Examples J in Lines

Click this to start.

Examples J in Lines

0. See if there is a particular point every line indicated by the equation below passes through for every pair of values of ***a*** and ***b***.

$$\frac{2x}{a} + \frac{y}{2b} = 1 \text{ where } a \text{ and } b \text{ are constant, and satisfy } \frac{1}{2a} + \frac{1}{b} = \frac{1}{5}.$$

And specify the particular point if there is.

1.0. See if there is a particular point every line indicated by the equation below passes through for every pair of values of ***a*** and ***b***.

$$5ax - (3b - 2)y + 2b = 8 \text{ where } a \text{ and } b \text{ are constant, and satisfy } 2a - b = 2.$$

And specify the particular point if there is.

1.1. Of all the lines that can be indicated by the equation above, find the one parallel to a line $2x + 5y = 10$.

Suggestions or Solutions
To the Problem in the Example 0

See if there is a particular point every line indicated by the equation below passes through for every pair of values of a and b. $\frac{2x}{a}+\frac{y}{2b}=1$ **where a and b are constant, and also, satisfy** $\frac{1}{2a}+\frac{1}{b}=\frac{1}{5}$**. And specify the particular point if there is.**

To begin with, set $t=\frac{1}{a}$ and $s=\frac{1}{b}$.

Then, t and s are constant, and thus, we get

$\frac{2x}{a}+\frac{y}{2b}=1 \Rightarrow 2tx+\frac{s}{2}y=1 \Rightarrow 4tx+sy-2=0$.

Also, we get $\frac{1}{2a}+\frac{1}{b}=\frac{1}{5} \Rightarrow \frac{t}{2}+s=\frac{1}{5} \Rightarrow t=\frac{2}{5}-2s$. Thus, we get

$4tx+sy-2=4(\frac{2}{5}-2s)x+sy-2=\frac{8}{5}x-8sx+sy-2=0 \Rightarrow s(y-8x)+\frac{8}{5}x-2=0$, which indicates a line for each value of s.

Suppose next, the line $y-8x=0$ meets the line $\frac{8}{5}x-2=0$ at (u, v).

Then, we get $v-8u=0$, and $\frac{8}{5}u-2=0$, so $s(v-8u)+\frac{8}{5}u-2=0$.

Therefore, if the two lines above meet at a particular point, every line indicated by the equation given includes the particular point, too. Finding the particular point, we get

$\frac{8}{5}x-2=0 \Rightarrow x=\frac{5}{4} \Rightarrow y-8x=y-8\cdot\frac{5}{4}=y-10=0 \Rightarrow y=10$.

Therefore, the particular point is $(\frac{5}{4},10)$.

If not quite sure of the idea behind the processes above, follow the steps below.

We can put the problem the way below.

"Choosing pairs of numbers for a and b in the expression $\frac{1}{2a}+\frac{1}{b}=\frac{1}{5}$, we can get as many lines as the number of the choices we make, and each line we can get is indicated by the equation $\frac{2x}{a}+\frac{y}{2b}=1$. Then, see if all the lines meet at a particular point altogether, and also, if it is the case, find the particular point."

Suppose however, the problem is put the way below.

"Find a particular point every line indicated by the equation below includes for every pair of values of $\boldsymbol{a}$ and $\boldsymbol{b}$. $\frac{2x}{a}+\frac{y}{2b}=1$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant, and satisfy $\frac{1}{2a}+\frac{1}{b}=\frac{1}{5}$."

Then, in this problem, too, all lines share one particular point for all values of $\boldsymbol{a}$ and $\boldsymbol{b}$ that satisfy $\frac{1}{2a}+\frac{1}{b}=\frac{1}{5}$.

So we can choose a couple of pairs of values for $\boldsymbol{a}$ and $\boldsymbol{b}$, get two lines, and then, find the point where the lines meet. Since $\boldsymbol{a}$ and $\boldsymbol{b}$ can be any real numbers that can satisfy the expression above, a quick choice for a pair of lines can be as follows.

$$a=\frac{1}{2}\Rightarrow\frac{1}{2a}+\frac{1}{b}=1+\frac{1}{b}=\frac{1}{5}\Rightarrow\frac{1}{b}=-\frac{4}{5}\Rightarrow b=-\frac{5}{4}\Rightarrow\frac{2x}{a}+\frac{y}{2b}=4x-\frac{2}{5}y=1\Rightarrow y=10x-\frac{5}{2}.$$

$$b=1\Rightarrow\frac{1}{2a}+\frac{1}{b}=\frac{1}{2a}+1=\frac{1}{5}\Rightarrow\frac{1}{2a}=-\frac{4}{5}\Rightarrow a=-\frac{5}{8}\Rightarrow\frac{2x}{a}+\frac{y}{2b}=-\frac{16}{5}x+\frac{1}{2}y=1.$$

