Examples K in Lines

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0. Suppose two lines meet each other at $P(s, t)$ where $s > 0$ and $t > 0$, and the two lines are as follows. $2x + y - 2 = 0$ and $2kx - y + k - 1 = 0$ where k is constant.

Then, find the extent of k.

1. Suppose A is a line $(2k - 1)x - (3k + 1)y + 5 + 2k = 0$ where k is constant, and B is a line segment $y = 2x$ for $|x| \leq 1$.

Then, determine the value of k so that the line A meets the line segment B.

Suggestions or Solutions
To the Problem in the Example 0

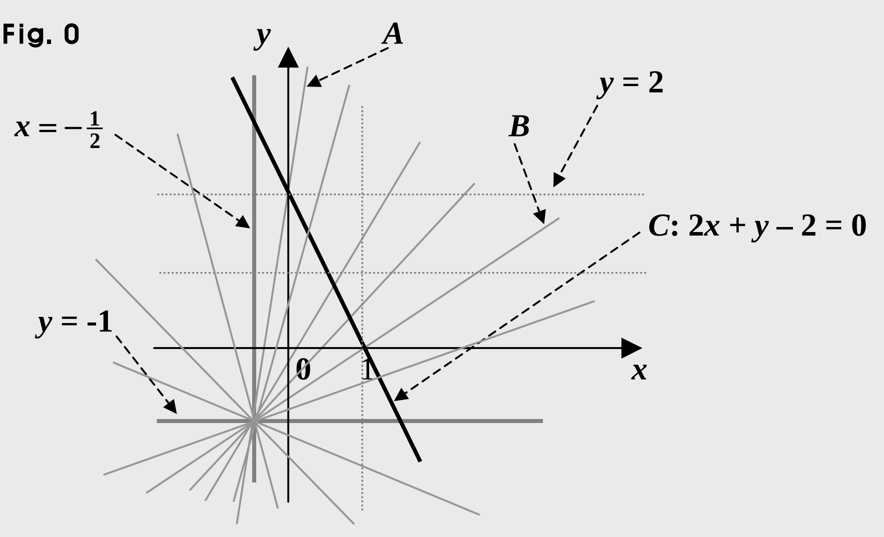
Suppose two lines meet each other at $P(s, t)$ where $s > 0$ and $t > 0$, and the two lines are as follows. $2x + y - 2 = 0$ and $2kx - y + k - 1 = 0$ where k is a constant.
Then, find the extent of k.

Suppose first, L is $2kx - y + k - 1 = 0$, and C is the line $2x + y - 2 = 0$.
Then, L is a line, and can be put this way: $2kx - y + k - 1 = 0 \Rightarrow k(2x + 1) - (y + 1) = 0$.

Next, finding the point where two lines $2x + 1 = 0$ and $y + 1 = 0$, we get
$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$. And $y + 1 = 0 \Rightarrow y = -1$. So the two lines meet at $(-\frac{1}{2}, -1)$.

So for every k, L includes $(-\frac{1}{2}, -1)$ where the line $2x + 1 = 0$ meets the line $y + 1 = 0$.
Putting in a graph the line C, together with some lines L represents, we get

Fig. 0



The line L has to meet a line segment, which is in C, and is in the first quadrant, because the line L meets C at a point $P(s, t)$ where $s > 0$ and $t > 0$, and the point P is in the line C.

In the line C, $2x + y - 2 = 0$, setting $x = 0$, we get $y = 2$, which is y-intercept, and setting $y = 0$, we get $x = 1$, which is the x-intercept.

Suppose of all the lines in the graph, ***A*** is the one where the ***y***-intercept is 2, and ***B*** is the one where 1 is the ***x***-intercept. Then, of all lines passing through the point $\left(-\frac{1}{2}, -1\right)$, only the ones between the lines ***A*** and ***B*** can meet the line segment.

