Examples I in Lines

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0. See if there is a particular point every line indicated by the equation below passes through for every value of $\boldsymbol{k}$.

$$\mathbf{3(\boldsymbol{k} - 1)\boldsymbol{x} + (2\boldsymbol{k} + 3)\boldsymbol{y} + 4\boldsymbol{k} + 5 = 0} \text{ where } \boldsymbol{k} \text{ is constant.}$$

And specify the particular point if there is.

1. See if there is a particular point every line indicated by the equation below passes through for every pair of values of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\boldsymbol{y = ax - b} \text{ where } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are constant, and satisfy } \mathbf{3\boldsymbol{a} + \boldsymbol{b} = 2}.$$

And specify the particular point if there is.

Suggestions or Solutions
To the Problem in the Example 0

See if there is a particular point every line indicated by the equation below passes through for every value of k. $3(k-1)x+(2k+3)y+4k+5=0$ where k is constant. And specify the particular point if there is.

First, we can get $\mathbf{3(k-1)x+(2k+3)y+4k+5=0 \Rightarrow 3kx-3x+2ky+3y+4k+5=0}$ $\mathbf{\Rightarrow 3kx+2ky+4k-3x+3y+5=0 \Rightarrow k(3x+2y+4)-(3x-3y-5)=0}$, which is an equation of a line for each value of k, and is a linear combination of two lines.

One is $\mathbf{3x+2y+4=0}$, and the other is $\mathbf{3x-3y-5=0}$.

Assuming next, the two lines meet at $\mathbf{(s, t)}$, we can get
$\mathbf{3s+2t+4=0}$, and $\mathbf{3s-3t-5=0}$, so we get $\mathbf{k(3s+2t+4)-(3s-3t-5)=0}$.

Therefore, if the two lines above meet at a particular point, every line indicated by the equation given includes the particular point, too. And finding the point, we get

$\mathbf{3x+2y+4-(3x-3y-5)=0 \Rightarrow 5y+9=0}$
$\mathbf{\Rightarrow y=-\frac{9}{5} \Rightarrow 3x+2y+4=3x-\frac{18}{5}+4=0 \Rightarrow 3x=-\frac{2}{5} \Rightarrow x=-\frac{2}{15}}$.
Therefore, the particular point is $\mathbf{(-\frac{2}{15}, -\frac{9}{5})}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Suppose now, that the problem given in this example is put the way below.
"Find a particular point every curve indicated by the equation below passes through for every value of k. $\mathbf{3(k-1)x+(2k+3)y+4k+5=0}$ where k is constant."

Then, the problem is saying that no matter what value k may have, every line passes through one particular point, and thus, is asking us to find the particular point.
How then, can we get the particular point?

We can choose a couple of values for k, get two lines, and then, find the point where the lines meet. Since k can be any real number, the quickest choice will be as follows.

$\boldsymbol{k = 1 \Rightarrow 3(k - 1)x + (2k + 3)y + 4k + 5 = 5y + 9 = 0 \Rightarrow y = -\frac{9}{5}}$, which is a line parallel to the $\boldsymbol{x}$-axis.

$\boldsymbol{k = -\frac{3}{2} \Rightarrow 3(k - 1)x + (2k + 3)y + 4k + 5 = -\frac{15}{2}x - 6 + 5 = 0 \Rightarrow x = -\frac{2}{15}}$, which is a line parallel to the $\boldsymbol{y}$-axis, and thus, is perpendicular to the line $\boldsymbol{y = -\frac{9}{5}}$.

So the two lines, $\boldsymbol{x = -\frac{2}{15}}$ and $\boldsymbol{y = -\frac{9}{5}}$ are from the equation given, and meet at $\boldsymbol{(-\frac{2}{15}, -\frac{9}{5})}$, which is therefore, the particular point.

In this problem however, we need to show first, the fact that there is a particular point not just some lines but *all* the lines for all the values of $\boldsymbol{k}$ pass through, and then, specify the particular point. How then, can we show the fact?

We can begin with a fact as follows.

• If two lines meet at a particular point, every linear combination of the two passes through the particular point, too.

What do we mean by the linear combination though?

We mean a linear combination of two equations.

