 = 0}$, and $\mathbf{x + \frac{3}{2}y - \frac{5}{2} = 0}$ indicate the same line. That's because they are multiples of one another.

Multiplying the first equation by 2, we get the second one. Also, the first one is twice the fourth one.

Besides, multiplying the first one by -1, we get the third one, multiplying the second one by $-\frac{1}{2}$, we get the third one, too, and multiplying by $-\frac{1}{2}$ again each of the constants in the third, we get the fourth. What then, do we get from the examples above?

Multiplying by the same number all the constants in an equation, we get a multiple of the equation. So the ratios between corresponding constants are the same.

Therefore, if all such ratios are the same, the equations are actually the same, and thus, indicate the same line. Of the examples above, let's take the first two, and compare them.

They are $\mathbf{2x + 3y - 5 = 0}$ and $\mathbf{4x + 6y - 10 = 0}$.

So the ratios between the corresponding coefficients or constant terms are as follows.

$\frac{2}{4} = \frac{3}{6} = \frac{-5}{-10} = \frac{1}{2}$. All the ratios are $\frac{1}{2}$, so both equations are the same as each other.

We can look at this problem the way below, too.

To begin with, since $\boldsymbol{abc \neq 0}$, and $\boldsymbol{uvw \neq 0}$, neither of the constants can be 0.
So division by any of the constants can be done.
Now, putting the given equations in the standard form, we get

$\boldsymbol{ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}}$, and $\boldsymbol{ux + vy + w = 0 \Rightarrow y = -\frac{u}{v}x - \frac{w}{v}}$.

If the two equations indicate the same line, we need to get the same slope and the same $\boldsymbol{y}$-intercept from the two equations.

That is, we need to get $-\frac{a}{b} = -\frac{u}{v}$, and $-\frac{c}{b} = -\frac{w}{v}$. Thus, we get

$$-\frac{a}{b} = -\frac{u}{v} \Rightarrow \frac{a}{b} = \frac{u}{v} \Rightarrow av = bu \Rightarrow \frac{a}{u} = \frac{b}{v}, \text{ and } -\frac{c}{b} = -\frac{w}{v} \Rightarrow \frac{c}{b} = \frac{w}{v} \Rightarrow cv = bw \Rightarrow \frac{b}{v} = \frac{c}{w}.$$

Thus, we get $\frac{a}{u} = \frac{b}{v}$, and $\frac{b}{v} = \frac{c}{w}$, so we get $\frac{a}{u} = \frac{b}{v} = \frac{c}{w}$.

Thus, if $ax + by + c = 0$ and $ux + vy + w = 0$ indicate the same line, we get $\frac{a}{u} = \frac{b}{v} = \frac{c}{w}$,

which is the connective equation between the constants.

So we get the same story, "The ratios are the same."
Thus, getting the same ratios between the corresponding constants, we get the same lines.

In short:

To begin with, since $abc \neq 0$, and $uvw \neq 0$, neither of the constants can be 0.
So division by any of the constants can be done. Thus, we can get

$$ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}, \text{ and } ux + vy + w = 0 \Rightarrow y = -\frac{u}{v}x - \frac{w}{v}.$$

If the two equations indicate the same line, we need to get the same slope and the same y-intercept from the two equations.

Thus, we get $\frac{a}{b} = \frac{u}{v}$, and $\frac{c}{b} = \frac{w}{v}$.

So we get $\frac{a}{b} = \frac{u}{v} \Rightarrow av = bu \Rightarrow \frac{a}{u} = \frac{b}{v}$, and $\frac{c}{b} = \frac{w}{v} \Rightarrow cv = bw \Rightarrow \frac{b}{v} = \frac{c}{w}$.

Thus, we get $\frac{a}{u} = \frac{b}{v}$, and $\frac{b}{v} = \frac{c}{w}$, so we get $\frac{a}{u} = \frac{b}{v} = \frac{c}{w}$, which is the connective equation.

Suggestions or Solutions
To the Problem in the Example 1

Suppose a, b, c, u, v, and w are constants, $abc \neq 0$, and $uvw \neq 0$. Suppose also, A is a line $ax + by + c = 0$, and B is a line $ux + vy + w = 0$. Then, assuming A is parallel to B, find a connective equation between the constants.

$ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$, and $ux + vy + w = 0 \Rightarrow y = -\frac{u}{v}x - \frac{w}{v}$.

The two equations indicate two parallel lines, if $\frac{a}{b} = \frac{u}{v}$, but $\frac{c}{b} \neq \frac{w}{v}$.