We know all the lines in the group include the point $\left(-\frac{1}{2}, -1\right)$. Also, we know

The line ***A*** includes the point where the line ***C*** meets the ***y***-axis, and the point is **(0, 2)**.
The line ***B*** includes the point where the line ***C*** meets the ***x***-axis, and the point is **(1, 0)**.

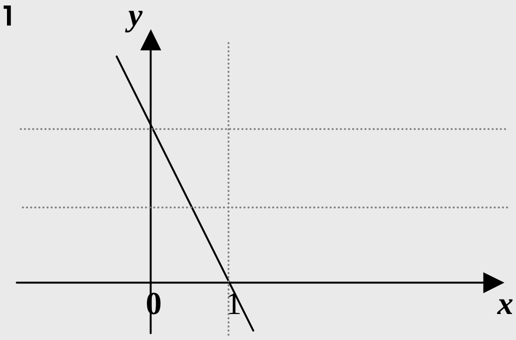
So the slope of the line ***A*** is $\frac{2-(-1)}{0-\left(-\frac{1}{2}\right)} = 6$, and that of the line ***B*** is $\frac{0-(-1)}{1-\left(-\frac{1}{2}\right)} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$.

Thus, we get $\frac{2}{3} < 2k < 6 \Rightarrow \frac{1}{3} < k < 3$.

If not quite sure of the idea behind the processes above, follow the steps below.

Putting in a graph the line $2x + y - 2 = 0$, we can see better whereabouts the solution is around. First, setting $x = 0$, we get $y = 2$, which is the ***y***-intercept, and next, setting $y = 0$, we see the ***x***-intercept is $x = 1$, so we can quickly put the line in a graph as shown below.

Fig. 1



Only the line segment in the first quadrant is applicable, since in $P(s, t)$, $s > 0$ and $t > 0$. That is, both coordinates are positive at the point ***P***.

Suppose now, ***D*** is the line $2kx - y + k - 1 = 0$. Then, ***D*** can be made of two lines, and includes a point ***Q*** if the two lines meet at the point ***Q***. What in the world is the point ***Q***?

It's a particular point in the ***x-y*** plane. What in the world are the two lines, then?

Any line passing through a particular point can be a linear combination of two lines passing through the particular point. So what the statement above is saying is that ***D*** can be a linear combination of two lines, and if it is the case, if the two lines meet at a point ***Q***, ***D*** includes the point ***Q***.

Now, for each value of ***k***, ***D*** indicates a line, so a line ***D*** can be said to represent a group of lines. Examining the equation of the line ***D***, we can see that ***D*** is a linear combination of two lines. And if the two lines meet at a point, all the lines in the group meet each other at the point, too.

Suppose next, ***U*** and ***V*** are the two lines. Then, we can set $D = kU + V$, and ***D*** can be called a linear combination of the two lines ***U*** and ***V***.

So let's find the two lines so that we can see where the group of the lines is around.

$2kx - y + k - 1 = 0 \Rightarrow k(2x + 1) - (y + 1) = 0$.

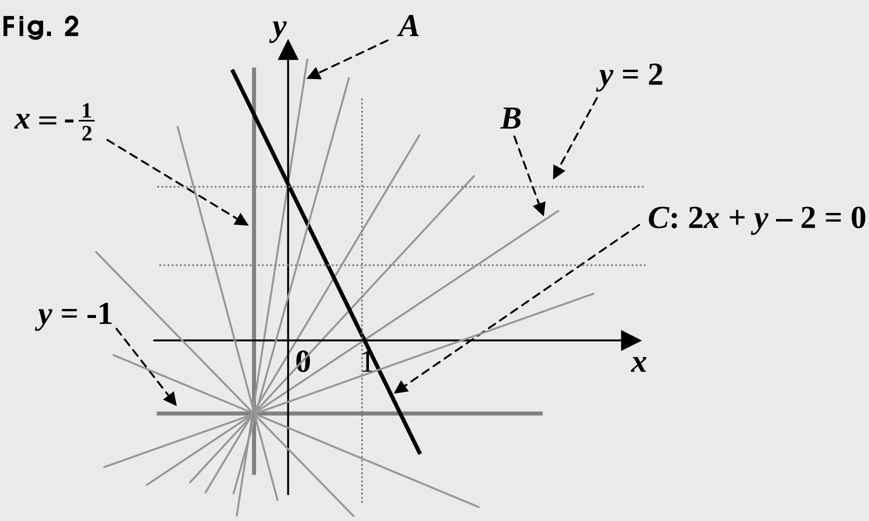
Then, the two lines are $2x + 1 = 0$ and $y + 1 = 0$, of which ***D*** is a linear combination.