A linear combination of two equations means a sum of any multiples of the two.

So for instance, taking the sum of twice an equation $\boldsymbol{U}$ and half an equation $\boldsymbol{V}$, we can say that the sum is a linear combination of the two equations $\boldsymbol{U}$ and $\boldsymbol{V}$.

That is to say that if $\boldsymbol{W = 2U + 0.5V}$, $\boldsymbol{W}$ can be called a linear combination of $\boldsymbol{U}$ and $\boldsymbol{V}$.

So in general, if $\boldsymbol{K = mU + nV}$ where $\boldsymbol{m}$ and $\boldsymbol{n}$ are constant, $\boldsymbol{K}$ can be called a linear combination of $\boldsymbol{U}$ and $\boldsymbol{V}$. So we can have $\boldsymbol{K = U}$, $\boldsymbol{2U}$, $\boldsymbol{-V}$, $\boldsymbol{U + 2V}$, $\boldsymbol{0.2U - 0.3V}$, etc.

Now, getting back to the fact stated above, we can put it the way below.

Suppose first, in the equations below, all the letters other than $\boldsymbol{x}$ and $\boldsymbol{y}$ are constant.
Suppose next,
$\boldsymbol{A}$ is an equation $\boldsymbol{m(x + ay + b) + n(x + cy + d) = 0}$, and $\boldsymbol{m + n \neq 0}$.
$\boldsymbol{B}$ is a line $\boldsymbol{x + ay + b = 0}$, and $\boldsymbol{C}$ is a line $\boldsymbol{x + cy + d = 0}$.
The line $\boldsymbol{B}$ meets the line $\boldsymbol{C}$ at a particular point.

Then, for each pair of values of $\boldsymbol{m}$ and $\boldsymbol{n}$, $\boldsymbol{A}$ is an equation of a line passing through the particular point where $\boldsymbol{B}$ meets $\boldsymbol{C}$. That is, $\boldsymbol{A}$ represents a group of lines each of which includes the particular point where the line $\boldsymbol{B}$ meets the line $\boldsymbol{C}$.   How?

First, rearranging the equation $\boldsymbol{A}$ above, we get

$$\boldsymbol{m(x + ay + b) + n(x + cy + d) = (m + n)x + (ma + nc)y + mb + nd = 0}.$$

In the equation above, all the letters other than $\boldsymbol{x}$ and $\boldsymbol{y}$ are constants, so their products and sums are constants, too. So the equation $\boldsymbol{A}$ indicates a line.

Next, suppose a point $\boldsymbol{(s, t)}$ is the particular point the two lines $\boldsymbol{B}$ and $\boldsymbol{C}$ meet each other.

Then, we need to show that the line $\boldsymbol{A}$ passes through the particular point, too.

To begin with, we get $\boldsymbol{s + at + b = 0}$ and $\boldsymbol{s + ct + d = 0}$, for the two lines $\boldsymbol{B}$ and $\boldsymbol{C}$ pass through the point $\boldsymbol{(s, t)}$, since the two lines meet at the point.

Thus, we get $\boldsymbol{m(s + at + b) + n(s + ct + d) = 0}$, because $\boldsymbol{m \cdot 0 + n \cdot 0 = 0}$.
So the line $\boldsymbol{A}$ passes through the particular point $\boldsymbol{(s, t)}$.

What if however, $\boldsymbol{m + n = 0}$?

Even if that is the case, the equation $\boldsymbol{m(x + ay + b) + n(x + cy + d) = 0}$ still indicates a line passing through the particular point $\boldsymbol{(s, t)}$.   How?

$\boldsymbol{m(x + ay + b) + n(x + cy + d) = (m + n)x + (am + cn)y + bm + dn}$
$\boldsymbol{= (am + cn)y + bm + dn = 0}$, since $\boldsymbol{m + n = 0}$.   So we get $\boldsymbol{y = \frac{bm+dn}{am+cn}}$ if $\boldsymbol{am + cn \neq 0}$.

Now, because the two lines $x + ay + b = 0$ and $x + cy + d = 0$ pass through the point (s, t), we get $s + at + b = 0$, and $s + ct + d = 0$. So we get $m(s + at + b) + n(s + ct + d) = 0$.

