

Examples 3 in Parabolas

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Examples 3

0. Suppose m and n are constants, A is a parabola $y = -4x^2 + 8mx - n(n + 2)$, and B is another parabola $y = 8x^2 - 2x + \frac{5}{4}$. Then, assuming the vertex of A is in B , find m and n .

1. Suppose b and c are constants, A is a parabola $y = x^2 + bx + c$, and B is another parabola $y = -2x^2 - x + 5$. Then, assuming the vertex of A is in B , and that of B , too, is in A , find b and c .

2. Suppose b and c are constants, A is a parabola $y = 2x^2 + bx + c$, and B is another parabola $y = -2x^2 + 8x - 5$. Then, assuming the vertex of A is in B , and that of B , too, is in A , find b and c .

3. Suppose b and c are constants, A is a parabola $y = x^2 + bx + c$, and B is another parabola $y = -2x^2 + 8x - 5$. Then, assuming the vertex of A is in B , and that of B , too, is in A , find b and c .

Suggestions or Solutions
To the Problem in the Example 0

Suppose m and n are constants, A is a parabola $y = -4x^2 + 8mx - n(n + 2)$, and B is a parabola $y = 8x^2 - 2x + \frac{5}{4}$. Then, assuming the vertex of A is in B , find m and n .

$$y = -4x^2 + 8mx - n(n + 2) = -4(x^2 - 2mx) - n(n + 2) = -4(x - m)^2 + 4m^2 - n(n + 2).$$

So the vertex of $A = \{m, 4m^2 - n(n + 2)\}$, and putting it into the parabola B , we get

$$y = 8x^2 - 2x + \frac{5}{4} \Rightarrow 4m^2 - n(n + 2) = 8m^2 - 2m + \frac{5}{4} \Rightarrow 4m^2 - 2m + n(n + 2) + \frac{5}{4} = 0.$$

$$4m^2 - 2m + n(n + 1) + \frac{5}{4} = 4m^2 - 2m + \frac{1}{4} + n^2 + 2n + 1 = (2m - \frac{1}{2})^2 + (n + 1)^2 = 0.$$

Thus, $2m - \frac{1}{2} = 0 \Rightarrow m = \frac{1}{4}$, and also, $n + 1 = 0 \Rightarrow n = -1$.

If not quite sure of the idea behind the processes above, follow the steps below.

It looks we need a system of two equations for m and n . How then can we get the two?

To begin with, we know B has the vertex of A . So finding the vertex of A , and putting it into B , we can get one of the two equations. How then can we get the vertex?

Putting A in the vertex form, we can get the vertex. So putting it in the form, we get

$$y = -4x^2 + 8mx - n(n + 2) = -4(x^2 - 2mx) - n(n + 2) = -4(x - m)^2 + 4m^2 - n(n + 2) \\ \Rightarrow y = -4(x - m)^2 + 4m^2 - n(n + 2). \quad \text{So the vertex is } \{m, 4m^2 - n(n + 2)\}.$$

So next, putting the vertex into the parabola B , we get

$$y = 8x^2 - 2x + \frac{5}{4} \Rightarrow 4m^2 - n(n + 2) = 8m^2 - 2m + \frac{5}{4} \Rightarrow 4m^2 - 2m + n^2 + 2n + \frac{5}{4} = 0,$$

which is one of two equations for m and n . So we need the other equation for m and n . However, we don't have any information on the other equation we need. What then?

It is not always the case where we can get two separate equations finding two unknowns.

That is to say that we can get the other equation from inside the one equation we've got.

Examining the expression above, we can see that both unknowns are squared. So we can use an algebraic property on real numbers as follows.

• $A^2 + B^2 = 0 \Rightarrow A$ and B both are 0 .

So in the real number space, if a sum of squares is 0, every square has to be 0.

In fact, $A^2 + B^2 + C^2 + \dots = 0 \Rightarrow A, B, C, \dots$ are all 0 .

And by the same token, $(A - a)^2 + (B - b)^2 + (C - c)^2 = 0 \Rightarrow A = a, B = b,$ and $C = c$.

Also, $A^4 + B^4 + C^4 + \dots = 0 \Rightarrow A, B, C, \dots$ are all 0 .