Finding the particular point, we get $y + 1 = 0 \Rightarrow y = -1$, and $2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$.

Therefore, all the lines in the group meet each other at $(-\frac{1}{2}, -1)$.

Let's now, add some lines in the group to the graph above. Then, we get

Fig. 2



The lines in gray belong to the group of the lines stated above. So we can see now, where the group of lines is around. We know ***D*** is a line representing all the lines in the group. Suppose now, ***C*** is the line $2x + y - 2 = 0$. Then, we can see what lines in the group can meet the line ***C***. Of course, the only line that cannot meet the line ***C*** is a line parallel to the line ***C***.

However, it is not true that $\boldsymbol{D}$ meets $\boldsymbol{C}$ at any point. The line $\boldsymbol{D}$ meets $\mathbf{C}$ at a point $\boldsymbol{P(s, t)}$ where $\boldsymbol{s > 0}$ and $\boldsymbol{t > 0}$, and the point $\boldsymbol{P}$ is in the line $\boldsymbol{C}$. So the line $\boldsymbol{D}$ has to meet a line segment, which is in the line $\boldsymbol{C}$, and is in the first quadrant.

Suppose in the group, $\boldsymbol{A}$ is a line where the $\boldsymbol{y}$-intercept is 2, and $\boldsymbol{B}$ is a line where 1 is the $\boldsymbol{x}$-intercept. Then, of all lines passing through the point $\mathbf{(-\frac{1}{2}, -1)}$, only the ones between the two lines $\boldsymbol{A}$ and $\boldsymbol{B}$ can meet the line segment since at the point $\boldsymbol{P(s, t)}$, $\boldsymbol{s > 0}$ and $\boldsymbol{t > 0}$.

Also, we can readily see that $\mathbf{2}\boldsymbol{k}$ is the slope of the line $\mathbf{2}\boldsymbol{kx} - \boldsymbol{y} + \boldsymbol{k} - \mathbf{1 = 0}$, and that in the graph, the slope $\mathbf{2}\boldsymbol{k}$ is the largest when the $\boldsymbol{y}$-intercept is 2, and is the smallest when the $\boldsymbol{x}$-intercept is 1.

We know all the lines in the group include the point $\mathbf{(-\frac{1}{2}, -1)}$. Also, we know

The line $\boldsymbol{A}$ includes the point where the line $\boldsymbol{C}$ meets the $\boldsymbol{y}$-axis, and the point is $\mathbf{(0, 2)}$.

The line $\boldsymbol{B}$ includes the point where the line $\boldsymbol{C}$ meets the $\boldsymbol{x}$-axis, and the point is $\mathbf{(1, 0)}$.

So the slope of the line $\boldsymbol{A}$ is $\mathbf{\frac{2-(-1)}{0-(-\frac{1}{2})} = 6}$, and that of the line $\boldsymbol{B}$ is $\mathbf{\frac{0-(-1)}{1-(-\frac{1}{2})} = \frac{1}{\frac{3}{2}} = \frac{2}{3}}$.

