

Examples 2 in Parabolas

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Examples 2

0. Assuming a and c are constant, a parabola $\frac{y}{2} = ax^2 - 2x + \frac{c}{2}$ includes a point $(\frac{1}{2}, -\frac{1}{2})$, and a line $y = x - 1$ includes the vertex, find a and c .

1. Assuming b and c are constant, a parabola $y = 6x^2 + bx + c$ includes a point $(2, 2)$, and a line $y = x - 1$ includes the vertex, find b and c .

2. Assuming a , b and c are constant, a parabola $y = ax^2 + bx + c$ has two points $(1, 1)$ and $(2, 3)$, and the vertex is in a line $y = 2x - 1$, find a , b , and c .

3. Assuming a , b and c are constant, and a parabola $y = ax^2 + bx + c$ is passing through two points $(1, 1)$ and $(2, 4)$, and has the vertex in a line $y = 2x - 1$, find a , b , and c .

Suggestions or Solutions
To the Problem in the Example 0

Assuming a and c are constant, a parabola $\frac{y}{2} = ax^2 - 2x + \frac{c}{2}$ includes a point $(\frac{1}{2}, -\frac{1}{2})$, and a line $y = x - 1$ includes the vertex, find a and c .

Putting first, the point $(\frac{1}{2}, -\frac{1}{2})$ into the parabola $\frac{y}{2} = ax^2 - 2x + \frac{c}{2}$, we get

$$\frac{1}{2}(-\frac{1}{2}) = a(\frac{1}{2})^2 - 2 \cdot \frac{1}{2} + \frac{c}{2} \Rightarrow -\frac{1}{4} = \frac{a}{4} - 1 + \frac{c}{2} \Rightarrow -1 = a - 4 + 2c \Rightarrow a + 2c = 3.$$

Next, finding the vertex, we get

$$\begin{aligned} \frac{y}{2} &= ax^2 - 2x + \frac{c}{2} \Rightarrow y = 2ax^2 - 4x + c = 2a(x^2 - \frac{2}{a}x) + c = 2a\{x^2 - \frac{2}{a}x + (\frac{1}{a})^2 - (\frac{1}{a})^2\} + c \\ &= 2a(x - \frac{1}{a})^2 - 2a(\frac{1}{a})^2 + c = 2a(x - \frac{1}{a})^2 - \frac{2}{a} + c = 2a(x - \frac{1}{a})^2 - \frac{2-ac}{a} \\ &\Rightarrow y - \frac{ac-2}{a} = 2a(x - \frac{1}{a})^2. \text{ So the vertex is } (\frac{1}{a}, \frac{ac-2}{a}). \end{aligned}$$

So next, putting the vertex in the line $y = x - 1$, we get

$$\begin{aligned} y = x - 1 &\Rightarrow \frac{ac-2}{a} = \frac{1}{a} - 1 \Rightarrow \frac{ac-2}{a} - \frac{1}{a} = -1 \Rightarrow \frac{ac-3}{a} = -1 \Rightarrow ac - 3 = -a \\ &\Rightarrow ac + a = 3 \Rightarrow a(c+1) = 3 \Rightarrow a = \frac{3}{c+1}. \text{ And we have this, too: } a + 2c = 3. \end{aligned}$$

So now, finding c first, we get

$$\begin{aligned} a + 2c &= \frac{3}{c+1} + 2c = 3 \Rightarrow 3 + 2c(c+1) = 3(c+1) \Rightarrow 3 + 2c^2 + 2c = 3c + 3 \\ &\Rightarrow 2c^2 - c = 0 \Rightarrow c(2c - 1) = 0 \Rightarrow c = 0 \text{ or } \frac{1}{2}. \text{ And next, finding } a, \text{ we get} \\ c = 0 &\Rightarrow a + 2c = a + 0 = 3 \Rightarrow a = 3, \text{ and } c = \frac{1}{2} \Rightarrow a + 2c = a + 1 = 3 \Rightarrow a = 2. \end{aligned}$$

If not quite sure of the idea behind the processes above, follow the steps below.

Since we have two unknowns a and c , we need to get two equations.

First, the parabola $\frac{y}{2} = ax^2 - 2x + \frac{c}{2}$ passes through the point $(\frac{1}{2}, -\frac{1}{2})$, so we get

$\frac{1}{2}(-\frac{1}{2}) = a(\frac{1}{2})^2 - 2 \cdot \frac{1}{2} + \frac{c}{2} \Rightarrow -\frac{1}{4} = \frac{a}{4} - 1 + \frac{c}{2} \Rightarrow -1 = a - 4 + 2c \Rightarrow a + 2c = 3$, which is one of the two equations.

And assuming A is the parabola $\frac{y}{2} = ax^2 - 2x + \frac{c}{2}$, we can put A the way below, too.

$$y = 2ax^2 - 4x + c.$$

So putting the point $(\frac{1}{2}, -\frac{1}{2})$ into $y = 2ax^2 - 4x + c$, we get this, too: $a + 2c = 3$.

What then about the other equation?

We know that the line given passes through the vertex, so getting the vertex and putting it into the line, we get the other equation. How then do we get the vertex?

The axis of symmetry has the vertex, and if a parabola is $y = ax^2 + bx + c$, the axis of symmetry is a line $x = -\frac{b}{2a}$. So?

So the x -coordinate of the vertex is $-\frac{b}{2a}$.

And putting the x -coordinate $-\frac{b}{2a}$ into $y = ax^2 + bx + c$, we get the y -coordinate, which is $-\frac{b^2-4ac}{4a}$. So the vertex of a parabola $y = ax^2 + bx + c$ is $(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})$. How?