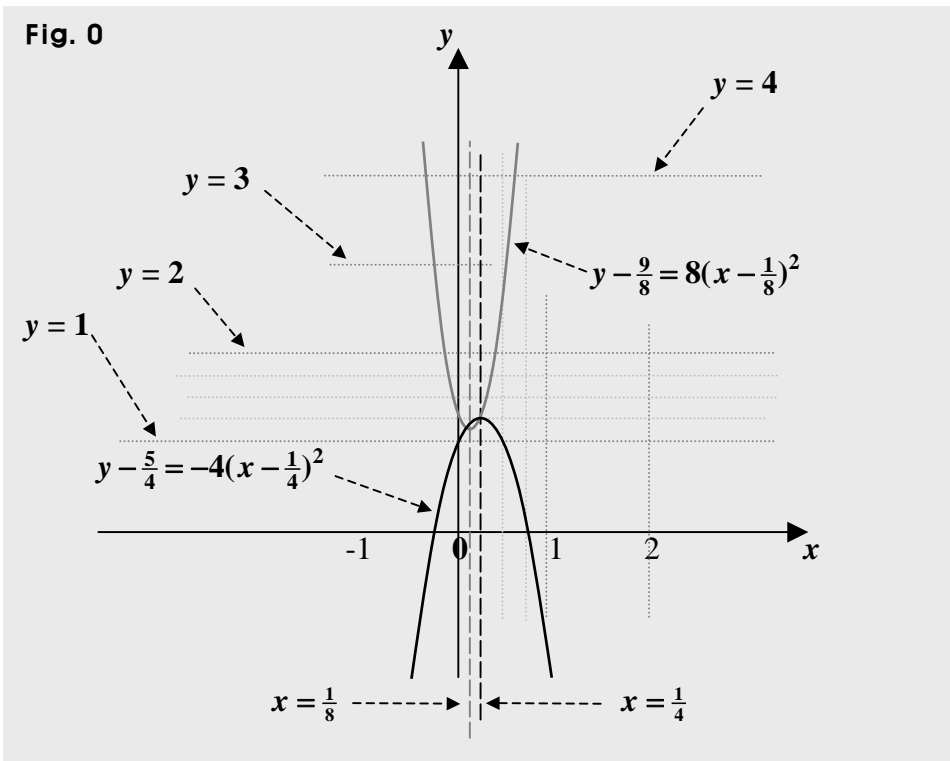
So if n is even, $A^n + B^n + C^n + \dots = 0 \Rightarrow A, B, C, \dots$ are all 0 .

So we may want to try convert the equation the way below.

$$4m^2 - 2m + n(n+1) + \frac{5}{4} = 4m^2 - 2m + \frac{1}{4} + n^2 + 2n + 1 = (2m - \frac{1}{2})^2 + (n+1)^2 = 0.$$

So we get $2m - \frac{1}{2} = 0 \Rightarrow m = \frac{1}{4}$, and also, $n + 1 = 0 \Rightarrow n = -1$.

Thus, sometimes, setting up a system of equations, we need to consider the conditions or constraints as the algebraic property explained above, and make use of them.



Suggestions or Solutions To the Problem in the Example 1

Suppose b and c are constants, A is a parabola $y = x^2 + bx + c$, and B is another parabola $y = -2x^2 - x + 5$. Then, assuming the vertex of A is in B , and that of B , too, is in A , find b and c .

To begin with, $y = x^2 + bx + c = x^2 + bx + (\frac{b}{2})^2 - (\frac{b}{2})^2 + c = (x + \frac{b}{2})^2 - (\frac{b}{2})^2 + c$
 $\Rightarrow y = (x + \frac{b}{2})^2 - \frac{b^2 - 4c}{4}$. So the vertex of A is $(-\frac{b}{2}, -\frac{b^2 - 4c}{4})$.

Next, putting the vertex into B , we get

$$-\frac{b^2 - 4c}{4} = -2(-\frac{b}{2})^2 - (-\frac{b}{2}) + 5 = -\frac{b^2}{2} + \frac{b}{2} + 5 \Rightarrow b^2 - 4c = 2b^2 - 2b - 20$$

$$\Rightarrow b^2 - 2b + 4c - 20 = 0.$$

Next, $y = -2x^2 - x + 5 = -2(x^2 + \frac{1}{2}x + \frac{1}{16} - \frac{1}{16}) + 5 = -2(x + \frac{1}{4})^2 + \frac{41}{8}$.

So the vertex of B is $(-\frac{1}{4}, \frac{41}{8})$.