So we get $\frac{a}{b} = \frac{u}{v} \Rightarrow av = bu \Rightarrow \frac{a}{u} = \frac{b}{v}$, and $\frac{c}{b} \neq \frac{w}{v} \Rightarrow cv \neq bw \Rightarrow \frac{b}{v} \neq \frac{c}{w}$.

Thus, we get $\frac{a}{u} = \frac{b}{v}$, and $\frac{b}{v} \neq \frac{c}{w}$, so we get $\frac{a}{u} = \frac{b}{v} \neq \frac{c}{w}$, which is the connective equation.

If not quite sure of the idea behind the processes above, follow the steps below.

Lines parallel to each other have the same slope, and lines having the same slope are parallel to each other. So if the two lines are parallel to each other, the lines need to have the same slope. What then, has to be different?

The $\boldsymbol{y}$-intercepts or the $\boldsymbol{x}$-intercepts need to be different. So having the same slopes but different $\boldsymbol{y}$-intercepts (or $\boldsymbol{x}$-intercepts), the two lines are parallel to each other. Technically though, the same lines can be said to be parallel to each other, too.

Now, putting the given equations in the standard form, we get

$ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$, and $ux + vy + w = 0 \Rightarrow y = -\frac{u}{v}x - \frac{w}{v}$.

Then, the equations indicate two lines parallel to each other if $-\frac{a}{b} = -\frac{u}{v}$, but $-\frac{c}{b} \neq -\frac{w}{v}$.

Thus, we get

$-\frac{a}{b} = -\frac{u}{v} \Rightarrow \frac{a}{b} = \frac{u}{v} \Rightarrow av = bu \Rightarrow \frac{a}{u} = \frac{b}{v}$, and $-\frac{c}{b} \neq -\frac{w}{v} \Rightarrow \frac{c}{b} \neq \frac{w}{v} \Rightarrow cv \neq bw \Rightarrow \frac{b}{v} \neq \frac{c}{w}$.

Thus, we get $\frac{a}{u} = \frac{b}{v}$, and $\frac{b}{v} \neq \frac{c}{w}$, so we get $\frac{a}{u} = \frac{b}{v} \neq \frac{c}{w}$.

So if $\boldsymbol{ax + by + c = 0}$ and $\boldsymbol{ux + vy + w = 0}$ indicate two parallel lines, we get $\frac{a}{u} = \frac{b}{v} \neq \frac{c}{w}$,

which is the connective equation between the constants.

That is, if the ratio between coefficients of $\boldsymbol{x}$ is the same as the one between those of $\boldsymbol{y}$, but is different from the one between the constant terms, the equations indicate two lines parallel to each other. In short, the ratio between the constant terms is different from those between coefficients, if parallel.

In short:

$\boldsymbol{ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}}$, and $\boldsymbol{ux + vy + w = 0 \Rightarrow y = -\frac{u}{v}x - \frac{w}{v}}$.

The two equations indicate two parallel lines, if $\frac{a}{b} = \frac{u}{v}$, but $\frac{c}{b} \neq \frac{w}{v}$.

So we get $\frac{a}{b} = \frac{u}{v} \Rightarrow \boldsymbol{av = bu} \Rightarrow \frac{a}{u} = \frac{b}{v}$, and $\frac{c}{b} \neq \frac{w}{v} \Rightarrow \boldsymbol{cv \neq bw} \Rightarrow \frac{b}{v} \neq \frac{c}{w}$.

Thus, we get $\frac{a}{u} = \frac{b}{v}$, and $\frac{b}{v} \neq \frac{c}{w}$, so we get $\frac{a}{u} = \frac{b}{v} \neq \frac{c}{w}$, which is the connective equation.

Suggestions or Solutions
To the Problem in the Example 2

Suppose *a*, *b*, *c*, *u*, *v*, and *w* are constants, *abc* ≠ 0, and *uvw* ≠ 0. Suppose also, *A* is a line *ax* + *by* + *c* = 0, and *B* is a line *ux* + *vy* + *w* = 0. Then, assuming that *A* and B are perpendicular to each other, find a connective equation between the constants.

We can do divisions by any of the constants since $abc \neq 0$, and $uvw \neq 0$. So we can get

$ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$, and $ux + vy + w = 0 \Rightarrow y = -\frac{u}{v}x - \frac{w}{v}$.

Then, the equations indicate two lines perpendicular to each other if $-\frac{a}{b}(-\frac{u}{v}) = -1$.