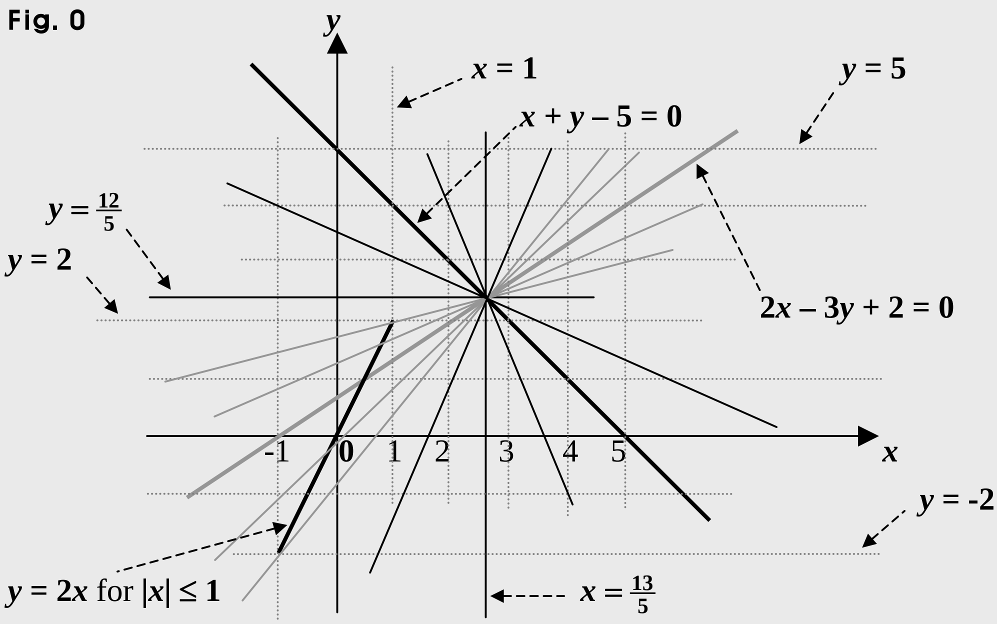
Thus, we get $\mathbf{\frac{2}{3}} < \mathbf{2}\boldsymbol{k} < \mathbf{6} \Rightarrow \mathbf{\frac{1}{3}} < \boldsymbol{k} < \mathbf{3}$.

Suggestions or Solutions
To the Problem in the Example 1

Suppose *A* is a line $(2k – 1)x – (3k + 1)y + 5 + 2k = 0$ where *k* is a constant, and *B* is a line segment $y = 2x$ for $|x| \le 1$.
Then, determine the value of *k* so that the line *A* can meet the line segment *B*.

To begin with, $(2k – 1)x – (3k + 1)y + 5 + 2k = 0 \Rightarrow k(2x – 3y + 2) – (x + y – 5) = 0$.
Next, finding the point where two lines $2x – 3y + 2 = 0$ and $x + y – 5 = 0$ meet, we get
$-2(x + y – 5) = 0 \Rightarrow -2x – 2y + 10 = 0$.
$2x – 3y + 2 + (-2x – 2y + 10) = 0 \Rightarrow -5y + 12 = 0 \Rightarrow y = \frac{12}{5}$
$\Rightarrow x + y - 5 = x + \frac{12}{5} - 5 = 0 \Rightarrow x = \frac{13}{5}$. Thus, the two lines meet at $(\frac{13}{5}, \frac{12}{5})$.
So for every *k*, every line *A* passes through $(\frac{13}{5}, \frac{12}{5})$.
Putting in a graph some lines *A* represents and the line segment *B*, we get

Fig. 0



The endpoints of the line segment *B* are **(-1, -2)** and **(1, 2)**.
The slope of the line having two points **(1, 2)** and $(\frac{13}{5}, \frac{12}{5})$ is $\frac{\frac{12}{5}-2}{\frac{13}{5}-1} = \frac{\frac{2}{5}}{\frac{8}{5}} = \frac{2}{8} = \frac{1}{4}$.

The slope of the line having two points **(-1, -2)** and $\left(\frac{13}{5}, \frac{12}{5}\right)$ is $\frac{\frac{12}{5}-(-2)}{\frac{13}{5}-(-1)} = \frac{\frac{22}{5}}{\frac{18}{5}} = \frac{22}{18} = \frac{11}{9}$.