Next, putting the vertex into A , we get

$$\frac{41}{8} = (-\frac{1}{4})^2 + b(-\frac{1}{4}) + c = \frac{1}{16} - \frac{b}{4} + c \Rightarrow 82 = 1 - 4b + 16c \Rightarrow 16c - 4b = 81 \Rightarrow 4c = b + \frac{81}{4}.$$

Thus, finding b and c , we get

$$b^2 - 2b + 4c - 20 = b^2 - 2b + b + \frac{81}{4} - 20 = b^2 - b + \frac{1}{4} = (b - \frac{1}{2})^2 = 0 \Rightarrow b = \frac{1}{2}.$$

$$\Rightarrow 4c = b + \frac{81}{4} = \frac{1}{2} + \frac{81}{4} = \frac{83}{4} \Rightarrow c = \frac{83}{16}.$$

If not quite sure of the idea behind the processes above, follow the steps below.

What we are after in this problem is of course, the values of the two constants b and c . Finding the parabola A is no other than finding b and c .

So let's see what parabola can be the parabola A .

The parabola A is subject to another parabola called B , which is $y = -2x^2 - x + 5$.

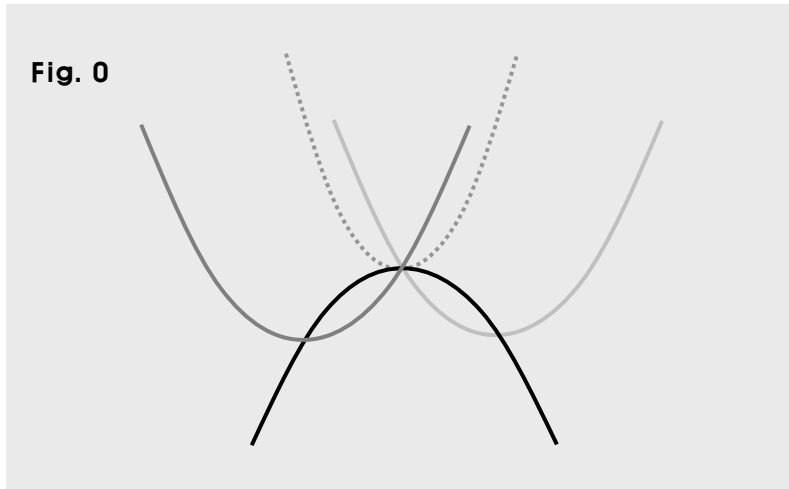
The vertex of A is a point in B , and that of B is a point in A .

So we may want to begin with how both can be positioned with respect to each other.

Note first, that the coefficient of x^2 is called the slope of the parabola in the examples.

The signs of the slope of the two parabolas, A and B are opposite of each other. Thus, so are their concavities. Viewing is better than just thinking, isn't it?

So we may want to put some parabolas in a graph so that we can see how they have to be positioned with respect to each other if they pass through each other's vertex.



Assuming in the figure above, A is one of the parabolas in gray, and B is the parabola in black, we can see that there seem to be three cases where the two parabolas can include each other's vertex.

In one case, A is the one on the left. In another case, A is the one on the right. And in the other case, A is at the top, that is, both vertices of A and B are the same.

That is, there seem to be one or two sets values of b and c . If A and B share one point as the vertex, the parabola dotted can be A ; otherwise, the parabolas solid in gray can be A .

It seems however, that in the figure above, all the parabolas themselves are the same.

They are in fact, all the same in magnitude.

That is to say that infinitely many parabolas can be the parabola A .

Otherwise, that is, if the magnitudes are different, the two parabolas A and B have the same vertex.

And doing this example, we will get to see in what case it can be the case.

Now, the foremost condition we have to use in this problem is the fact that A and B pass through each other's vertex.

So to begin with, we are going to get the vertex of A , and then, put the vertex into B .

$$y = x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

$$\Rightarrow y = \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{4}. \text{ So the vertex of } A \text{ is } \left(-\frac{b}{2}, -\frac{b^2 - 4c}{4}\right).$$

Next, B passes through the vertex above, so we get

$$-\frac{b^2 - 4c}{4} = -2\left(-\frac{b}{2}\right)^2 - \left(-\frac{b}{2}\right) + 5 = -\frac{b^2}{2} + \frac{b}{2} + 5 \Rightarrow b^2 - 4c = 2b^2 - 2b - 20$$

$\Rightarrow b^2 - 2b + 4c - 20 = 0$, which is now, one of two equations we can expect, since we have two unknowns.

Also, the equation just found is quadratic with respect to b , which means two roots, so we can expect two parabolas can be A . It may not be the case though, since the solution can be a double-root. And there can be some other case, too, of course.