Thus, we get $-\frac{a}{b}(-\frac{u}{v}) = -1 \Rightarrow \frac{au}{bv} = -1 \Rightarrow au = -bv \Rightarrow au + bv = 0$, which is the connective equation.

If not quite sure of the idea behind the processes above, follow the steps below.

What is the condition under which two lines are perpendicular to each other?

The product of the slopes is -1.

Now, none of the constants is 0 since $abc \neq 0$, and $uvw \neq 0$.

So we can do divisions by any of the constants, and thus, putting the given equations in the standard form, we get

$ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$, and $ux + vy + w = 0 \Rightarrow y = -\frac{u}{v}x - \frac{w}{v}$.

Then, the equations indicate two lines perpendicular to each other if $-\frac{a}{b}(-\frac{u}{v}) = -1$.

Then, we get $-\frac{a}{b}(-\frac{u}{v}) = -1 \Rightarrow \frac{au}{bv} = -1 \Rightarrow au = -bv \Rightarrow au + bv = 0$.

So if $ax + by + c = 0$ and $ux + vy + w = 0$ indicate lines perpendicular to each other, we get $au + bv = 0$, which is the connective equation between the constants.

That is, if the sum of the product of the coefficients of x and the product of those of y is 0, the equations indicate lines perpendicular to each other. And the product of the slopes is -1 if the two lines are perpendicular to each other.

Suggestions or Solutions
To the Problem in the Example 3

Suppose *a*, *b*, *c*, *u*, *v*, and *w* are constants, *abc* ≠ 0, and *uvw* ≠ 0. Suppose also, *A* is a line *ax* + *by* + *c* = 0, and *B* is a line *ux* + *vy* + *w* = 0. Then, assuming *A* meets *B* at a point, find a connective equation between the constants.

$ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$, and $ux + vy + w = 0 \Rightarrow y = -\frac{u}{v}x - \frac{w}{v}$.

The two equations indicate lines meeting at a point, if $-\frac{a}{b} \neq -\frac{u}{v}$.

Then, we get $-\frac{a}{b} \neq -\frac{u}{v} \Rightarrow \frac{a}{b} \neq \frac{u}{v} \Rightarrow av \neq bu \Rightarrow \frac{a}{u} \neq \frac{b}{v}$.

Thus, we get $\frac{a}{u} \neq \frac{b}{v}$, which is the connective equation.

If not quite sure of the idea behind the processes above, follow the steps below.

If parallel to each other, two lines have the same slope, and do not meet each other.
So if the slopes are different, two lines meet each other at a point.

Therefore, meeting each other at a point, two lines have different slopes.
What then, about the intercepts? The ***y***-intercepts or ***x***-intercepts?

If the ***x***-intercepts are the same, the two lines have to meet at (a point in) the ***x***-axis, and if the ***y***-intercepts are the same, the two lines have to meet at (a point in) the ***y***-axis.

Also, it's not necessary for the ***x***-intercepts to be the same even if two line meet at a point. The same is true for the ***y***-intercepts, too.

So what matters is not the intercept but the slope.

Now, putting the given equations in the standard form, we get

$ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$, and $ux + vy + w = 0 \Rightarrow y = -\frac{u}{v}x - \frac{w}{v}$.

Then, the two equations indicate two lines meeting each other at a point if $-\frac{a}{b} \neq -\frac{u}{v}$.

Then, we get $-\frac{a}{b} \neq -\frac{u}{v} \Rightarrow \frac{a}{b} \neq \frac{u}{v} \Rightarrow av \neq bu \Rightarrow \frac{a}{u} \neq \frac{b}{v}$.

So if $ax + by + c = 0$ and $ux + vy + w = 0$ meet each other at a point, we get

$\frac{a}{u} \neq \frac{b}{v}$, which is the connective equation between the constants.

That is, if the ratio between coefficients of x is not the same as the one between those of y, the equations indicate lines meeting each other at a point.

In short:

$ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$, and $ux + vy + w = 0 \Rightarrow y = -\frac{u}{v}x - \frac{w}{v}$.

The two equations indicate lines meeting at a point, if $-\frac{a}{b} \neq -\frac{u}{v}$.

Then, we get $-\frac{a}{b} \neq -\frac{u}{v} \Rightarrow \frac{a}{b} \neq \frac{u}{v} \Rightarrow av \neq bu \Rightarrow \frac{a}{u} \neq \frac{b}{v}$.