Putting $y = ax^2 + bx + c$ in the vertex form, we get $y = a(x + \frac{b}{2a})^2 - \frac{b^2-4ac}{4a}$, which shows the vertex. And the conversion goes the way as follows.

$$\begin{aligned} y &= ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a\{x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2\} + c \\ \Rightarrow y &= a(x + \frac{b}{2a})^2 - a(\frac{b}{2a})^2 + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c = a(x + \frac{b}{2a})^2 - \frac{b^2-4ac}{4a} \\ \Rightarrow y + \frac{b^2-4ac}{4a} &= a(x + \frac{b}{2a})^2. \end{aligned}$$

So putting the parabola $y = 2ax^2 - 4x + c$ in the vertex form, we can get the vertex.

$$\begin{aligned} y &= 2ax^2 - 4x + c = 2a(x^2 - \frac{2}{a}x) + c = 2a\{x^2 - \frac{2}{a}x + (\frac{1}{a})^2 - (\frac{1}{a})^2\} + c \\ &= 2a(x - \frac{1}{a})^2 - 2a(\frac{1}{a})^2 + c = 2a(x - \frac{1}{a})^2 - \frac{2}{a} + c = 2a(x - \frac{1}{a})^2 - \frac{2-ac}{a} \\ \Rightarrow y &= 2a(x - \frac{1}{a})^2 + \frac{ac-2}{a}. \end{aligned}$$

So the vertex we want is $(\frac{1}{a}, \frac{ac-2}{a})$.

So next, putting the vertex above into the line $y = x - 1$, we get the other equation, and we can get it the way below.

$$y = x - 1 \Rightarrow \frac{ac-2}{a} = \frac{1}{a} - 1 \Rightarrow \frac{ac-2}{a} - \frac{1}{a} = -1 \Rightarrow \frac{ac-3}{a} = -1 \Rightarrow ac - 3 = -a$$

$$\Rightarrow ac + a = 3 \Rightarrow a(c + 1) = 3 \Rightarrow a = \frac{3}{c+1}, \text{ which is the other equation.}$$

So we now get to solve a system of two equations, $a + 2c = 3$, and $a = \frac{3}{c+1}$.

Thus next, solving the system above, and finding c first, we get

$$a + 2c = \frac{3}{c+1} + 2c = 3 \Rightarrow 3 + 2c(c+1) = 3(c+1) \Rightarrow 3 + 2c^2 + 2c = 3c + 3$$

$$\Rightarrow 2c^2 - c = 0 \Rightarrow c(2c - 1) = 0 \Rightarrow c = 0 \text{ or } \frac{1}{2}. \text{ So next, finding } a, \text{ we get}$$

$$c = 0 \Rightarrow a + 2c = a + 0 = 3 \Rightarrow a = 3, \text{ and } c = \frac{1}{2} \Rightarrow a + 2c = a + 1 = 3 \Rightarrow a = 2.$$

So we can see that two parabolas can be the solution.

$$\text{One is } \frac{y}{2} = 3x^2 - 2x, \text{ and the other is } \frac{y}{2} = 2x^2 - 2x + \frac{1}{4}.$$

And we can put them this way, too, of course $y = 6x^2 - 4x$, and $y = 4x^2 - 4x + \frac{1}{2}$.

And let's now put the two parabolas and the line in one graph.

Putting first, the two parabolas in the vertex form, we can see the vertex, so we can easily put the parabolas in the graph. So converting the first one, we get

$$y = 6x^2 - 4x = 6\left(x^2 - \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right) = 6\left(x - \frac{1}{3}\right)^2 - \frac{2}{3} \Rightarrow y = 6\left(x - \frac{1}{3}\right)^2 - \frac{2}{3}.$$

So the vertex is $\left(\frac{1}{3}, -\frac{2}{3}\right)$.

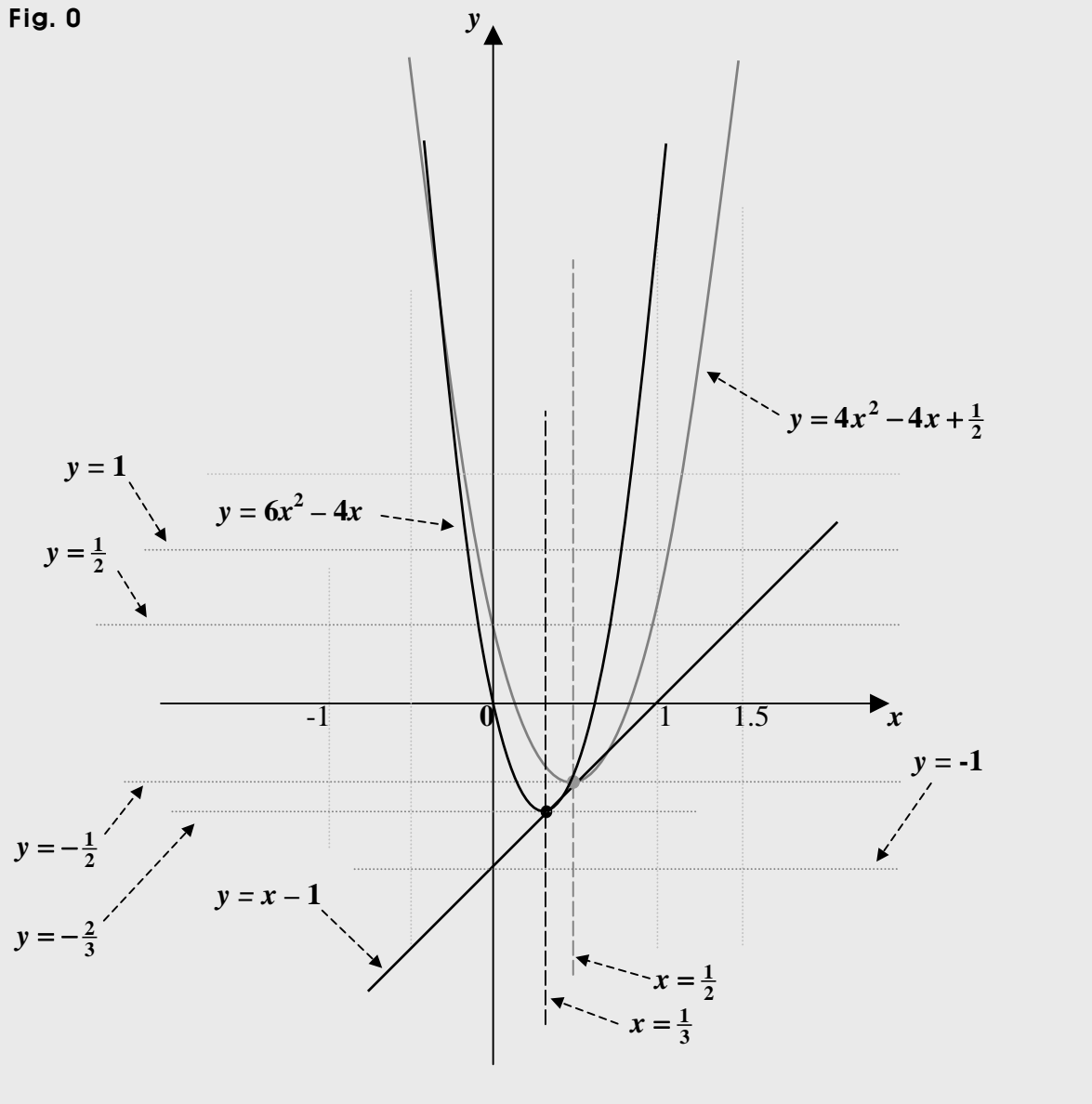
$$\text{And next, converting the second, we get } y = 4x^2 - 4x + \frac{1}{2} = 4(x^2 - x) + \frac{1}{2}$$