Anyway next, since the vertex of B is in A , let's find the vertex, too, and put it into A .

$$y = -2x^2 - x + 5 = -2\left(x^2 + \frac{1}{2}x + \frac{1}{16} - \frac{1}{16}\right) + 5 = -2\left(x + \frac{1}{4}\right)^2 + \frac{41}{8}.$$

So the vertex of B is $\left(-\frac{1}{4}, \frac{41}{8}\right)$. So next, putting the vertex into A , we get

$$\frac{41}{8} = \left(-\frac{1}{4}\right)^2 + b\left(-\frac{1}{4}\right) + c = \frac{1}{16} - \frac{b}{4} + c \Rightarrow 82 = 1 - 4b + 16c \Rightarrow 16c - 4b = 81 \Rightarrow 4c = b + \frac{81}{4}.$$

Now, we get to solve the system as follows: $b^2 - 2b + 4c - 20 = 0$, and $4c = b + \frac{81}{4}$.

So finding b first, we get

$$b^2 - 2b + 4c - 20 = b^2 - 2b + b + \frac{81}{4} - 20 = b^2 - b + \frac{1}{4} = \left(b - \frac{1}{2}\right)^2 = 0 \Rightarrow b = \frac{1}{2}, \text{ which means a double root. So we can see now, that only one parabola can be } A.$$

And next, finding c using b above, we get $4c = b + \frac{81}{4} = \frac{1}{2} + \frac{81}{4} = \frac{83}{4} \Rightarrow c = \frac{83}{16}$.

Therefore, the parabola A is $y = x^2 + \frac{1}{2}x + \frac{83}{16}$.

Thus, only one parabola can be A , so A meets B in such a way as follows.



So both parabolas have the same vertex.

Let's now check though, to see if it is the case indeed.

The vertex of A is $(-\frac{b}{2}, -\frac{b^2-4c}{4})$.

And we have $b = \frac{1}{2}$, and $c = \frac{83}{16}$.

So we get $-\frac{b}{2} = \frac{1}{2}(-\frac{1}{2}) = -\frac{1}{4}$, and $-\frac{b^2-4c}{4} = -\frac{1}{4}\{(-\frac{1}{2})^2 - 4 \cdot \frac{83}{16}\} = -\frac{1}{4}(\frac{1}{4} - \frac{83}{4}) = \frac{1}{4} \cdot \frac{82}{4} = \frac{41}{8}$.

Thus, the vertex of A is $(-\frac{1}{4}, \frac{41}{8})$, and so is the vertex of B .

Therefore, the two parabolas, A and B have the same vertex.

Suggestions or Solutions To the Problem in the Example 2

Suppose b and c are constants, A is a parabola $y = 2x^2 + bx + c$, and B is another parabola $y = -2x^2 + 8x - 5$. Then, assuming the vertex of A is in B , and that of B , too, is in A , find b and c .

To begin with, $y = 2x^2 + bx + c = 2\{x^2 + \frac{b}{2}x + (\frac{b}{4})^2 - (\frac{b}{4})^2\} + c = 2(x + \frac{b}{4})^2 - \frac{b^2 - 8c}{8}$.

So the vertex is $(-\frac{b}{4}, -\frac{b^2 - 8c}{8})$. Putting the vertex into B , we get

$$-\frac{b^2 - 8c}{8} = -2(-\frac{b}{4})^2 + 8(-\frac{b}{4}) - 5 = -\frac{b^2}{8} - 2b - 5 \Rightarrow b^2 - 8c = b^2 + 16b + 40$$

$$\Rightarrow 16b + 8c + 40 = 0 \Rightarrow 2b + c + 5 = 0.$$

Next, $y = -2x^2 + 8x - 5 = -2(x^2 - 4x) - 5 = -2(x - 2)^2 + 3$.

So the vertex of B is $(2, 3)$, and putting it into A , we get $3 = 8 + 2b + c \Rightarrow 2b + c = -5$.

Thus, all the pairs of values of b and c that satisfy the expression $2b + c + 5 = 0$ can be the solution.

If not quite sure of the idea behind the processes above, follow the steps below.

This example is almost the same as the one presented earlier. So we are going to first, find the vertex of the parabola A , and then, put the vertex into the parabola B .

$$y = 2x^2 + bx + c = 2\{x^2 + \frac{b}{2}x + (\frac{b}{4})^2 - (\frac{b}{4})^2\} + c = 2(x + \frac{b}{4})^2 - \frac{b^2 - 8c}{8}.$$

So the vertex is $(-\frac{b}{4}, -\frac{b^2 - 8c}{8})$. Let's now, put the vertex into B . Then, we get

$$-\frac{b^2 - 8c}{8} = -2(-\frac{b}{4})^2 + 8(-\frac{b}{4}) - 5 = -\frac{b^2}{8} - 2b - 5 \Rightarrow b^2 - 8c = b^2 + 16b + 40$$

$$\Rightarrow 16b + 8c + 40 = 0 \Rightarrow 2b + c + 5 = 0.$$

Also, the vertex of the parabola B is in the parabola A , so let's find the vertex of B , and put it into A . Then, we get