So finding the particular point, we get

$$-\frac{16}{5}x+\frac{1}{2}y=-\frac{16}{5}x+5x-\frac{5}{4}=\frac{9}{5}x-\frac{5}{4}=1\Rightarrow x=\frac{5}{4}\Rightarrow y=10x-\frac{5}{2}=\frac{25}{2}-\frac{5}{2}=10.$$

Therefore, the two lines above meet at $(\frac{5}{4},10)$, which is therefore, the particular point.

Note that $\boldsymbol{a}$ and $\boldsymbol{b}$ are subject to the expression where $\frac{1}{2a}+\frac{1}{b}=\frac{1}{5}$, so $\boldsymbol{a}$ and $\boldsymbol{b}$ cannot be 0 since denominators cannot be 0.

In this problem, too, however, we need to support the fact that there is a particular point where not just some lines but *all* lines for all values of $\boldsymbol{a}$ and $\boldsymbol{b}$ pass through, and then, specify the particular point.

So we may want to begin with a fact as follows.

• If two lines meet at a particular point, a linear combination of the two passes through the particular point, too.

And putting the fact above more specifically, we can put it the way below.

Suppose first, in the equations below, all the letters other than x and y indicate constants.
Suppose next,
A is an equation $m(x + ay + b) + n(x + cy + d) = 0$, where $m^2 + n^2 \neq 0$.
B is a line $x + ay + b = 0$, and C is a line $x + cy + d = 0$.
The line B meets the line C at a particular point.

Then, for each pair of values of m and n, A is an equation of a line passing through the particular point where B and C meet each other. That is, A represents a group of lines that share the particular point where the line B meets the line C.

• So if two lines B and C meet at a point (s, t), then for every pair of values of m and n, every line $A = mB + nC$ passes through the point (s, t), too.

In particular, a line $A = mB + nC$ is called a linear combination of two lines B and C.

••• Now, let's get to the solution to this problem.

To begin with, suppose D is the equation $\frac{2x}{a} + \frac{y}{2b} = 1$.

Then, we want to show first, that D is a linear combination of two lines.

Then, we can put the equation D in the form of $m(px + qy + r) + n(ux + vy + w) = 0$.
Suppose now, D is put in the form of $m(px + qy + r) + n(ux + vy + w) = 0$.

Then, D will be shown as a linear combination of two lines.
Then, we need to check to see if the two lines meet each other.
If that is the case, we can find the point where the two lines meet each other.
Then, the point is the particular point.

To begin with, let's set $t = \frac{1}{a}$ and $s = \frac{1}{b}$.

Then, t and s are constant, too, since a and b are constant, and thus, we get

$\frac{2x}{a} + \frac{y}{2b} = 1 \Rightarrow 2tx + \frac{s}{2}y = 1 \Rightarrow 4tx + sy - 2 = 0$, which however, has too many constants.

So we want to eliminate either of s and t.
We can do so by means of the expression where $\frac{1}{2a}+\frac{1}{b}=\frac{1}{5}$. Let's get rid of s.

We have $t=\frac{1}{a}$ and $s=\frac{1}{b}$. So we get $\frac{1}{2a}+\frac{1}{b}=\frac{1}{5}\Rightarrow\frac{t}{2}+s=\frac{1}{5}\Rightarrow s=\frac{1}{5}-\frac{t}{2}$. Then, we get

$$4tx+sy-2=4tx+(\tfrac{1}{5}-\tfrac{t}{2})y-2=4tx+\tfrac{1}{5}y-\tfrac{t}{2}y-2=0\Rightarrow t(4x-\tfrac{1}{2}y)+1(\tfrac{1}{5}y-2)=0.$$

Then, in the form stated above, we can say that $m = t$ and $n = 1$.

Thus, the two equations $4x-\frac{1}{2}y=0$ and $\frac{1}{5}y-2=0$ indicate the two lines.

So next, we need to find the point where the two lines meet each other.

$\frac{1}{5}y-2=0\Rightarrow y=10$, and thus, we get $4x-\frac{1}{2}y=0\Rightarrow x=\frac{1}{8}y=\frac{10}{8}=\frac{5}{4}$.