Thus, since $m + n = 0$, we get $(am + cn)t + bm + dn = 0 \Rightarrow t = \frac{bm+dn}{am+cn}$.

So $am + cn \neq 0$ because the point (s, t) exists since the two lines meet at the point.

Thus, we get $y = \frac{bm+dn}{am+cn} = t \Rightarrow y = t$, which is a line parallel to the x-axis, and is passing through the particular point (s, t). Not quite sure?

The y-coordinate of every point in the line $y = t$ is always the same, and is t only, but the x-coordinate is always different, and is a real number, which can be s, too, since s is a real number, also. So one of all the points in the line $y = t$ is (s, t).

Therefore, even if $m + n = 0$, the equation $m(x + ay + b) + n(x + cy + d) = 0$ is still an equation of a line, which is $y – t = 0$, which is $y = t$, and also, the line includes (s, t) at which the two lines B and C ($x + ay + b = 0$ and $x + cy + d = 0$) meet each other.

• So if two lines B and C meet at a point (s, t), then for every pair of values of m and n, every line $A = mB + nC$ passes through the point (s, t), too.

In particular, a line $A = mB + nC$ is called a linear combination of two lines B and C.

••• Now, let's get this problem done.

Suppose now, D is $3(k – 1)x + (2k + 3)y + 4k + 5 = 0$.
Then, we want to show first, that D is a linear combination of two lines. How?

We can put the equation D in terms of k. The constant k is like the constants m and n in the expression of the linear combination, $A = mB + nC$.

Suppose now, D is put in terms of k.
Then, D will be shown as a linear combination of two lines.
Then, we need to check to see if the two lines meet each other.
If that is the case, we can find the point where the two lines meet each other.

Then, the point is the particular point. How then, do we put the equation D in terms of k?

Put the equation $\mathbf{3(}\boldsymbol{k}\mathbf{ - 1)}\boldsymbol{x}\mathbf{ + (2}\boldsymbol{k}\mathbf{ + 3)}\boldsymbol{y}\mathbf{ + 4}\boldsymbol{k}\mathbf{ + 5 = 0}$ in the form as follows.

$\boldsymbol{m(px + qy + r) + n(ux + vy + w) = 0}$. Why not in $\boldsymbol{m(x + ay + b) + n(x + cy + d) = 0}$?

The two forms above are practically the same.
Dividing by $\boldsymbol{p}$ both sides of $\boldsymbol{px + qy + r = 0}$, we get $\boldsymbol{x + \frac{q}{p} y + \frac{r}{p} = 0}$, which is in the form of $\boldsymbol{x + ay + b = 0}$. And the same idea applies to $\boldsymbol{ux + vy + w = 0}$, too.

So we get $\mathbf{3(}\boldsymbol{k}\mathbf{ - 1)}\boldsymbol{x}\mathbf{ + (2}\boldsymbol{k}\mathbf{ + 3)}\boldsymbol{y}\mathbf{ + 4}\boldsymbol{k}\mathbf{ + 5 = 0} \Rightarrow \mathbf{3}\boldsymbol{kx}\mathbf{ - 3}\boldsymbol{x}\mathbf{ + 2}\boldsymbol{ky}\mathbf{ + 3}\boldsymbol{y}\mathbf{ + 4}\boldsymbol{k}\mathbf{ + 5 = 0}$
$\Rightarrow \mathbf{3}\boldsymbol{kx}\mathbf{ + 2}\boldsymbol{ky}\mathbf{ + 4}\boldsymbol{k}\mathbf{ - 3}\boldsymbol{x}\mathbf{ + 3}\boldsymbol{y}\mathbf{ + 5 = 0} \Rightarrow \boldsymbol{k}\mathbf{(3}\boldsymbol{x}\mathbf{ + 2}\boldsymbol{y}\mathbf{ + 4) - (3}\boldsymbol{x}\mathbf{ - 3}\boldsymbol{y}\mathbf{ - 5) = 0}$, which is in the form of $\boldsymbol{m(px + qy + r) + n(ux + vy + w) = 0}$.