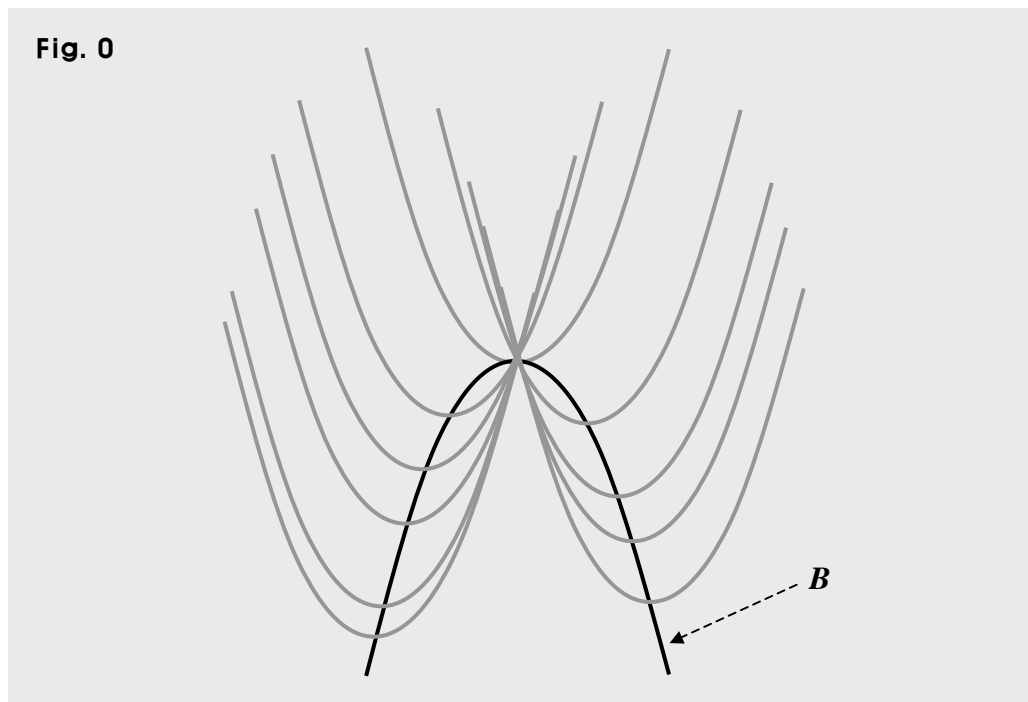
$$y = -2x^2 + 8x - 5 = -2(x^2 - 4x) - 5 = -2(x - 2)^2 + 3. \quad \text{So the vertex of } B \text{ is } (2, 3).$$

So next, putting it into A , we get $3 = 2(2^2) + 2b + c = 8 + 2b + c \Rightarrow 2b + c = -5$.

Now, we have two unknowns, yet one equation only exists.

$2b + c = -5$ is the same as $2b + c + 5 = 0$. What else then can we do?

Nothing else. In fact, there are infinitely many parabolas that can be A . In other words, all the values of b and c that satisfy the expression $2b + c + 5 = 0$ can be the solution. So the problem is done, now. Therefore, the solution is all the pairs of values of b and c for which $2b + c + 5 = 0$. In other words, every point in the line $2b + c + 5 = 0$ in the b - c plane is the solution. That is, at each and every point in the line, the b -coordinate is the value of b , and the value of c is the c -coordinate.



All the parabolas in gray above can be the parabola A .

In fact, finding the equation of the trace the vertex $(-\frac{b}{4}, -\frac{b^2-8c}{8})$ makes as the values of b and c change, we get the equation of the parabola B . That is, the trace is B . How?

The connective equation between b and c is: $2b + c + 5 = 0$.

Since the vertex is $(-\frac{b}{4}, -\frac{b^2-8c}{8})$, we have $x = -\frac{b}{4}$, and $y = -\frac{b^2-8c}{8}$.