$$= 4\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right) + \frac{1}{2} = 4\left(x - \frac{1}{2}\right)^2 - 1 + \frac{1}{2} = 4\left(x - \frac{1}{2}\right)^2 - \frac{1}{2} \Rightarrow y = 4\left(x - \frac{1}{2}\right)^2 - \frac{1}{2}.$$

So the vertex is $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

And now, we can put the two parabolas and the line in one graph the way below.

Fig. 0



Suggestions or Solutions To the Problem in the Example 1

Assuming b and c are constant, a parabola $y = 6x^2 + bx + c$ includes a point $(2, 2)$, and a line $y = x - 1$ includes the vertex, find b and c .

First, assuming (p, q) is the vertex, we can get $y = 6x^2 + bx + c \Rightarrow y - q = 6(x - p)^2$.

So next, we can get $(2, 2) \Rightarrow 2 - q = 6(2 - p)^2$.

And also, the vertex (p, q) is in the line $y = x - 1$, too, so we get $q = p - 1$.

Then, we get $q = p - 1 \Rightarrow 2 - q = 3 - p$, and $6(2 - p)^2 = 6(4 - 4p + p^2) = 24 - 24p + 6p^2$.

So first, $3 - p = 24 - 24p + 6p^2 \Rightarrow 6p^2 - 23p + 21 = (2p - 3)(3p - 7) = 0 \Rightarrow p = \frac{3}{2}$ or $\frac{7}{3}$.

So next, we get $p = \frac{3}{2} \Rightarrow q = p - 1 = \frac{3}{2} - 1 = \frac{1}{2}$, and $p = \frac{7}{3} \Rightarrow q = p - 1 = \frac{7}{3} - 1 = \frac{4}{3}$.

Thus first, $y - \frac{1}{2} = 6(x - \frac{3}{2})^2 = 6(x^2 - 3x + \frac{9}{4}) = 6x^2 - 18x + \frac{27}{2}$

$\Rightarrow y = 6x^2 - 18x + \frac{27}{2} + \frac{1}{2} = 6x^2 - 18x + 14 \Rightarrow b = -18$ and $c = 14$.

Next, $y - \frac{4}{3} = 6(x - \frac{7}{3})^2 = 6(x^2 - \frac{14}{3}x + \frac{49}{9}) = 6x^2 - 28x + \frac{98}{3}$

$\Rightarrow y = 6x^2 - 28x + \frac{98}{3} + \frac{4}{3} = 6x^2 - 28x + 34 \Rightarrow b = -28$ and $c = 34$.

If not quite sure of the idea behind the processes above, follow the steps below.

Assuming first, D is the parabola $y = 6x^2 + bx + c$, we can put D in the vertex form the way as follows: $y = a(x - p)^2 + q$. Then, (p, q) is the vertex, and we get $a = 6$.

Why do we put it that way though?

That's because not only D but the line given is passing through the vertex, so we can get a system of two equations for m and n . And solving the system, and simplifying the equation in the vertex form, we can get the values of b and c .

So to begin with, putting D in the vertex form, we get $y = 6(x - p)^2 + q$.

And we can put it this way, too: $y - q = 6(x - p)^2$.

So next, putting the point $(2, 2)$ into D , we get $2 - q = 6(2 - p)^2$, which is now, one of the two equations for m and n .

And the equation is quadratic. So we can expect two parabolas that can be D . We'll see it shortly.

Next, the vertex (p, q) is in the line $y = x - 1$, too, so we get $q = p - 1$, which is the other of the two equations.

So we are now all set to go and get the vertex, then the equation of the parabola D .

First, we can get $q = p - 1 \Rightarrow -q = 1 - p \Rightarrow 2 - q = 3 - p$.

So next, we can get $2 - q = 6(2 - p)^2 \Rightarrow 3 - p = 6(2 - p)^2$.