Then, we have $\boldsymbol{m = k}$ and $\boldsymbol{n} \mathbf{= -1}$; also, $\boldsymbol{p}\mathbf{ = 3}$, $\boldsymbol{q}\mathbf{ = 2}$, $\boldsymbol{r}\mathbf{ = 4}$, $\boldsymbol{u}\mathbf{ = 3}$, $\boldsymbol{v}\mathbf{ = -3}$, and $\boldsymbol{w}\mathbf{ = -5}$.

So the equation $\mathbf{3(}\boldsymbol{k}\mathbf{ - 1)}\boldsymbol{x}\mathbf{ + (2}\boldsymbol{k}\mathbf{ + 3)}\boldsymbol{y}\mathbf{ + 4}\boldsymbol{k}\mathbf{ + 5 = 0}$ is a linear combination of two lines $\mathbf{3}\boldsymbol{x}\mathbf{ + 2}\boldsymbol{y}\mathbf{ + 4 = 0}$ and $\mathbf{3}\boldsymbol{x}\mathbf{ - 3}\boldsymbol{y}\mathbf{ - 5 = 0}$, and for every $\boldsymbol{k}$, it indicates a line passing through the point where the line $\mathbf{3}\boldsymbol{x}\mathbf{ + 2}\boldsymbol{y}\mathbf{ + 4 = 0}$ meets the line $\mathbf{3}\boldsymbol{x}\mathbf{ - 3}\boldsymbol{y}\mathbf{ - 5 = 0}$.

(Note that $\mathbf{3}\boldsymbol{x}\mathbf{ + 2}\boldsymbol{y}\mathbf{ + 4 = 0}$ and $\boldsymbol{x + \frac{2}{3} y + \frac{4}{3} = 0}$ indicate the same line, and are equivalent.)

Next, solving the system of the two equations above, we can get the point where the two lines meet. That is, if the solution exists, there is a particular point every line indicated by the equation $\boldsymbol{D}$ passes through for every value of $\boldsymbol{k}$.

So now, solving the system, we get

$$\mathbf{3}\boldsymbol{x}\mathbf{ + 2}\boldsymbol{y}\mathbf{ + 4 - (3}\boldsymbol{x}\mathbf{ - 3}\boldsymbol{y}\mathbf{ - 5) = 0} \Rightarrow \mathbf{5}\boldsymbol{y}\mathbf{ + 9 = 0} \;\Rightarrow \boldsymbol{y} = -\frac{9}{5}$$
$$\Rightarrow 3x + 2y + 4 = 3x - \frac{18}{5} + 4 = 0 \Rightarrow 3x = -\frac{2}{5} \Rightarrow x = -\frac{2}{15}.$$

So the two lines meet at $\left(-\frac{2}{15}, -\frac{9}{5}\right)$. Therefore, not only the two lines but also all the lines indicated by the equation $\boldsymbol{D}$ include the point $\left(-\frac{2}{15}, -\frac{9}{5}\right)$.

In short:

First, we can get $\mathbf{3(}\boldsymbol{k}\mathbf{ - 1)}\boldsymbol{x}\mathbf{ + (2}\boldsymbol{k}\mathbf{ + 3)}\boldsymbol{y}\mathbf{ + 4}\boldsymbol{k}\mathbf{ + 5 = 0} \Rightarrow \mathbf{3}\boldsymbol{kx}\mathbf{ - 3}\boldsymbol{x}\mathbf{ + 2}\boldsymbol{ky}\mathbf{ + 3}\boldsymbol{y}\mathbf{ + 4}\boldsymbol{k}\mathbf{ + 5 = 0}$ $\Rightarrow \mathbf{3}\boldsymbol{kx}\mathbf{ + 2}\boldsymbol{ky}\mathbf{ + 4}\boldsymbol{k}\mathbf{ - 3}\boldsymbol{x}\mathbf{ + 3}\boldsymbol{y}\mathbf{ + 5 = 0} \Rightarrow \boldsymbol{k}\mathbf{(3}\boldsymbol{x}\mathbf{ + 2}\boldsymbol{y}\mathbf{ + 4) - (3}\boldsymbol{x}\mathbf{ - 3}\boldsymbol{y}\mathbf{ - 5) = 0}$, which is an equation of a line for each value of $\boldsymbol{k}$, and is a linear combination of two lines.