For each k, $(2k - 1)x - (3k + 1)y + 5 + 2k = 0$ indicates each line passing through the point $\left(\frac{13}{5}, \frac{12}{5}\right)$. So the slope of each of the lines can be indicated by $\frac{2k-1}{3k+1}$. Then,

$$\frac{2k-1}{3k+1} = \frac{1}{4} \Rightarrow 2k - 1 = 4(3k+1) = 12k + 4 \Rightarrow 10k = -5 \Rightarrow k = -\frac{1}{2}.$$

$$\frac{2k-1}{3k+1} = \frac{11}{9} \Rightarrow 9(2k-1) = 11(3k+1) \Rightarrow 18k - 9 = 33k + 11 \Rightarrow 15k = -20 \Rightarrow k = -\frac{4}{3}.$$

Therefore, we get $-\frac{4}{3} \le k \le -\frac{1}{2}$.

If not quite sure of the idea behind the processes above, follow the steps below.

The line A, $(2k - 1)x - (3k + 1)y + 5 + 2k = 0$ can be made of two lines, and includes a point Q if the two lines meet at the point Q.

We get a different line for each value of k, so the line A represents a group of lines, and if the two lines meet at a point, all the lines in the group meet each other at the point, too.

Suppose now, U and V are the two lines. Then, we can set $A = kU + V$, and the line A can be called a linear combination of the two lines U and V.

So let's find the two lines so that we can see where the group is around.

$(2k - 1)x - (3k + 1)y + 5 + 2k = 0 \Rightarrow k(2x - 3y + 2) - (x + y - 5) = 0$.

Thus, the two lines are $2x - 3y + 2 = 0$ and $x + y - 5 = 0$.

Next, find the point where the two lines meet each other. Multiply the second equation by -2, add the product to the first equation to eliminate x, and then, find the value of y, then that of x.

$$-2(x + y - 5) = 0 \Rightarrow -2x - 2y + 10 = 0.$$

$$2x - 3y + 2 + (-2x - 2y + 10) = 0 \Rightarrow -5y + 12 = 0 \Rightarrow y = \frac{12}{5}$$

$$\Rightarrow x + y - 5 = x + \frac{12}{5} - 5 = 0 \Rightarrow x = \frac{13}{5}.$$

Therefore, the two lines meet at $(\frac{13}{5}, \frac{12}{5})$, so for every k, every line A includes $(\frac{13}{5}, \frac{12}{5})$.

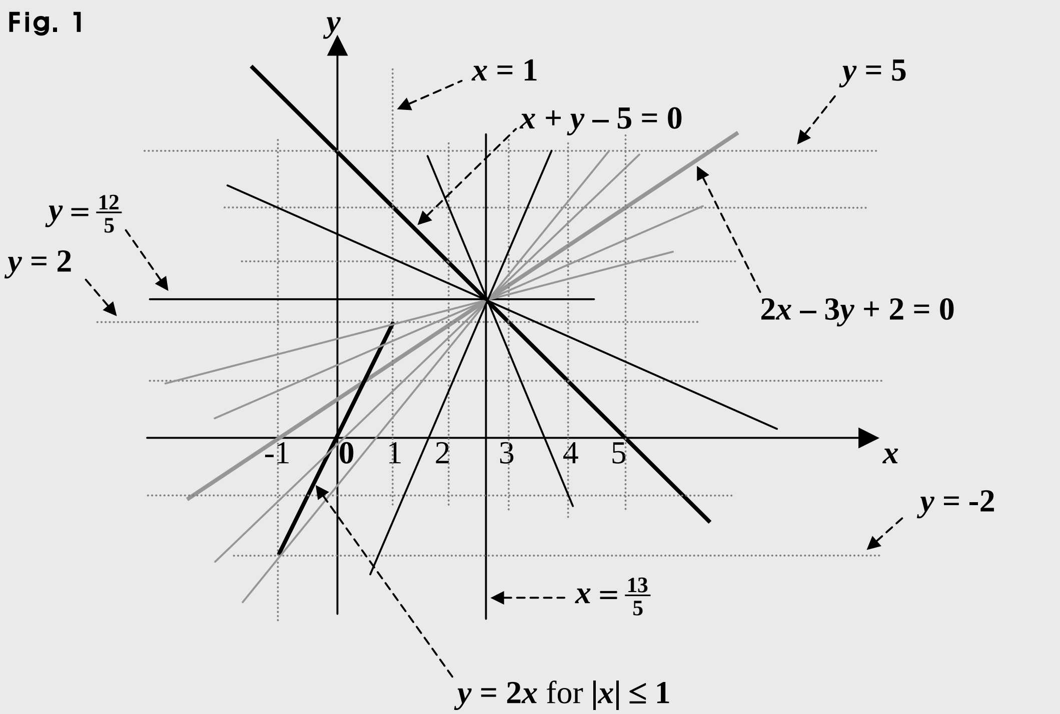
That's because the two lines $2x - 3y + 2 = 0$ and $x + y - 5 = 0$ meet at the point above, and A is $k(2x - 3y + 2) - (x + y - 5) = 0$.