And we can put it the way below, too, of course.

$$2 - q = 6(2 - p)^2 \Rightarrow 2 - (p - 1) = 6(2 - p)^2 \Rightarrow 3 - p = 6(2 - p)^2.$$

So anyway, we can get

$$3 - p = 6(2 - p)^2 = 6(4 - 4p + p^2) = 24 - 24p + 6p^2 \Rightarrow 3 - p = 24 - 24p + 6p^2$$

$$\Rightarrow 6p^2 - 23p + 21 = 0 \Rightarrow (2p - 3)(3p - 7) = 0 \Rightarrow p = \frac{3}{2} \text{ or } \frac{7}{3}.$$

So next, we get $p = \frac{3}{2} \Rightarrow q = p - 1 = \frac{3}{2} - 1 = \frac{1}{2}$, and $p = \frac{7}{3} \Rightarrow q = p - 1 = \frac{7}{3} - 1 = \frac{4}{3}$.

So two parabolas can be the parabola D , and the two are as follows.

$$y - \frac{1}{2} = 6(x - \frac{3}{2})^2, \text{ and } y - \frac{4}{3} = 6(x - \frac{7}{3})^2.$$

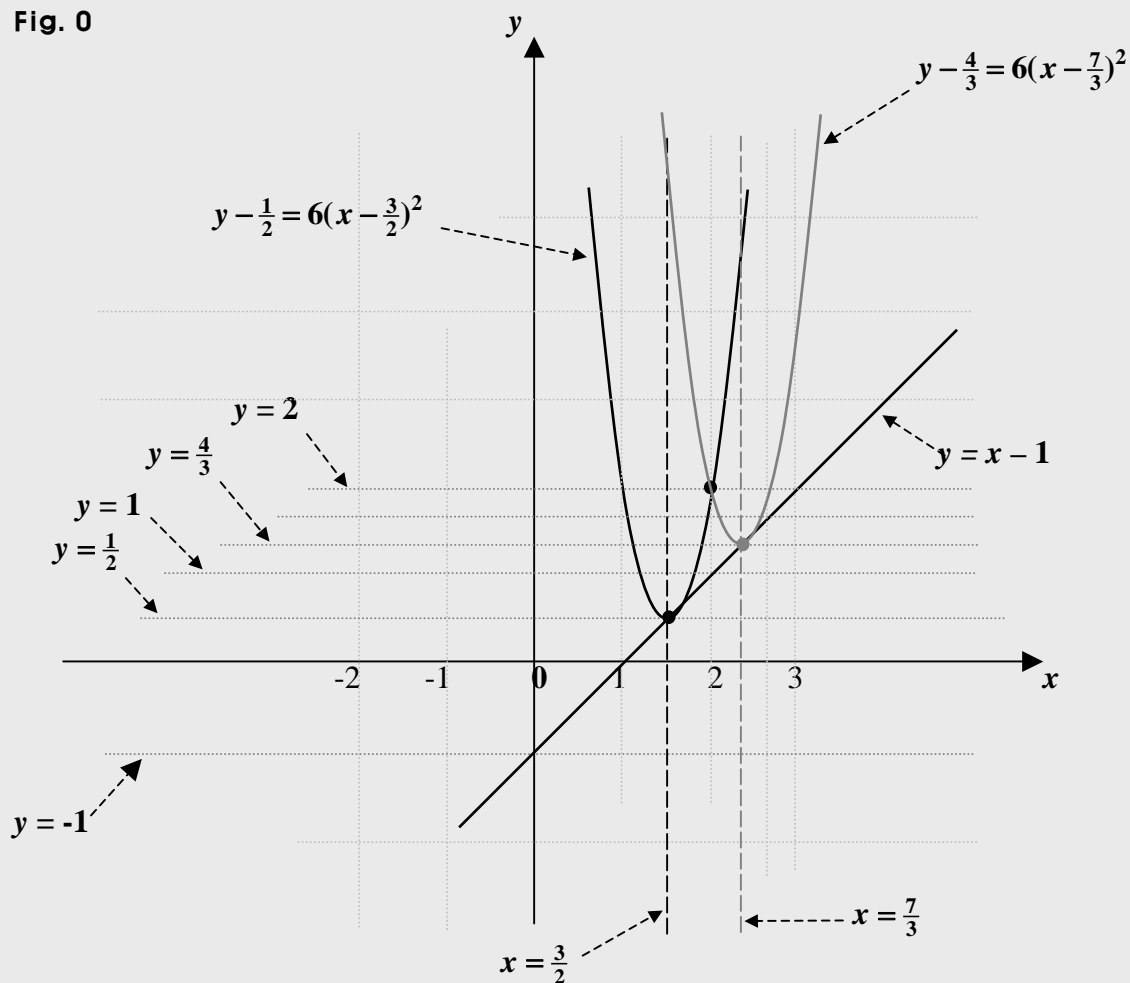
$$\text{Then first, } y - \frac{1}{2} = 6(x - \frac{3}{2})^2 = 6(x^2 - 3x + \frac{9}{4}) = 6x^2 - 18x + \frac{27}{2}$$

$$\Rightarrow y = 6x^2 - 18x + \frac{27}{2} + \frac{1}{2} = 6x^2 - 18x + 14 \Rightarrow b = -18, \text{ and } c = 14.$$

$$\text{Next, } y - \frac{4}{3} = 6\left(x - \frac{7}{3}\right)^2 = 6\left(x^2 - \frac{14}{3}x + \frac{49}{9}\right) = 6x^2 - 28x + \frac{98}{3}$$

$$\Rightarrow y = 6x^2 - 28x + \frac{98}{3} + \frac{4}{3} = 6x^2 - 28x + 34 \Rightarrow b = -28, \text{ and } c = 34.$$

Fig. 0



Suggestions or Solutions
To the Problem in the Example 2

Assuming a , b and c are constant, a parabola $y = ax^2 + bx + c$ has two points $(1, 1)$ and $(2, 3)$, and the vertex is in a line $y = 2x - 1$, find a , b , and c .