Thus, we get $\frac{a}{u} \neq \frac{b}{v}$, which is the connective equation.

Suggestions or Solutions
To the Problem in the Example 4

Suppose a, b, c, u, v, and w are constants, $abc \neq 0$, and $uvw \neq 0$. Suppose also, A is a line $ax + by + c = 0$, and B is a line $ux + vy + w = 0$. Then, assuming again, the two lines meet at a point, find another line passing through the point.

Suppose the two lines given in this problem meet each other at a point (s, t).

Then, we get $as + bt + c = 0$ and $us + vt + w = 0$.

So we get $m(as + bt + c) + n(us + vt + w) = 0$.

Thus, if $m^2 + n^2 \neq 0$, $m(ax + by + c) + n(ux + vy + w) = 0$ indicates a line passing through the point (s, t).

Thus, $m(ax + by + c) + n(ux + vy + w) = 0$, where $(m - 1)(n - 1) \neq 0$, and $m^2 + n^2 \neq 0$, can be the solution to this problem.

If not quite sure of the idea behind the processes above, follow the steps below.

How many lines can include a point where two lines meet each other?

Infinitely many can do so. So of such many lines, coming up with just one other than A and B, we get the solution. Where in the world though, is the point where the two lines meet each other?

We can of course, take for the point the solution to the system of two equations below.
$ax + by + c = 0$, and $ux + vy + w = 0$.

Then, we can use a slope other than those of A and B.
So assuming the point is (s, t), and the slope is q, we get $y - t = q(x - s)$ as the other line.

Suppose this time, the two lines given in this problem meet each other at a point (s, t).

Then, since the two lines pass through the point (s, t), we get
$as + bt + c = 0$ and $us + vt + w = 0$.

So we get $m(as + bt + c) + n(us + vt + w) = 0$ where m and n are constants.

Thus, if m and $n \neq 0$ at the same time, $m(ax + by + c) + n(ux + vy + w) = 0$, too, is an equation of a line passing through the point (s, t). Why?

First, $m(ax + by + c) + n(ux + vy + w) = (am + nu)x + (bm + nv)y + cm + nw = 0$.

We know $am + nu$, $bm + nv$, and $cm + nw$ are constant because a, b, c, u, v, w, m, and n are constant.

Therefore, $(am + nu)x + (bm + nv)y + cm + nw = 0$ is an equation of a line, and thus, $m(ax + by + c) + n(ux + vy + w) = 0$, too, is an equation of a line.

Next, if $m(as + bt + c) + n(us + vt + w) = 0$, the line $m(ax + by + c) + n(ux + vy + w) = 0$ includes the point (s, t).

So $m(ax + by + c) + n(ux + vy + w) = 0$ indicates a line passing through the point (s, t) where the two lines $ax + by + c = 0$ and $ux + vy + w = 0$ meet each other.

In particular, if $(m - 1)(n - 1) \neq 0$, and $m^2 + n^2 \neq 0$, $m(ax + by + c) + n(ux + vy + w) = 0$ indicates a line passing through the point (s, t) where the two lines $ax + by + c = 0$ and $ux + vy + w = 0$ meet each other, but is not the same as either of the two lines.

That's because,

if $(m - 1)(n - 1) \neq 0$, neither of m and n cant be 1.
And if $m^2 + n^2 \neq 0$, both of m and n cannot be 0 at the same time.

Thus, $m(ax + by + c) + n(ux + vy + w) = 0$, where $(m - 1)(n - 1) \neq 0$, and $m^2 + n^2 \neq 0$, can be the solution to this problem.

And the same is true for the lines parallel to each other, too.

So suppose now, two lines $ax + by + c = 0$ and $ux + vy + w = 0$ are parallel to each other.

Then, $m(ax + by + c) + n(ux + vy + w) = 0$ is an equation of a line parallel to both of the two lines $ax + by + c = 0$ and $ux + vy + w = 0$.

In short:

Suppose the two lines given in this problem meet each other at a point (s, t).

Then, we get $as + bt + c = 0$ and $us + vt + w = 0$.

So we get $m(as + bt + c) + n(us + vt + w) = 0$.

Thus, if $m^2 + n^2 \neq 0$, $m(ax + by + c) + n(ux + vy + w) = 0$ indicates a line passing through the point (s, t).

Thus, $m(ax + by + c) + n(ux + vy + w) = 0$, where $(m - 1)(n - 1) \neq 0$, and $m^2 + n^2 \neq 0$, can be the solution to this problem.