So every line in the group stated above passes through the point $(\frac{13}{5}, \frac{12}{5})$.

Next, let's move on to the line segment B, $y = 2x$ for $|x| \leq 1$.

What we want to do is to find the extent of k so that all the lines in the group can meet the line segment B. Putting in a graph some of the lines in the group, together with the line segment, we can see better whereabouts the solution is around.

Fig. 1



In the graph above, we can see that the lines in gray only can meet the line segment. The topmost line in gray includes two points $(1, 2)$ and $(\frac{13}{5}, \frac{12}{5})$.

So the slope of the topmost line is $\frac{\frac{12}{5}-2}{\frac{13}{5}-1} = \frac{\frac{2}{5}}{\frac{8}{5}} = \frac{2}{8} = \frac{1}{4}$.

The bottommost line in gray includes two points $(-1, -2)$ and $(\frac{13}{5}, \frac{12}{5})$.

So the slope of the bottommost line is $\frac{\frac{12}{5}-(-2)}{\frac{13}{5}-(-1)}=\frac{\frac{22}{5}}{\frac{18}{5}}=\frac{22}{18}=\frac{11}{9}$.

We know that for each $\boldsymbol{k}$, $\mathbf{(2}\boldsymbol{k}\mathbf{-1)}\boldsymbol{x}\mathbf{-(3}\boldsymbol{k}\mathbf{+1)}\boldsymbol{y}\mathbf{+5+2}\boldsymbol{k}\mathbf{=0}$ indicates each line in the group. So the slope of each line in the group can be indicated by $\frac{2k-1}{3k+1}$. Thus, we get

$$\frac{2k-1}{3k+1}=\frac{1}{4}\Rightarrow 2k-1=4(3k+1)=12k+4\Rightarrow 10k=-5\Rightarrow k=-\frac{1}{2}.$$

$$\frac{2k-1}{3k+1}=\frac{11}{9}\Rightarrow 9(2k-1)=11(3k+1)\Rightarrow 18k-9=33k+11\Rightarrow 15k=-20\Rightarrow k=-\frac{4}{3}.$$

Therefore, we get $-\frac{4}{3}\le k\le -\frac{1}{2}$.

In short:

To begin with, $(2k-1)x-(3k+1)y+5+2k=0\Rightarrow k(2x-3y+2)-(x+y-5)=0$.

Next, finding the point where two lines $2x-3y+2=0$ and $x+y-5=0$ meet, we get

$-2(x+y-5)=0\Rightarrow -2x-2y+10=0$.