The vertex of a parabola $y = ax^2 + bx + c$ is $(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})$.

So first, since the vertex is in the line $y = 2x - 1$, we get $-\frac{b^2-4ac}{4a} = 2(-\frac{b}{2a}) - 1$.

Next, the two points $(1, 1)$ and $(2, 3)$ are in the parabola, so we get

$$(1, 1) \Rightarrow 1 = a + b + c, \text{ and also, } (2, 3) \Rightarrow 3 = 4a + 2b + c.$$

To begin with, $-\frac{b^2-4ac}{4a} = 2(-\frac{b}{2a}) - 1 = -\frac{b}{a} - 1 = -\frac{b+a}{a} \Rightarrow b^2 - 4ac = 4(a + b)$.

Next, $a + b + c = 1 \Rightarrow a + b = 1 - c$, and thus, we get $b^2 - 4ac = 4(1 - c)$.

$$\begin{aligned} \text{Next, } 4a + 2b + c = 3 &\Rightarrow c(4a + 2b + c) = 3c \Rightarrow 4ac + 2bc + c^2 = 3c \\ &\Rightarrow 4ac = 3c - 2bc - c^2. \end{aligned}$$

Thus, $b^2 - 4ac = b^2 - (3c - 2bc - c^2) = b^2 - 3c + 2bc + c^2 = 4(1 - c) = 4 - 4c$.

$$\text{So } b^2 - 3c + 2bc + c^2 = 4 - 4c \Rightarrow b^2 + 2bc + c^2 + c - 4 = 0 \Rightarrow (b + c)^2 + c - 4 = 0.$$

Next, we have $4a + 2b + c = 3$ and $a + b + c = 1$.

$$\text{So we get } 4(a + b + c) - (4a + 2b + c) = 4 - 3 \Rightarrow 2b + 3c = 1.$$

$$\text{Meanwhile, } 2b + 3c = 1 \Rightarrow 2(b + c) + c = 1 \Rightarrow 2(b + c) = 1 - c \Rightarrow b + c = \frac{1-c}{2}.$$

$$\text{So we get } (b + c)^2 + c - 4 = 0 \Rightarrow \frac{(1-c)^2}{4} + c - 4 = 0 \Rightarrow (1-c)^2 + 4c - 16 = 0.$$

$$\text{Thus, } 1 - 2c + c^2 + 4c - 16 = 0 \Rightarrow c^2 + 2c - 15 = 0 \Rightarrow (c + 5)(c - 3) = 0 \Rightarrow c = -5 \text{ or } 3.$$

So first, $c = -5$, $a + b + c = 1 \Rightarrow a + b = 6$, and $4a + 2b + c = 3 \Rightarrow 2a + b = 4$.

$$\text{Then, } 2a + b - (a + b) = 4 - 6 = -2 \Rightarrow a = -2 \Rightarrow a + b = -2 + b = 6 \Rightarrow b = 8.$$

And next, $c = 3$, $a + b + c = 1 \Rightarrow a + b = -2$, and $4a + 2b + c = 3 \Rightarrow 2a + b = 0$.

$$\text{Then, } 2a + b - (a + b) = 0 - (-2) = 2 \Rightarrow a = 2 \Rightarrow a + b = 2 + b = -2 \Rightarrow b = -4.$$

Thus, we get two cases.

One is $a = -2$, $b = 8$, and $c = -5$. And the other is $a = 2$, $b = -4$, and $c = 3$.

If not quite sure of the idea behind the processes above, follow the steps below.

First, the vertex of the parabola $y = ax^2 + bx + c$ is $(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})$, and we can quickly get it putting the parabola in the vertex form.

And the vertex is in the line $y = 2x - 1$, too, so we get $-\frac{b^2-4ac}{4a} = 2(-\frac{b}{2a}) - 1$.

Next, the two points $(1, 1)$ and $(2, 3)$ are in the parabola $y = ax^2 + bx + c$, so we get $(1, 1) \Rightarrow 1 = a \cdot 1^2 + b \cdot 1 + c = a + b + c$, and also, $(2, 3) \Rightarrow 3 = 4a + 2b + c$.

So we now have a system of three equations.

One is $-\frac{b^2-4ac}{4a} = 2(-\frac{b}{2a}) - 1$. Another is $1 = a + b + c$. And the other is $3 = 4a + 2b + c$.

So we've got to do some algebra. Let's now, get rid of a first.

Beginning with the first equation, we can put it the way below.

$-\frac{b^2-4ac}{4a} = 2(-\frac{b}{2a}) - 1 = -\frac{b}{a} - 1 = -\frac{b+a}{a} \Rightarrow b^2 - 4ac = 4(a+b)$, which is not exactly though, the same as the original equation. So keep in mind that a cannot be 0, because it is originally in the denominator $4a$, and in $2a$, too, and a denominator cannot be 0.

Next, moving on to $a + b + c = 1$, we can see $a + b$, which is in the first equation above. And we can put it this way: $a + b = 1 - c$. Then, we can get this: $b^2 - 4ac = 4(1 - c)$.

Next, looking at $4a + 2b + c = 3$, we can notice that we can make $4ac$, which belongs to the equation above.

So making it, we get $c(4a + 2b + c) = 3c \Rightarrow 4ac + 2bc + c^2 = 3c \Rightarrow 4ac = 3c - 2bc - c^2$.

Then, we get $b^2 - 4ac = b^2 - (3c - 2bc - c^2) = b^2 - 3c + 2bc + c^2$.

And we can get this: $4(1 - c) = 4 - 4c$.