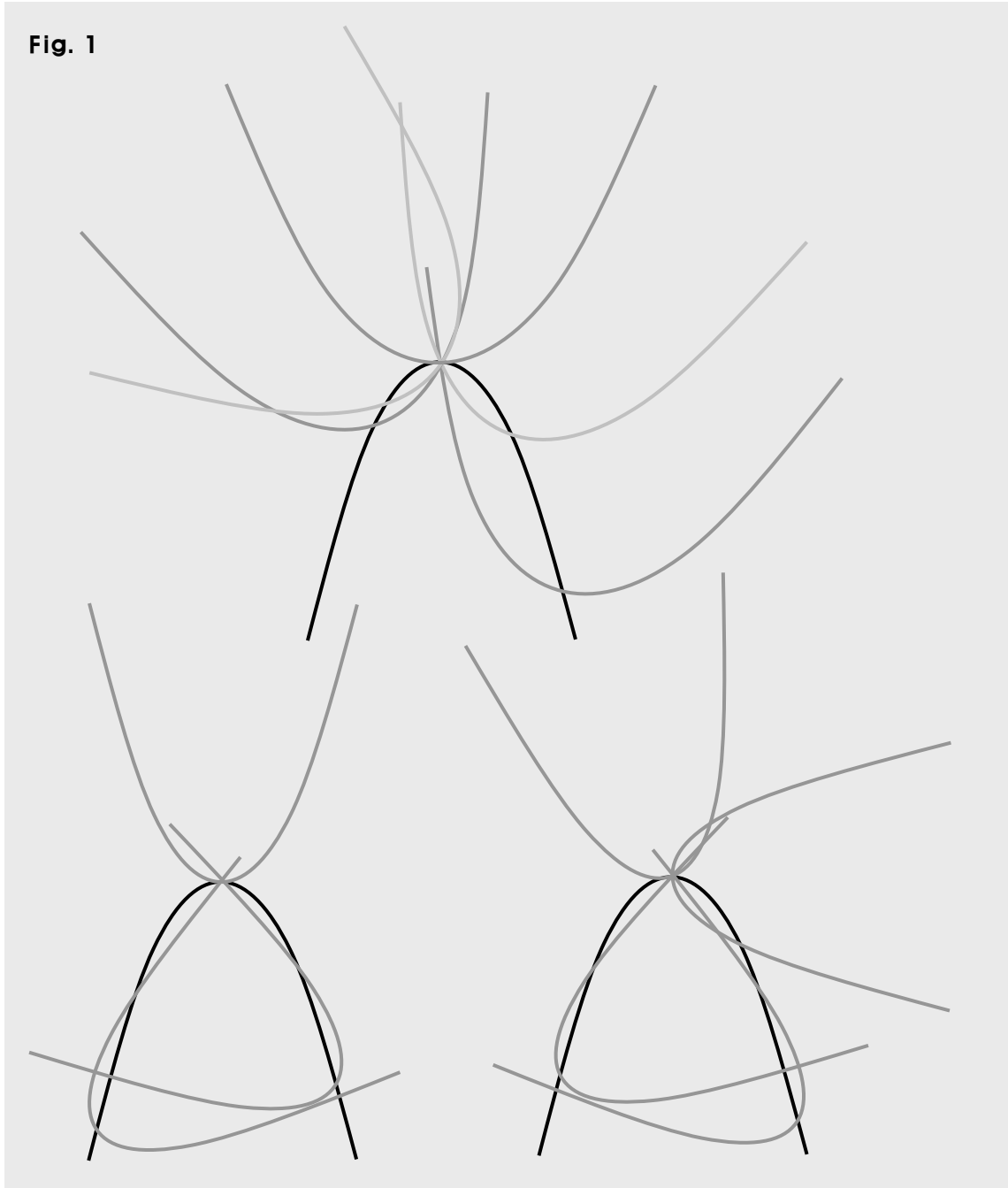
Thus, we get $x = -\frac{b}{4} \Rightarrow b = -4x$, so we get $y = -\frac{b^2-8c}{8} = -\frac{16x^2-8c}{8} = -2x^2 + c$.

Meanwhile, $2b + c + 5 = 0 \Rightarrow c = -2b - 5 = -2(-4x) - 5 = 8x - 5$.

Thus, we get $y = -2x^2 + c = -2x^2 + 8x - 5$, which is the parabola **B**.

And of course, there are infinitely many other cases where two parabolas can have each other's vertex, and more than two parabolas can have each other's vertex, too.

Fig. 1



Suggestions or Solutions
To the Problem in the Example 3

Suppose b and c are constants, A is a parabola $y = x^2 + bx + c$, and B is another parabola $y = -2x^2 + 8x - 5$. Then, assuming the vertex of A is in B , and that of B , too, is in A , find b and c .

To begin with, $y = x^2 + bx + c = (x + \frac{b}{2})^2 - \frac{b^2-4c}{4}$. So the vertex is $(-\frac{b}{2}, -\frac{b^2-4c}{4})$.