Thus, the two lines meet each other at the point $(\frac{5}{4},10)$, which is the particular point.

In short:

To begin with, set $t=\frac{1}{a}$ and $s=\frac{1}{b}$. Then, t and s are constant, and thus, we get

$\frac{2x}{a}+\frac{y}{2b}=1\Rightarrow 2tx+\frac{s}{2}y=1\Rightarrow 4tx+sy-2=0$.

Also, we get $\frac{1}{2a}+\frac{1}{b}=\frac{1}{5}\Rightarrow\frac{t}{2}+s=\frac{1}{5}\Rightarrow t=\frac{2}{5}-2s$. Thus, we get

$4tx+sy-2=4(\frac{2}{5}-2s)x+sy-2=\frac{8}{5}x-8sx+sy-2=0\Rightarrow s(y-8x)+\frac{8}{5}x-2=0$, which indicates a line for each value of s.

Suppose the line $y-8x=0$ meets the line $\frac{8}{5}x-2=0$ at (u, v).

Then, we get $v-8u=0$, and $\frac{8}{5}u-2=0$, so $s(v-8u)+\frac{8}{5}u-2=0$.

Therefore, if the two lines above meet at a particular point, every line indicated by the equation given includes the particular point, too. Finding the particular point, we get

$$\tfrac{8}{5}x-2=0\Rightarrow x=\tfrac{5}{4}\Rightarrow y-8x=y-8\cdot\tfrac{5}{4}=y-10=0\Rightarrow y=10.$$

Therefore, the particular point is $(\frac{5}{4},10)$.

Suggestions or Solutions
To the Problem 0 in the Example 1

See if there is a particular point every line indicated by the equation below passes through for every pair of values of *a* and *b*. $5ax - (3b - 2)y + 2b = 8$ where *a* and *b* are constant, and satisfy $2a - b = 2$. And specify the particular point if there is.

$$5ax - (3b-2)y + 2b = 8 \Rightarrow 5ax - 3by + 2y + 2b - 8 = 0.$$

$$2a - b = 2 \Rightarrow b = 2a - 2 \Rightarrow 5ax - 3by + 2y + 2b - 8 = 5ax - (6a-6)y + 2y + (4a-4) - 8$$

$= 5ax - 6ay + 6y + 2y + 4a - 12 = a(5x - 6y + 4) + 8y - 12 = 0$, which is a line for each a.

Suppose the line $5x - 6y + 4 = 0$ meets the line $8y - 12 = 0$ at (s, t).
Then, $5s - 6t + 4 = 0$, and $8t - 12 = 0$, so $a(5s - 6t + 4) + 8t - 12 = 0$.

Therefore, if the two lines above meet at a particular point, every line indicated by the equation given includes the particular point, too. Finding the particular point, we get

$$8y - 12 = 0 \Rightarrow y = \tfrac{3}{2} \Rightarrow 5x - 6y + 4 = 5x - 6\cdot\tfrac{3}{2} + 4 = 5x - 5 = 0 \Rightarrow x = 1.$$

Therefore, the particular point is $(1, \frac{3}{2})$.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's begin with a fact as follows.

• If two lines B and C meet at a point (s, t), then for every pair of values of m and n, every line $A = mB + nC$ passes through the point (s, t), too.

In particular, a line $A = mB + nC$ is called a linear combination of two lines B and C.

Now, let's get to the solution to this problem.

Suppose now, D is the equation $5ax - (3b - 2)y + 2b = 8$.
Then, we want to show first, that D is a linear combination of two lines.

We can put the equation $\boldsymbol{D}$ in the form of $\boldsymbol{m(px + qy + r) + n(ux + vy + w) = 0}$.

Then, we need to check to see if the two lines meet each other.

If that is the case, we can find the point where the two lines meet each other.
Then, the point is the particular point. Now, to begin with, we get

$\boldsymbol{5ax - (3b - 2)y + 2b = 8 \Rightarrow 5ax - 3by + 2y + 2b - 8 = 0}$, which however, has too many constants. So eliminating $\boldsymbol{a}$ by means of the expression $\boldsymbol{2a - b = 2}$, we get

$$\boldsymbol{2a - b = 2 \Rightarrow a = \tfrac{2+b}{2} = 1 + \tfrac{b}{2} \Rightarrow 5ax - 3by + 2y + 2b - 8 = (5 + \tfrac{5b}{2})x - 3by + 2y + 2b - 8}$$

$$\boldsymbol{= 5x + \tfrac{5b}{2}x - 3by + 2y + 2b - 8 = b(\tfrac{5}{2}x - 3y + 2) + 5x + 2y - 8 = 0}.$$

Then in the form above, we can see that $\boldsymbol{m = b}$ and $\boldsymbol{n = 1}$, so $\boldsymbol{D}$ is a linear combination of two lines $\boldsymbol{\tfrac{5}{2}x - 3y + 2 = 0}$ and $\boldsymbol{5x + 2y - 8 = 0}$.