So we get $b^2 - 4ac = 4(1 - c) \Rightarrow b^2 - 3c + 2bc + c^2 = 4 - 4c \Rightarrow b^2 + 2bc + c^2 + c - 4 = 0 \Rightarrow (b + c)^2 + c - 4 = 0$. So finally, we can see no a . So it's gone for now.

And we can quickly get another equation for b and c only.

We have $4a + 2b + c = 3$ and $a + b + c = 1$.

So we can get $4(a + b + c) - (4a + 2b + c) = 4 - 3 \Rightarrow 2b + 3c = 1$.

So we now have $(b + c)^2 + c - 4 = 0$, and $2b + 3c = 1$.

Then again, looking at closely the second equation above, we can see that it has $b + c$, which is in this: $(b + c)^2 + c - 4 = 0$. So it looks like this time, b wants to go away.

So now, beginning with the second equation, we can get:

$$2b + 3c = 1 \Rightarrow 2(b + c) + c = 1 \Rightarrow 2(b + c) = 1 - c \Rightarrow b + c = \frac{1-c}{2}.$$

And next, putting in into the first equation, we can get

$$(b + c)^2 + c - 4 = 0 \Rightarrow \frac{(1-c)^2}{4} + c - 4 = 0 \Rightarrow (1-c)^2 + 4c - 16 = 0, \text{ which is now free of } b,$$

and is a simple quadratic equation for c only. So solving it, we get

$$1 - 2c + c^2 + 4c - 16 = 0 \Rightarrow c^2 + 2c - 15 = 0 \Rightarrow (c + 5)(c - 3) = 0 \Rightarrow c = -5 \text{ or } 3.$$

So next, we want to get a and b . Getting them, we want to use c , of course.

So let's begin with the case where $c = -5$.

First, we have $a + b + c = 1$, so we get $a + b = 6$.

Next, we have $4a + 2b + c = 3$, so we get $2a + b = 4$.

And next, subtracting the second from the first, we get

$$(2a + b) - (a + b) = 4 - 6 = -2 \Rightarrow a = -2 \Rightarrow a + b = -2 + b = 6 \Rightarrow b = 8.$$

Let's next, move on to the case where $c = -5$.

Since $a + b + c = 1$, we get $a + b = -2$, and since $4a + 2b + c = 3$, we get $2a + b = 0$.

So we now have $a + b = -2$, and $2a + b = 0$.

Thus, we get $(2a + b) - (a + b) = 0 - (-2) = 2 \Rightarrow a = 2 \Rightarrow a + b = 2 + b = -2 \Rightarrow b = -4$.

Therefore, we get to have two parabolas.

One is $y = -2x^2 + 8x - 5$, and the others is $y = 2x^2 - 4x + 3$.

And we can do this problem the way we did the problem in the example 1, too. That is, we can do it beginning with putting $y = ax^2 + bx + c$ in the vertex form

So putting it in the vertex form, we get $y = a(x - m)^2 + n$. Then (m, n) is the vertex.

So next, putting the point $(1, 1)$ into the equation above, we get $1 = a(1 - m)^2 + n$.

Next, the vertex (m, n) is in the line $y = 2x - 1$, too, so we can get this: $n = 2m - 1$.

Next, the other point $(2, 3)$ is in the parabola, too, so we get $3 = a(2 - m)^2 + n$.

So we are now all set to go and get the vertex and a , then the parabola.

First, we have $n = 2m - 1$, so we get

$$1 - n = a(1 - m)^2 \Rightarrow 1 - (2m - 1) = a(1 - m)^2 \Rightarrow 2 - 2m = a(1 - m)^2 \dots\dots (1)$$

$$3 - n = a(2 - m)^2 \Rightarrow 3 - (2m - 1) = a(2 - m)^2 \Rightarrow 4 - 2m = a(2 - m)^2 \dots\dots (2)$$

Next, subtracting the first equation from the second, we get

$$(4 - 2m) - (2 - 2m) = a(1 - m)^2 - a(2 - m)^2.$$

So next, using the factorization identity, $A^2 - B^2 = (A + B)(A - B)$, we get

$$2 = a(2 - m + 1 - m)(2 - m - 1 + m) = a(3 - 2m) \Rightarrow 2 = a(3 - 2m) \dots\dots (3)$$

Next, dividing the equation (1) by the equation (3) above, we get

$$\frac{2 - 2m}{2} = \frac{a(1 - m)^2}{a(3 - 2m)} \Rightarrow 1 - m = \frac{(1 - m)^2}{3 - 2m} \Rightarrow (1 - m)(3 - 2m) = (1 - m)^2, \text{ so we get}$$

$$3 - 2m - 3m + 2m^2 = 1 - 2m + m^2 \Rightarrow m^2 - 3m + 2 = (m - 1)(m - 2) = 0 \Rightarrow m = 1 \text{ or } 2.$$

And if $m = 1$ or 2 , we get $3 - 2m \neq 0$. So we get $m = 1$ or 2 .

Next, finding a and n , we get

$$\text{First, we get } m = 1 \Rightarrow n = 2m - 1 = 2 - 1 = 1, \text{ and also, } 2 = a(3 - 2m) = a \cdot 1 \Rightarrow a = 2.$$

$$\text{Next, we get } m = 2 \Rightarrow n = 2m - 1 = 4 - 1 = 3, \text{ and also, } 2 = a(3 - 2m) = a \cdot (-1) \Rightarrow a = -2.$$

Therefore, we get to have two parabolas as follows.