Putting the vertex into B , we get

$$-\frac{b^2-4c}{4} = -2(-\frac{b}{2})^2 + 8(-\frac{b}{2}) - 5 = -\frac{b^2}{2} - 4b - 5 \Rightarrow b^2 - 4c = 2b^2 + 16b + 20$$

$$\Rightarrow b^2 + 16b + 4c + 20 = 0.$$

Next, $y = -2x^2 + 8x - 5 = -2(x - 2)^2 + 3$. So the vertex of B is $(2, 3)$.

Putting it into A , we get $3 = (2^2) + 2b + c = 4 + 2b + c \Rightarrow c = -1 - 2b \Rightarrow 4c = -4 - 8b$.

$$\text{Thus, } b^2 + 16b + 4c + 20 = b^2 + 16b - 4 - 8b + 20 = b^2 + 8b + 16 = (b + 4)^2 = 0$$

$$\Rightarrow b = -4 \Rightarrow 4c = -4 - 8b = -4 + 32 = 28 \Rightarrow c = 7.$$

If not quite sure of the idea behind the processes above, follow the steps below.

This example, too, is almost the same as the previous ones.

So we are going to first, find the vertex of the parabola A , and then, put the vertex into the parabola B .

To begin with, $y = x^2 + bx + c = (x + \frac{b}{2})^2 - \frac{b^2-4c}{4}$. So the vertex is $(-\frac{b}{2}, -\frac{b^2-4c}{4})$.

Next, putting the vertex into B , we get

$$-\frac{b^2-4c}{4} = -2(-\frac{b}{2})^2 + 8(-\frac{b}{2}) - 5 = -\frac{b^2}{2} - 4b - 5 \Rightarrow b^2 - 4c = 2b^2 + 16b + 20$$

$\Rightarrow b^2 + 16b + 4c + 20 = 0$, which is now, one of two equations we can expect, since we have two unknowns.

Also, the equation just found is quadratic with respect to b , which means two roots, so we can expect two parabolas can be A . It may not be the case though, since the solution can be a double-root. And of course, there can be no root, too.

Now next, the vertex of B is in A , too, so let's now find B 's vertex, and put it into A .

Finding the vertex of B , we can put B this way: $y = -2x^2 + 8x - 5 = -2(x - 2)^2 + 3$.

So the vertex of B is $(2, 3)$.

And putting it into A , we get $3 = 2^2 + 2b + c = 4 + 2b + c \Rightarrow c = -1 - 2b \Rightarrow 4c = -4 - 8b$, which is the other of the two equations in the system for b and c .

So now, solving the system, and finding b first, we get

$$b^2 + 16b + 4c + 20 = b^2 + 16b - 4 - 8b + 20 = b^2 + 8b + 16 = (b + 4)^2 = 0 \Rightarrow b = -4,$$

which means a double root, so only one parabola can be A .

And next, finding c using b above, we get $4c = -4 - 8b = -4 + 32 = 28 \Rightarrow c = 7$.

Therefore, the parabola A is $y = x^2 - 4x + 7$. Meanwhile, $y = x^2 - 4x + 7 = (x - 2)^2 + 3$.

So the vertex of A is $(2, 3)$, which is the same as the vertex of B .

Suppose for one more instance, A is $y = x^2 + bx + c$, and B is $y = -2x^2 + 12x - 15$. Then, assuming the vertex of A is in B , and that of B , too, is in A , let's find again, b and c .

To begin with, we know the vertex of A is $(-\frac{b}{2}, -\frac{b^2-4c}{4})$.

So putting the vertex into B , we get

$$-\frac{b^2-4c}{4} = -2(-\frac{b}{2})^2 + 12(-\frac{b}{2}) - 15 = -\frac{b^2}{2} - 6b - 15 \Rightarrow b^2 - 4c = 2b^2 + 24b + 60$$

$$\Rightarrow b^2 + 24b + 4c + 60 = 0.$$

Next, the vertex of the parabola B is in the parabola A , too, so let's now find B 's vertex, and put it into A .

Finding the vertex of B , we can put B this way: $y = -2x^2 + 12x - 15 = -2(x - 3)^2 + 3$.

So the vertex of B is $(3, 3)$.

Putting it into A , we get $3 = 3^2 + 3b + c = 9 + 3b + c \Rightarrow c = -6 - 3b \Rightarrow 4c = -24 - 12b$, which is the other of the two equations in the system for b and c .

So now, solving the system, and finding b first, we get

$b^2 + 24b + 4c + 60 = b^2 + 24b - 24 - 12b + 60 = b^2 + 12b + 36 = (b + 6)^2 = 0$, which means a double root.

