Examples 7 in Parabolas

Click this to start.

Examples 7

Note: this set of examples is for those students who want to study parabolas in depth.

The examples here are about graphing.
Graphing matters. And it is particularly the case when we do word problems. Solving word problems, we often need to keep track of some change. Keeping track of it, we can put it in a graph so that we can actually see it. Then, we call it a curve. Of all kinds in curves, one of the most often used is a parabola. In this set of examples, we are going to do some practices on putting a parabola in a graph.

0. Assuming $y = f(x) = 2x^2 + bx + c$, and $0 \le y \le 2$ for $0 \le x \le 1$, find b and c.

1. Assuming $y = f(x) = ax^2 + bx + 1$, and $0 \le y \le 2$ for $0 \le x \le 1$, find a and b.

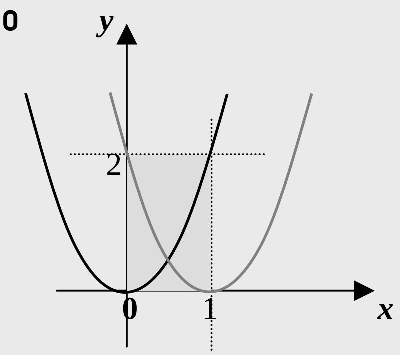
2. Assuming $y = f(x) = ax^2 + bx + c$, and $0 \le y \le 2$ for $0 \le x \le 1$, find a, b, and c.

Suggestions or Solutions
To the Problem in the Example 0

Assuming *b* and *c* are constant, $y = f(x) = 2x^2 + bx + c$, and $0 \le y \le 2$ for $0 \le x \le 1$, find *b* and *c*.

We have $\boldsymbol{y = f(x) = 2x^2 + bx + c}$ for $\mathbf{0 \le x \le 1}$ and $\mathbf{0 \le y \le 2}$.
Then, the curve of the function ***f*** is a parabola, and is either of the two as follows.

Fig. 0



In the parabola on the left, the two points are at the origin and **(1, 2)**.

So first, **(0, 0)** $\Rightarrow \mathbf{0 = 0 + 0 + \mathit{c} = \mathit{c}} \Rightarrow \mathbf{\mathit{c} = 0}$.

And next, **(1, 2)** $\Rightarrow \mathbf{2 = 2 + \mathit{b} + 0 = 2 + \mathit{b}} \Rightarrow \mathbf{\mathit{b} = 0}$.

Next, in the parabola on the right, the two points are at **(0, 2)** and **(1, 0)**.

So first, **(0, 2)** $\Rightarrow \mathbf{2 = 0 + 0 + \mathit{c} = \mathit{c}} \Rightarrow \mathbf{\mathit{c} = 2}$.

And next, **(1, 0)** $\Rightarrow \mathbf{0 = 2 + \mathit{b} + 2 = 4 + \mathit{b}} \Rightarrow \mathbf{\mathit{b} = -4}$.

Therefore, $\mathbf{\mathit{b} = 0}$ and $\mathbf{\mathit{c} = 0}$, or $\mathbf{\mathit{b} = -4}$ and $\mathbf{\mathit{c} = 2}$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, what do we mean by “$\mathbf{0 \le \mathit{y} \le 2}$ for $\mathbf{0 \le \mathit{x} \le 1}$.”?

We have a function $y = f(x)$, so the function f produces values from 0 to 2 for $0 \le x \le 1$.
So what?

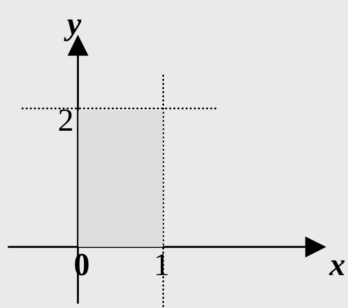
The function $f(x)$ is subject to the constants b and c, so we want to determine (find) their values so that f produces values from 0 to 2 when $0 \le x \le 1$.

So if we know how f should behave in a region, it can be easier to determine the values of the constants. We can see how f behaves looking at the curve of f, which is a parabola.

What region though?

The region is an area rectangular specified by $0 \le x \le 1$ and $0 \le y \le 2$ in the x-y plane.
Note that the region is the entire rectangle, together with the boundary.
So the region is the gray rectangle in the x-y plane as shown below.