So finding the point where the two lines meet, we get the particular point.

$$\boldsymbol{\tfrac{5}{2}x - 3y + 2 = 0 \Rightarrow 5x = 6y - 4 \Rightarrow 5x + 2y - 8 = 6y - 4 + 2y - 8 = 8y - 12 = 0 \Rightarrow y = \tfrac{3}{2}}$$

$$\boldsymbol{\Rightarrow 5x = 6y - 4 = 6 \cdot \tfrac{3}{2} - 4 = 9 - 4 = 5 \Rightarrow x = 1}.$$

Therefore, the particular point is $\boldsymbol{(1, \tfrac{3}{2})}$.

In short:

$$\boldsymbol{5ax - (3b - 2)y + 2b = 8 \Rightarrow 5ax - 3by + 2y + 2b - 8 = 0}.$$

$$\boldsymbol{2a - b = 2 \Rightarrow b = 2a - 2 \Rightarrow 5ax - 3by + 2y + 2b - 8 = 5ax - (6a - 6)y + 2y + (4a - 4) - 8}$$

$\boldsymbol{= 5ax - 6ay + 6y + 2y + 4a - 12 = a(5x - 6y + 4) + 8y - 12 = 0}$, which is a line for each $\boldsymbol{a}$.

Suppose the line $\boldsymbol{5x - 6y + 4 = 0}$ meets the line $\boldsymbol{8y - 12 = 0}$ at $\boldsymbol{(s, t)}$.
Then, $\boldsymbol{5s - 6t + 4 = 0}$, and $\boldsymbol{8t - 12 = 0}$, so $\boldsymbol{a(5s - 6t + 4) + 8t - 12 = 0}$.
Therefore, if the two lines above meet at a particular point, every line indicated by the equation given includes the particular point, too. Finding the particular point, we get

$$\boldsymbol{8y - 12 = 0 \Rightarrow y = \tfrac{3}{2} \Rightarrow 5x - 6y + 4 = 5x - 6 \cdot \tfrac{3}{2} + 4 = 5x - 5 = 0 \Rightarrow x = 1}.$$

Therefore, the particular point is $\boldsymbol{(1, \tfrac{3}{2})}$.