One is $\mathbf{3}\boldsymbol{x}\mathbf{ + 2}\boldsymbol{y}\mathbf{ + 4 = 0}$, and the other is $\mathbf{3}\boldsymbol{x}\mathbf{ - 3}\boldsymbol{y}\mathbf{ - 5 = 0}$.

Assuming next, the two lines meet at $\mathbf{(}\boldsymbol{s}\mathbf{,}\ \boldsymbol{t}\mathbf{)}$, we can get
$\mathbf{3}\boldsymbol{s}\mathbf{ + 2}\boldsymbol{t}\mathbf{ + 4 = 0}$, and $\mathbf{3}\boldsymbol{s}\mathbf{ - 3}\boldsymbol{t}\mathbf{ - 5 = 0}$, so we get $\boldsymbol{k}\mathbf{(3}\boldsymbol{s}\mathbf{ + 2}\boldsymbol{t}\mathbf{ + 4) - (3}\boldsymbol{s}\mathbf{ - 3}\boldsymbol{t}\mathbf{ - 5) = 0}$.

Therefore, if the two lines above meet at a particular point, every line indicated by the equation given includes the particular point, too. And finding the point, we get

$\mathbf{3}\boldsymbol{x}\mathbf{ + 2}\boldsymbol{y}\mathbf{ + 4 - (3}\boldsymbol{x}\mathbf{ - 3}\boldsymbol{y}\mathbf{ - 5) = 0} \Rightarrow \mathbf{5}\boldsymbol{y}\mathbf{ + 9 = 0}$
$\Rightarrow \boldsymbol{y} = -\frac{9}{5} \Rightarrow 3\boldsymbol{x} + 2\boldsymbol{y} + 4 = 3\boldsymbol{x} - \frac{18}{5} + 4 = 0 \Rightarrow 3\boldsymbol{x} = -\frac{2}{5} \Rightarrow \boldsymbol{x} = -\frac{2}{15}$.
Therefore, the particular point is $(-\frac{2}{15}, -\frac{9}{5})$.

Note:

In fact, any two lines passing through the point above can be used as the two lines above. Any line passing through the point can be a linear combination of any two lines passing through the point. If not sure, make any two lines passing through the point, and see if it is the case. A line of slope $\boldsymbol{a}$ passing a point $\mathbf{(}\boldsymbol{s}\mathbf{,}\ \boldsymbol{t}\mathbf{)}$ is $\boldsymbol{y}\mathbf{ - }\boldsymbol{t}\mathbf{ = }\boldsymbol{a}\mathbf{(}\boldsymbol{x}\mathbf{ - }\boldsymbol{s}\mathbf{)}$. We've got the point already, so making a line has the point, we have only to choose the slope. Then, make a linear combination of the two lines made. The simplest will be the sum of the two lines. Then, see if the sum passes through the point. Try some other combinations, too.

Suggestions or Solutions
To the Problem in the Example 1

See if there is a particular point every curve indicated by the equation below passes through for every pair of values of *a* and *b*. *y* = *ax* – *b* where *a* and *b* are constant, and satisfy 3*a* + *b* = 2. And specify the particular point if there is.

$\boldsymbol{y = ax - b \Rightarrow ax - y - b = 0}$.

$\boldsymbol{3a + b = 2 \Rightarrow a = \frac{2-b}{3}}$.

So $\boldsymbol{y = ax - b = \frac{2-b}{3}x - b = \frac{2}{3}x - \frac{b}{3}x - b \Rightarrow y - \frac{2}{3}x + b(\frac{1}{3}x + 1) = 0}$, which indicates a line for each value of $\boldsymbol{b}$.

Suppose the line $\boldsymbol{y - \frac{2}{3}x = 0}$ meets the line $\boldsymbol{\frac{1}{3}x + 1 = 0}$ at $\boldsymbol{(s, t)}$.

Then, we get $\boldsymbol{t - \frac{2}{3}s = 0}$, and $\boldsymbol{\frac{1}{3}s + 1 = 0}$, so $\boldsymbol{t - \frac{2}{3}s + b(\frac{1}{3}s + 1) = 0}$.