$$y = 2(x - 1)^2 + 1, \text{ and } y = -2(x - 2)^2 + 3.$$

Next, putting the two above in the non-vertex form, we get

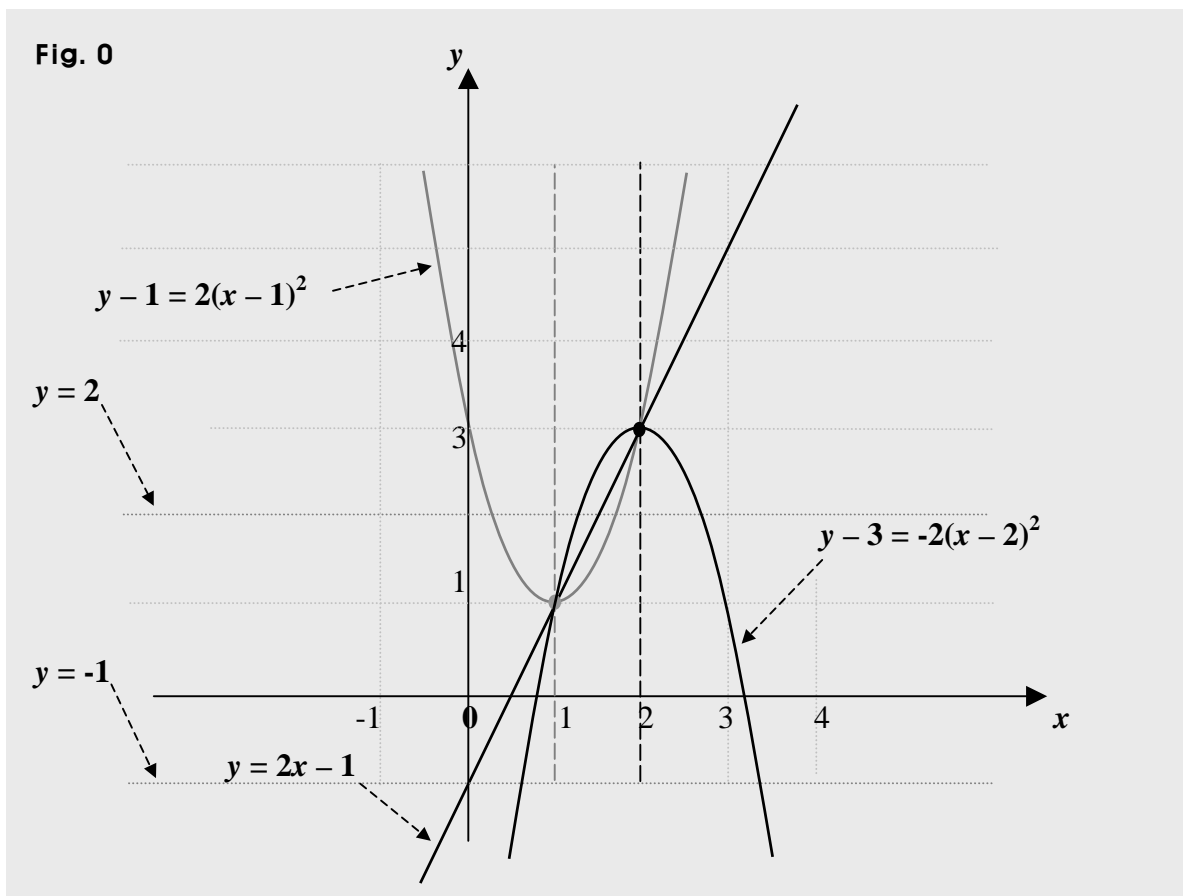
$$y - 1 = 2(x - 1)^2 = 2(x^2 - 2x + 1) \Rightarrow y = 2x^2 - 4x + 3 \Rightarrow a = 2, b = -4, \text{ and } c = 3.$$

$$y - 3 = -2(x - 2)^2 = -2(x^2 - 4x + 4) \Rightarrow y = -2x^2 + 8x - 5 \Rightarrow a = -2, b = 8, \text{ and } c = -5.$$

Let's now, put in a graph the parabolas and the line.

$$y = -2x^2 + 8x - 5 = -2(x^2 - 4x) - 5 = -2(x^2 - 4x + 4 - 4) - 5 = -2(x - 2)^2 + 3 \\ \Rightarrow y - 3 = -2(x - 2)^2.$$

$$y = 2x^2 - 4x + 3 = 2(x^2 - 2x) + 3 = 2(x - 1)^2 + 1 \Rightarrow y - 1 = 2(x - 1)^2.$$



Suggestions or Solutions
To the Problem in the Example 3

Assuming that a , b and c are constant, and that a parabola $y = ax^2 + bx + c$ passes through $(1, 1)$ and $(2, 4)$, and has the vertex in a line $y = 2x - 1$, find a , b , and c .

This example is no other than the previous example. The only difference is the y -coordinate at one of the two points given. This example has $(2, 4)$ instead of $(2, 3)$.

The purpose of this example is how the solution changes with that difference, and you can do some more practice on algebra about a system of equations.

So to begin with, getting the vertex of $y = ax^2 + bx + c$, we get $(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})$.

So next, since the vertex is in the line $y = 2x - 1$, too, we can get an equation as below.

$$-\frac{b^2-4ac}{4a} = 2(-\frac{b}{2a}) - 1.$$

Next, the two points $(1, 1)$ and $(2, 4)$ are in the parabola, so we get:

$$(1, 1) \Rightarrow 1 = a + b + c, \text{ and also, } (2, 4) \Rightarrow 4 = 4a + 2b + c.$$

So we need to solve the system of the three equations, which are as follows.

$$-\frac{b^2-4ac}{4a} = 2(-\frac{b}{2a}) - 1 \dots\dots\dots (1) \quad 1 = a + b + c \dots\dots\dots (2) \quad 4 = 4a + 2b + c \dots\dots\dots (3)$$

And beginning with the equation (1), we can get

$$-\frac{b^2-4ac}{4a} = 2(-\frac{b}{2a}) - 1 = -\frac{b}{a} - 1 = -\frac{b+a}{a} \Rightarrow b^2 - 4ac = 4(a+b) \dots\dots\dots (4)$$

Next, moving on to the equation (2), we can get $a + b + c = 1 \Rightarrow a + b = 1 - c$, and thus, putting it into the equation (4) above, we get $b^2 - 4ac = 4(1 - c) \dots\dots\dots (5)$

Next, looking at the equation (3), we can make $4ac$ out of it. So next, we can get

$$4a + 2b + c = 4 \Rightarrow c(4a + 2b + c) = 4ac + 2bc + c^2 = 4c \Rightarrow 4ac = 4c - 2bc - c^2.$$

Thus, we get $b^2 - 4ac = b^2 - 4c + 2bc + c^2$. And we can have $4(1 - c) = 4 - 4c$.