And putting the lines in a graph in each case, we can get

Fig. 0

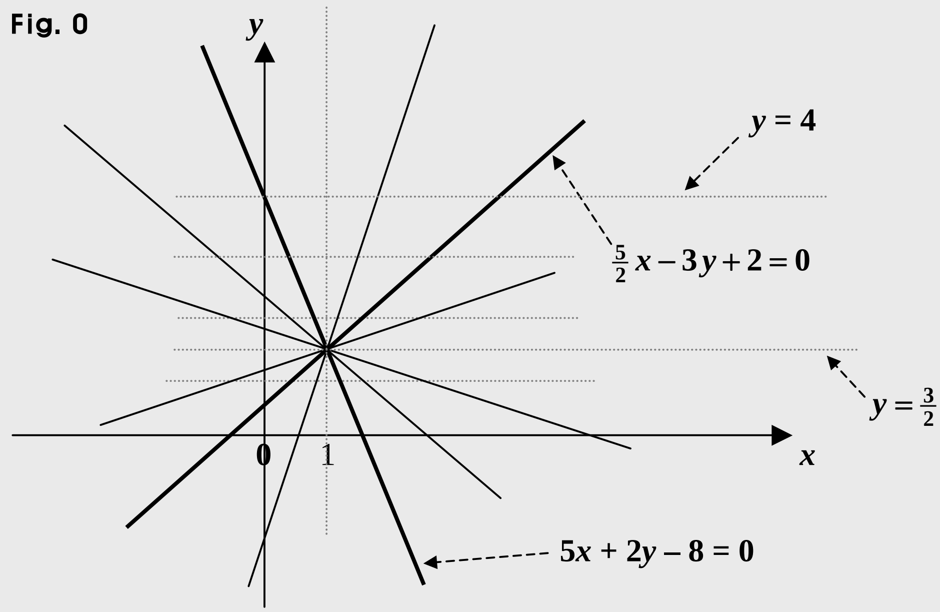
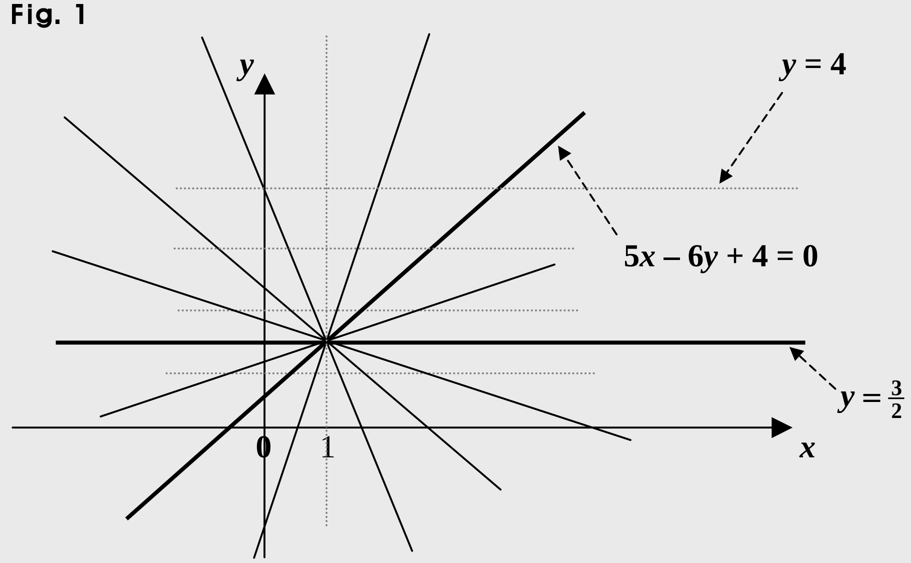


Fig. 1



Suggestions or Solutions
To the Problem 1 in the Example 1

Of all the lines that can be indicated by the equation below, find the one parallel to a line $2x + 5y = 10$.
$5ax - (3b - 2)y + 2b = 8$ where a and b are constant, and satisfy $2a - b = 2$.

The slope of $2x + 5y = 10$ is $-\frac{2}{5}$, and that of $5ax-(3b-2)y+2b=8$ is $\frac{5a}{3b-2}$.
Then, since the slopes have to be the same, we get

$\frac{5a}{3b-2} = -\frac{2}{5} \Rightarrow 5a = -\frac{2(3b-2)}{5} \Rightarrow a = \frac{2(2-3b)}{25} = \frac{4-6b}{25}$.

Also, a and b have to satisfy the expression, $2a - b = 2$.

So we get $2a-b = \frac{8-12b}{25} - b = \frac{8-12b-25b}{25} = \frac{8-37b}{25} = 2 \Rightarrow 8-37b = 50 \Rightarrow b = -\frac{42}{37}$

$\Rightarrow 2a-b = 2a+\frac{42}{37} = 2 \Rightarrow 2a = 2-\frac{42}{37} \Rightarrow a = 1-\frac{21}{37} = \frac{37-21}{37} = \frac{16}{37}$.

Thus, we get $5ax-(3b-2)y+2b=8 \Rightarrow 5\cdot\frac{16}{37}\cdot x-\{3\cdot(-\frac{42}{37})-2\}y+2\cdot(-\frac{42}{37})$

$= \frac{80}{37}x+(\frac{126}{37}+2)y-\frac{84}{37} = 8 \Rightarrow 80x+200y+380=0$.

Thus, $4x + 10y + 19 = 0$ is the line parallel to $2x + 5y = 10$.

If not quite sure of the idea behind the processes above, follow the steps below.

For each pair of values of a and b in $2a - b = 2$, we get a line $5ax - (3b - 2)y + 2b = 8$.

So we need to find the values of a and b so that the line $5ax - (3b - 2)y + 2b = 8$ is parallel to the line $2x + 5y = 10$.

If parallel to each other, lines have the same slopes. So the slope of the line we are after is the same as that of the line $2x + 5y = 10$.

Let's put the given equations in the point-slope form to find the slopes.

To begin with, we get $2x+5y=10 \Rightarrow y = -\frac{2}{5}x+2$. So the slope $= -\frac{2}{5}$.