Therefore, if the two lines above meet at a particular point, every line indicated by the equation given includes the particular point, too. Finding the point, we get

$\boldsymbol{y - \frac{2}{3}x = 0 \Rightarrow y = \frac{2}{3}x}$, and thus, $\boldsymbol{\frac{1}{3}x + 1 = 0 \Rightarrow x = -3 \Rightarrow y = \frac{2}{3}x = \frac{2}{3}(-3) = -2}$.

Therefore, the particular point is **(-3, -2)**.

If not quite sure of the idea behind the processes above, follow the steps below.

We can put the problem the way below, too.

“Choosing pairs of numbers for ***a*** and ***b*** that satisfy **3*a* + *b* = 2**, we get as many curves as the number of the choices, and each of all the curves we get is indicated by the equation ***y* = *ax* – *b***. Then, see if all the curves include a particular point, and also, if that is the case, find the particular point.”

Suppose however, the problem is put this way:

“Find a particular point every curve of the equation below passes through for every pair of values of ***a*** and ***b***. ***y* = *ax* – *b*** where ***a*** and ***b*** are constant, and satisfy **3*a* + *b* = 2**.”

Of course, the curve stated in this problem, too, is a line. A curve in math doesn't have to be a curved line as a parabola or circle, and thus, can be a straight line, too.

Then, in this problem, too, all lines include one particular point for all values of $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying $\mathbf{3}\boldsymbol{a} + \boldsymbol{b} = \mathbf{2}$. So we can choose a couple of pairs of values for $\boldsymbol{a}$ and $\boldsymbol{b}$, get two lines, and then, find the point where the lines meet. Since $\boldsymbol{a}$ and $\boldsymbol{b}$ both can be any real numbers that satisfy $\mathbf{3}\boldsymbol{a} + \boldsymbol{b} = \mathbf{2}$, the quickest choice will be as follows.

$a = 0 \Rightarrow 3a + b = b = 2 \Rightarrow y = ax - b = -2 \Rightarrow y = -2$, which is one of the two lines.
$b = 0 \Rightarrow 3a + b = 3a = 2 \Rightarrow a = \frac{2}{3} \Rightarrow y = ax + b = ax = \frac{2}{3}x \Rightarrow y = \frac{2}{3}x$, which is the other.

So finding the particular point, we get $-2 = \frac{2}{3}x \Rightarrow x = -3$.

Therefore, the two lines above meet at $(-3, -2)$, which is therefore, the particular point. Why is -2 the $\boldsymbol{y}$-coordinate though?

Since the line $y = -2$ meets the other line at the particular point, -2 is the $\boldsymbol{y}$-coordinate at the particular point.

In this problem, too, however, we need to show that there is a particular point not just some lines but *all* the lines for all the values of $\boldsymbol{a}$ and $\boldsymbol{b}$ pass through, and then, specify the particular point. So we may want to begin with a fact as follows.

• If two lines meet at a particular point, a linear combination of the two passes through the particular point, too.

Putting the fact above more specifically, we can put it the way below.

Suppose first, in the equations below, all the letters other than $\boldsymbol{x}$ and $\boldsymbol{y}$ indicate constants.
Suppose next,
$\boldsymbol{A}$ is an equation $m(x + ay + b) + n(x + cy + d) = 0$, where $m^2 + n^2 \neq 0$.
($m^2 + n^2 \neq 0 \Rightarrow m$ and $\boldsymbol{n}$ are not 0 at the same time.)
$\boldsymbol{B}$ is a line $x + ay + b = 0$, and $\boldsymbol{C}$ is a line $x + cy + d = 0$.
The line $\boldsymbol{B}$ meets the line $\boldsymbol{C}$ at a particular point.

Then, for each pair of values of $\boldsymbol{m}$ and $\boldsymbol{n}$, $\boldsymbol{A}$ is an equation of a line passing through the particular point where $\boldsymbol{B}$ and $\boldsymbol{C}$ meet each other. That is, $\boldsymbol{A}$ represents a group of lines each of which includes the particular point where the line $\boldsymbol{B}$ meets the line $\boldsymbol{C}$.