Fig. 1



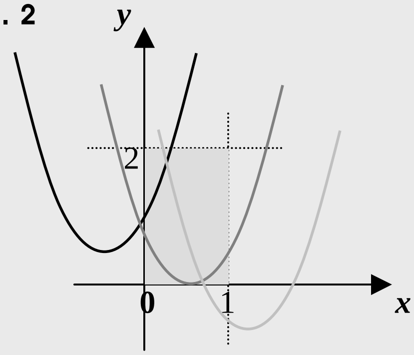
So let's now try putting a parabola in a graph so that it fits the region,

Don't just try with one parabola only though.
Try some parabolas and not just one, and keep trying with more until a parabola covers all the values from 0 to 1 along the x-axis, and the values from 0 to 2 along the y-axis.

It is often the case we cannot find such a parabola with just a few parabolas.
Just keep trying until one is found or you can be sure that there is none.
And don't forget to try with some extreme cases, too.

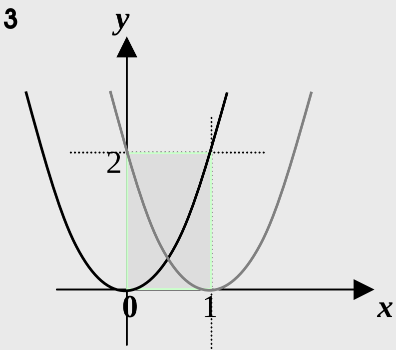
So let's now put some parabolas in a graph.

Fig. 2



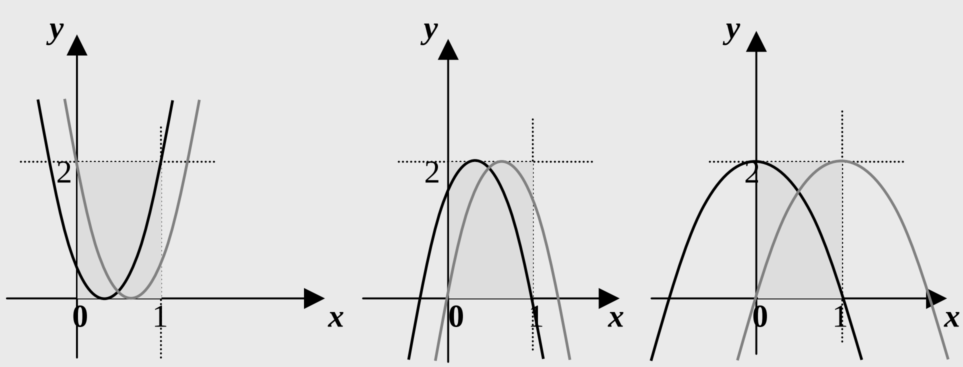
None of the above is appropriate, because no parabola covers all the values in both the intervals, $\mathbf{0 \le x \le 1}$ and $\mathbf{0 \le y \le 2}$.   So we want to try again, and can get the ones below.

Fig. 3



This time, we've got the right parabolas. Both parabolas above fit.
Each covers all the values specified.
So we can now start finding the values of the two constants $\boldsymbol{b}$ and $\boldsymbol{c}$.
It looks however, the parabolas below fit, too. So what about the ones below?

Fig. 4

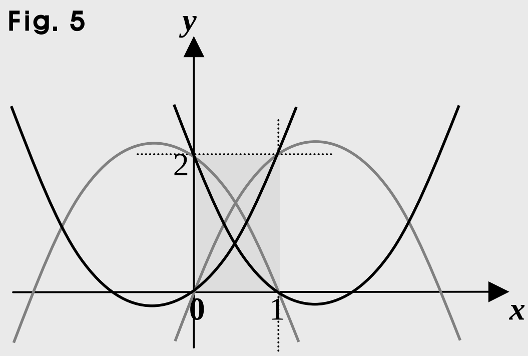


Every parabola above, too, covers all the values.
What matters is however, not only those values. What else, then?

But the steepness, too, that is, the coefficient of x^2 matters, too.
In the given equation $y = 2x^2 + bx + c$, we can see that the steepness of the parabola is 2, but in each of the parabolas shown in the graph above, the steepness is not 2.