Next, we get $5ax-(3b-2)y+2b=8 \Rightarrow y = \frac{5a}{3b-2}x+\frac{8-2b}{3b-2}$. So the slope $= \frac{5a}{3b-2}$.

(Of course, getting the slopes only, we don't have to put them in the standard form. In the general from, the slope is the negative ratio of the coefficient of $\boldsymbol{x}$ to that of $\boldsymbol{y}$.)

Now, since the slopes have to be the same, we get $\frac{5a}{3b-2} = -\frac{2}{5}$.

Next, solving the system where $\frac{5a}{3b-2} = -\frac{2}{5}$, and $\mathbf{2}\boldsymbol{a} - \boldsymbol{b} = \mathbf{2}$, we get the values of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Then, putting the values into $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbf{5}\boldsymbol{ax} - (\mathbf{3}\boldsymbol{b} - \mathbf{2})\boldsymbol{y} + \mathbf{2}\boldsymbol{b} = \mathbf{8}$, we get the line parallel to the line $\mathbf{2}\boldsymbol{x} + \mathbf{5}\boldsymbol{y} = \mathbf{10}$.

Eliminating $\boldsymbol{a}$, we get $\frac{5a}{3b-2} = -\frac{2}{5} \Rightarrow 5a = -\frac{2(3b-2)}{5} \Rightarrow a = \frac{2(2-3b)}{25} = \frac{4-6b}{25}$.

Thus, $2a - b = \frac{8-12b}{25} - b = \frac{8-12b-25b}{25} = \frac{8-37b}{25} = 2 \Rightarrow 8 - 37b = 50 \Rightarrow b = -\frac{42}{37}$

$\Rightarrow 2a - b = 2a + \frac{42}{37} = 2 \Rightarrow 2a = 2 - \frac{42}{37} \Rightarrow a = 1 - \frac{21}{37} = \frac{37-21}{37} = \frac{16}{37}$.

So we get $5ax - (3b-2)y + 2b = 8 \Rightarrow 5 \cdot \frac{16}{37} \cdot x - \{3 \cdot (-\frac{42}{37}) - 2\}y + 2 \cdot (-\frac{42}{37})$

$= \frac{80}{37}x + (\frac{126}{37} + 2)y - \frac{84}{37} = 8 \Rightarrow 80x + 200y + 380 = 0 \Rightarrow 4x + 10y + 19 = 0$, which is the line parallel to $\mathbf{2}\boldsymbol{x} + \mathbf{5}\boldsymbol{y} = \mathbf{10}$.

In short:

The slope of $\mathbf{2}\boldsymbol{x} + \mathbf{5}\boldsymbol{y} = \mathbf{10}$ is $-\frac{2}{5}$, and that of $5ax - (3b-2)y + 2b = 8$ is $\frac{5a}{3b-2}$.
Then, since the slopes have to be the same, we get

$\frac{5a}{3b-2} = -\frac{2}{5} \Rightarrow 5a = -\frac{2(3b-2)}{5} \Rightarrow a = \frac{2(2-3b)}{25} = \frac{4-6b}{25}$.

Also, $\boldsymbol{a}$ and $\boldsymbol{b}$ have to satisfy the expression, $\mathbf{2}\boldsymbol{a} - \boldsymbol{b} = \mathbf{2}$.

So we get $2a - b = \frac{8-12b}{25} - b = \frac{8-12b-25b}{25} = \frac{8-37b}{25} = 2 \Rightarrow 8 - 37b = 50 \Rightarrow b = -\frac{42}{37}$

$\Rightarrow 2a - b = 2a + \frac{42}{37} = 2 \Rightarrow 2a = 2 - \frac{42}{37} \Rightarrow a = 1 - \frac{21}{37} = \frac{37-21}{37} = \frac{16}{37}$.

Thus, we get $5ax - (3b-2)y + 2b = 8 \Rightarrow 5 \cdot \frac{16}{37} \cdot x - \{3 \cdot (-\frac{42}{37}) - 2\}y + 2 \cdot (-\frac{42}{37})$

$= \frac{80}{37}x + (\frac{126}{37} + 2)y - \frac{84}{37} = 8 \Rightarrow 80x + 200y + 380 = 0$.

Thus, $\mathbf{4}\boldsymbol{x} + \mathbf{10}\boldsymbol{y} + \mathbf{19} = \mathbf{0}$ is the line parallel to $\mathbf{2}\boldsymbol{x} + \mathbf{5}\boldsymbol{y} = \mathbf{10}$.