• So if two lines $\boldsymbol{B}$ and $\boldsymbol{C}$ meet at a point $\boldsymbol{(s, t)}$, then for every pair of values of $\boldsymbol{m}$ and $\boldsymbol{n}$, every line $\boldsymbol{A = mB + nC}$ passes through the point $\boldsymbol{(s, t)}$, too.

In particular, a line $\boldsymbol{A = mB + nC}$ is called a linear combination of two lines $\boldsymbol{B}$ and $\boldsymbol{C}$.

••• Now, let's get to the solution to this problem.

Suppose now, $\boldsymbol{D}$ is the equation $\boldsymbol{y = ax - b}$, that is, $\boldsymbol{ax - y - b = 0}$.

Then, we want to show first, that $\boldsymbol{D}$ is a linear combination of two lines. How?

We can put the equation $\boldsymbol{D}$ in the form of $\boldsymbol{m(px + qy + r) + n(ux + vy + w) = 0}$.

That is, we put the equation $\boldsymbol{ax - y - b = 0}$ in such a form as above.

The constants $\boldsymbol{m}$ and $\boldsymbol{n}$ in the form are the same as those used in $\boldsymbol{A = mB + nC}$.
Suppose now, $\boldsymbol{D}$ is put in the form of $\boldsymbol{m(px + qy + r) + n(ux + vy + w) = 0}$.

Then, $\boldsymbol{D}$ will be shown as a linear combination of two lines.

Then, we need to check to see if the two lines meet each other.

If that is the case, we can find the point where the two lines meet each other.

Then, the point is the particular point.

How then, do we put the equation $\boldsymbol{D}$ in the form of $\boldsymbol{m(px + qy + r) + n(ux + vy + w) = 0}$?

The equation $\boldsymbol{ax - y - b = 0}$ has too many constants, and the constants are $\boldsymbol{a}$ and $\boldsymbol{b}$.

So we want to eliminate either of the two constants $\boldsymbol{a}$ and $\boldsymbol{b}$. We can do so by means of the expression, $\boldsymbol{3a + b = 2}$. Let's get rid of $\boldsymbol{a}$.

Then, we get $\boldsymbol{3a + b = 2 \Rightarrow a = \frac{2-b}{3}}$.

So we get $\boldsymbol{y = ax - b = \frac{2-b}{3}x - b = \frac{2}{3}x - \frac{b}{3}x - b \Rightarrow 1(y - \frac{2}{3}x) + b(\frac{1}{3}x + 1) = 0}$.

Then, in the form above, we can say that $\boldsymbol{m = 1}$ and $\boldsymbol{n = b}$.

Thus, two equations, $\boldsymbol{y - \frac{2}{3}x = 0}$ and $\boldsymbol{\frac{1}{3}x + 1 = 0}$ indicate the two lines stated above.

So next, we need to find the point where the two lines meet each other.

$\boldsymbol{y - \frac{2}{3}x = 0 \Rightarrow y = \frac{2}{3}x}$, and thus, $\boldsymbol{\frac{1}{3}x + 1 = 0 \Rightarrow x = \text{-}3 \Rightarrow y = \frac{2}{3}x = \frac{2}{3}(\text{-}3) = \text{-}2}$.

So the two lines meet each other at **(-3, -2)**, which is the particular point we want.

What if we eliminate ***b*** instead of ***a*** in the expression $\boldsymbol{3a + b = 2}$?

We should get the same result, of course. Let's see if it is the case, though.
To begin with, we get

$\boldsymbol{3a + b = 2 \Rightarrow b = 2 - 3a \Rightarrow y = ax - b = ax - (2 - 3a) = ax - 2 + 3a = a(x + 3) - 2}$

$\boldsymbol{\Rightarrow a(x + 3) - 2 - y = 0 \Rightarrow a(x + 3) - (2 + y) = 0}$.

Thus, the two equations $\boldsymbol{x + 3 = 0}$ and $\boldsymbol{2 + y = 0}$ indicate the two lines.

What are the two lines though?

We can get $\boldsymbol{x + 3 = 0 \Rightarrow x = \text{-}3}$, and $\boldsymbol{2 + y = 0 \Rightarrow y = \text{-}2}$.