So getting back to the equation (5) above, we can get

$$b^2 - 4ac = 4(1 - c) \Rightarrow b^2 - 4c + 2bc + c^2 = 4 - 4c \Rightarrow b^2 + 2bc + c^2 - 4 = 0 \Rightarrow (b + c)^2 = 4.$$

Thus, we get $b + c = 2$ or -2 .

So next, using the result above, we can find a , since we have $a + b + c = 1$.

So finding a , we get

$$b + c = 2 \Rightarrow a + b + c = a + 2 = 1 \Rightarrow a = -1.$$

$$b + c = -2 \Rightarrow a + b + c = a - 2 = 1 \Rightarrow a = 3.$$

So we now have two cases.

One is $a = -1$, and $b + c = 2$. And the other is $a = 3$, and $b + c = -2$.

Next, in each case, using $4a + 2b + c = 4$, we can get b and c .

So beginning with the case where $a = -1$, and $b + c = 2$, we get this first:

$$a = -1 \Rightarrow 4a + 2b + c = -4 + 2b + c = 4 \Rightarrow 2b + c = 8.$$

And next, we have $b + c = 2$. So we get $2b + c = b + b + c = b + 2 = 8 \Rightarrow b = 6$.

And since $b = 6$, we can get $b + c = 2 \Rightarrow 6 + c = 2 \Rightarrow c = -4$.

So we now have $a = -1$, $b = 6$, and $c = -4$, which means the parabola is $y = -x^2 + 6x - 4$.

Let's next, move on to the case where $a = 3$, and $b + c = -2$.

Then first, since $a = 3$, and $4a + 2b + c = 4$, we can get $2b + c = 4 - 4a = 4 - 12 = -8$.

That is, we get $2b + c = -8$.

Next, since $b + c = -2$, and $2b + c = -8$, we get $2b + c = b + b + c = b - 2 = -8 \Rightarrow b = -6$.

So next, getting back to $b + c = -2$, we get $-6 + c = -2 \Rightarrow c = 4$.

So we now have $a = 3$, $b = -6$, and $c = 4$, which means the parabola is $y = 3x^2 - 6x + 4$.

Therefore, we get to have two parabolas as follows.

$$y = -x^2 + 6x - 4, \text{ and } y = 3x^2 - 6x + 4.$$

Fig. 0

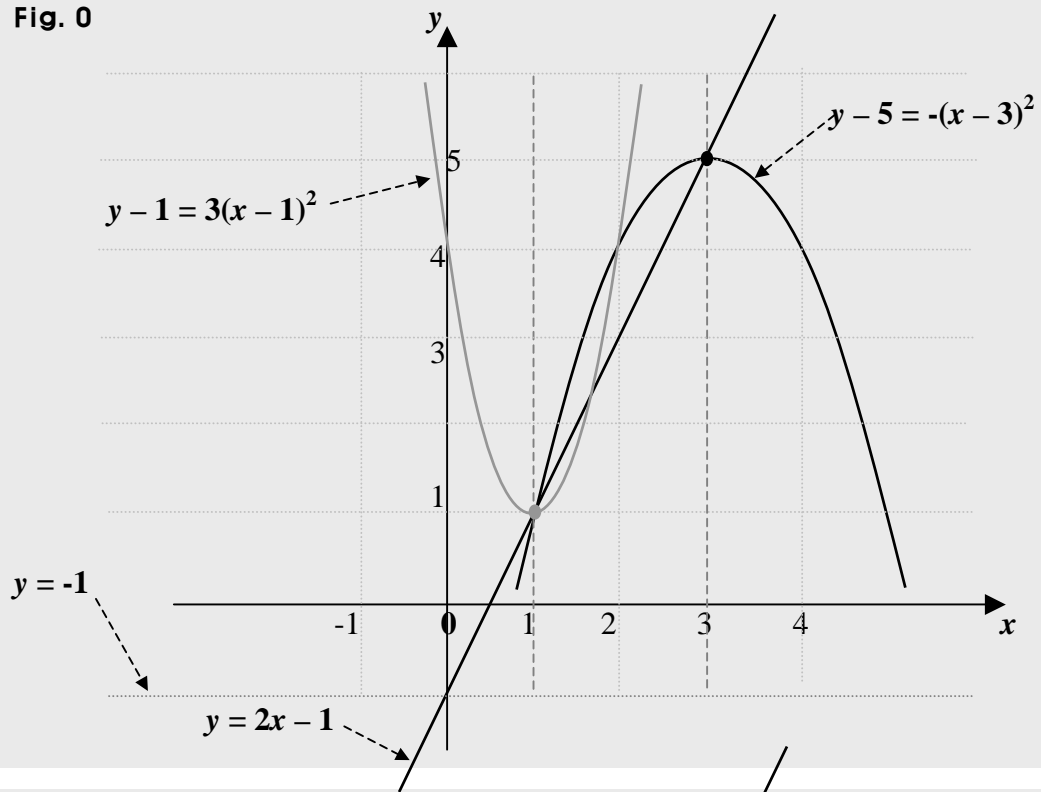


Fig. 1