Thus again, we get only one set of values of b and c , so only one parabola can be A .

- Now, we can make a summary the way as follows.

We can put it in two cases.

One case is as follows.

Suppose we are given a particular parabola B , and want to find another parabola A , the vertex of A is in B , and that of B , too, is in A .

Then, infinitely many parabolas can be A .

The other case is as follows.

Suppose again, we are given a particular parabola B , and want to find another parabola A , the vertex of A is in B , and that of B , too, is in A .

Suppose also, the axes of symmetry of A and B are parallel to each other.

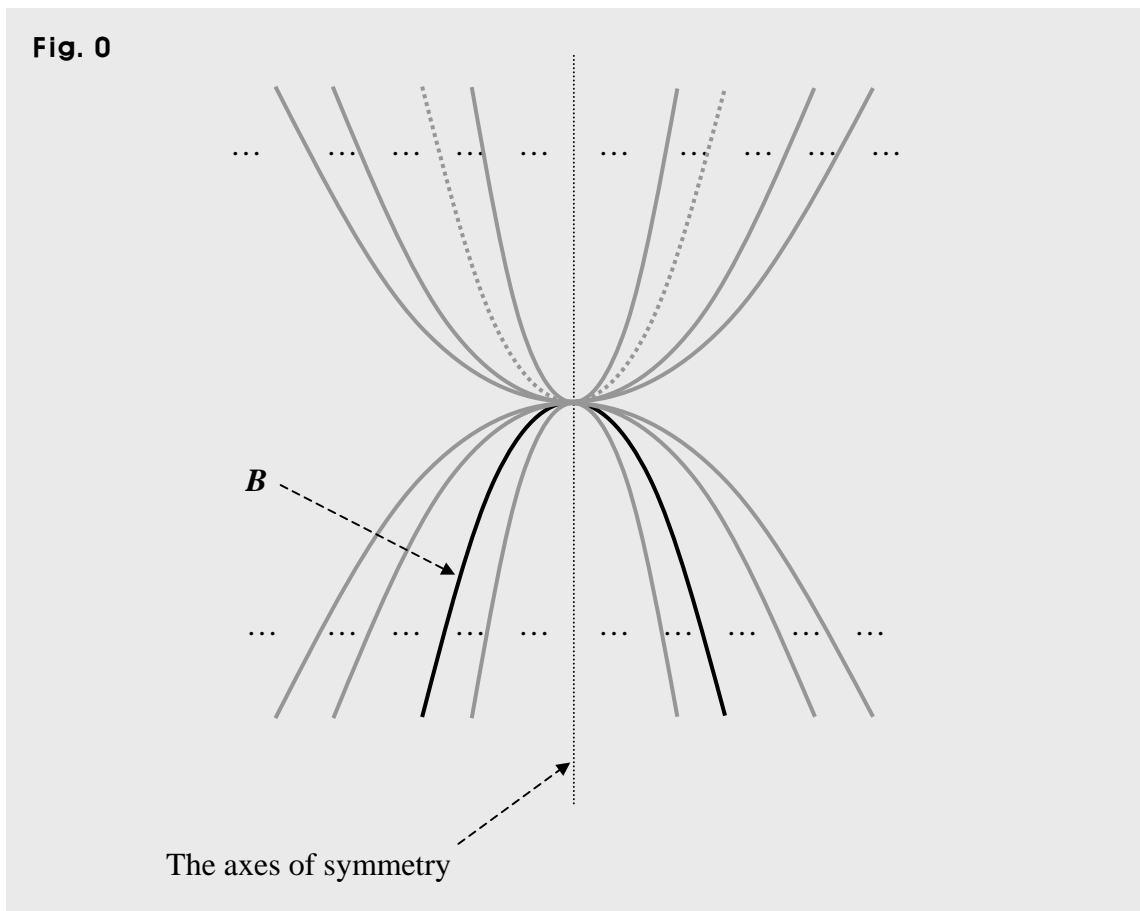
Then, A and B both share the same axis of symmetry, and share the same vertex.

Also, if the magnitude of the slope of A is different from that of B , only one parabola can be A ; otherwise, infinitely many parabolas can be A .

Let's now put in a graph the parabola B and all the parabolas that can be A .

In the figure below, B is the parabola in black. All the parabolas solid in gray can be A if the axes of symmetry of A and B are parallel to each other and the magnitudes of the slope of A and B are different from each other.

Only the parabola dotted in gray is A if the axes of symmetry of A and B are parallel to each other and the magnitudes of the slope of A and B are the same as each other.



All the parabolas above share the same axis of symmetry and the same vertex.