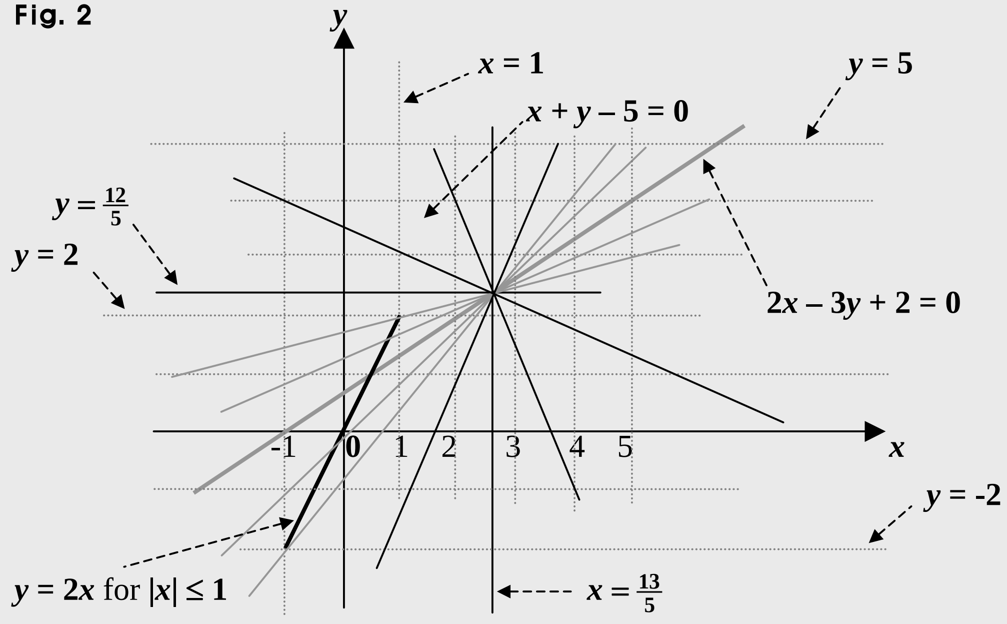
$2x-3y+2+(-2x-2y+10)=0\Rightarrow -5y+12=0\Rightarrow y=\frac{12}{5}$

$\Rightarrow x+y-5=x+\frac{12}{5}-5=0\Rightarrow x=\frac{13}{5}$. Thus, the two lines meet at $(\frac{13}{5},\frac{12}{5})$.

So for every $\boldsymbol{k}$, every line $\boldsymbol{A}$ passes through $(\frac{13}{5},\frac{12}{5})$.

Putting in a graph some lines $\boldsymbol{A}$ represents and the line segment $\boldsymbol{B}$, we get

Fig. 2



The endpoints of the line segment ***B*** are **(-1, -2)** and **(1, 2)**.

The slope of the line having two points **(1, 2)** and $\left(\frac{13}{5}, \frac{12}{5}\right)$ is $\frac{\frac{12}{5}-2}{\frac{13}{5}-1} = \frac{\frac{2}{5}}{\frac{8}{5}} = \frac{2}{8} = \frac{1}{4}$.

The slope of the line having two points **(-1, -2)** and $\left(\frac{13}{5}, \frac{12}{5}\right)$ is $\frac{\frac{12}{5}-(-2)}{\frac{13}{5}-(-1)} = \frac{\frac{22}{5}}{\frac{18}{5}} = \frac{22}{18} = \frac{11}{9}$.

For each k, $(2k - 1)x - (3k + 1)y + 5 + 2k = 0$ indicates each line passing through the point $\left(\frac{13}{5}, \frac{12}{5}\right)$. So the slope of each of the lines can be indicated by $\frac{2k-1}{3k+1}$. Then,

$$\frac{2k-1}{3k+1} = \frac{1}{4} \Rightarrow 2k - 1 = 4(3k+1) = 12k + 4 \Rightarrow 10k = -5 \Rightarrow k = -\frac{1}{2}.$$

$$\frac{2k-1}{3k+1} = \frac{11}{9} \Rightarrow 9(2k-1) = 11(3k+1) \Rightarrow 18k - 9 = 33k + 11 \Rightarrow 15k = -20 \Rightarrow k = -\frac{4}{3}.$$

Therefore, we get $-\frac{4}{3} \le k \le -\frac{1}{2}$.