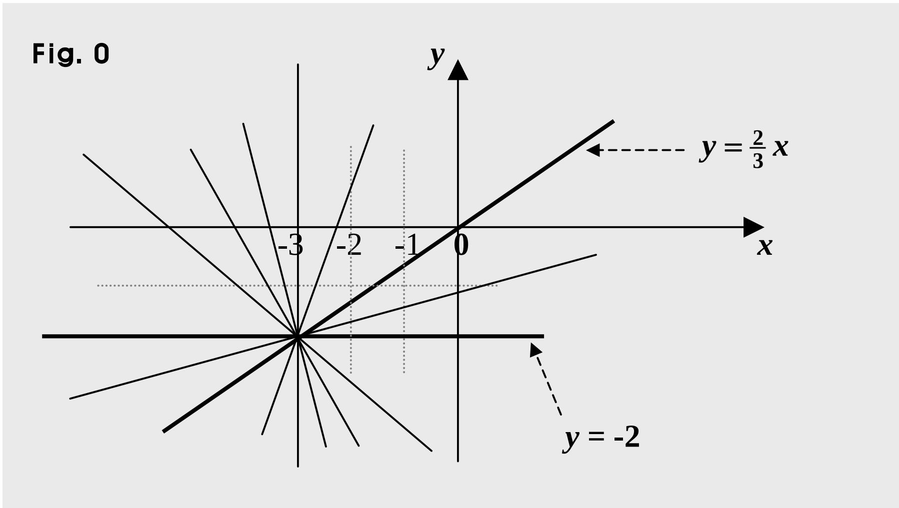
So the two lines are $\boldsymbol{x = \text{-}3}$, and $\boldsymbol{y = \text{-}2}$, and the two meet each other at **(-3, -2)**, which is the particular point.

In fact, any two lines passing through **(-3, -2)** can be used as such two lines as above, too.

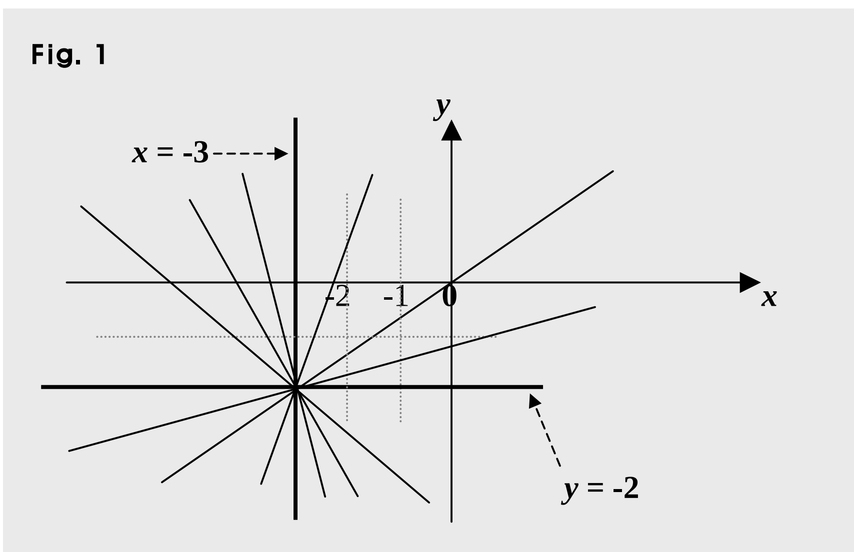
That is, any line passing through **(-3, -2)** is a linear combination of any two lines passing through **(-3, -2)**.

It's not a bad idea to make some examples and check to see if it is the case.

A line passing through **(-3, -2)** is $\boldsymbol{y + 2 = q(x + 3)}$ where $\boldsymbol{q}$ is the slope. Choosing a number for $\boldsymbol{q}$, we get a line passing through **(-3, -2)**.



Any line passing through **(-3, -2)** can be made of two lines $\boldsymbol{y = -2}$ and $\boldsymbol{y = \frac{2}{3}x}$.



Any line passing through **(-3, -2)** is a linear combination of two lines $\boldsymbol{x = -3}$ and $\boldsymbol{y = -2}$.