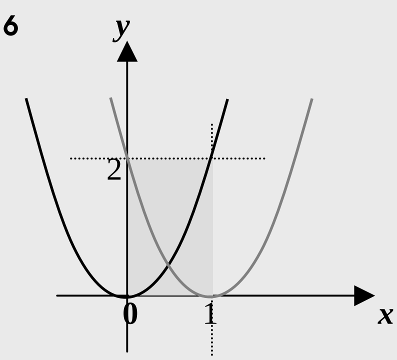
Suppose however, the steepness is a constant, and therefore, is another one to be found. Then, we should be able to consider the parabolas shown in the graph above, too.
And if that is the case, we want to consider the parabolas shown below, also.

Fig. 5



Now, we have found the parabolas that fit. So we can now begin finding the values of the two constants b and c. So getting back to the parabolas that fit, we have

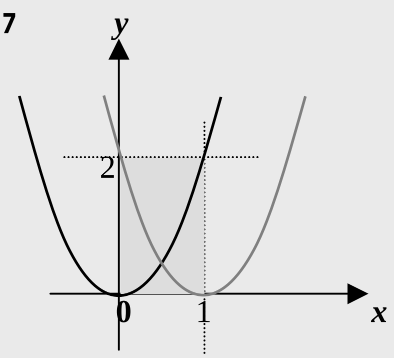
Fig. 6



We have $y = f(x) = 2x^2 + bx + c$ for $0 \leq x \leq 1$ and $0 \leq y \leq 2$.
In the graph above, we can see that each parabola has two critical points at the border of, that is, the boundary of the region. Why are they critical, and where are they?

Going back to the parabolas again, we have

Fig. 7



Now, the reason that the two points are critical is that without the two, we cannot find the values of $\boldsymbol{b}$ and $\boldsymbol{c}$. So we have to use the two. How then, can we use the two points?

Looking at the parabola on the left, we can see the two points are at the origin and **(1, 2)**. That is to say that the parabola that can have the two points is the parabola that can fit in the region, so we want to find that parabola. And we can find the parabola using the two points. That is, putting each into the equation given, we get an equation for the constants.

So let's find the parabola now. The equation given is $\boldsymbol{y = 2x^2 + bx + c}$.

So first, putting the point **(0, 0)** into the equation, we get $\boldsymbol{0 = 0 + 0 + c = c \Rightarrow c = 0}$.

Next, putting the point **(1, 2)** into the equation, we get $\boldsymbol{2 = 2 + b + 0 = 2 + b \Rightarrow b = 0}$.

So the parabola is $\boldsymbol{y = 2x^2}$.

Next, in the parabola on the right, the two points are at **(0, 2)** and **(1, 0)**.
So using the two points, we can the parabola on the right.

To begin with, the equation given is $\boldsymbol{y = 2x^2 + bx + c}$. Next, using the two points we get

$\boldsymbol{(0, 2) \Rightarrow 2 = 0 + 0 + c = c \Rightarrow c = 2}$.

$\boldsymbol{(1, 0) \Rightarrow 0 = 2 + b + 2 = 4 + b \Rightarrow b = -4}$.

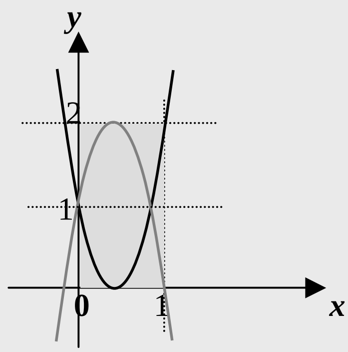
Thus, the parabola on the right is $\boldsymbol{y = 2x^2 - 4x + 2 = 2(x - 1)^2}$.

Suggestions or Solutions
To the Problem in the Example 1

Assuming a and b are constant, $y = f(x) = ax^2 + bx + 1$, and $0 \le y \le 2$ for $0 \le x \le 1$, find a and b.

The curve of f is a parabola, and it needs to be either of the two as follows.

Fig. 0



Beginning with the parabola concave-down, and using the point **(1, 0)**, we get $\mathbf{0 = a + b + 1 \Rightarrow a = -(1 + b)}$.

Next, finding the vertex of $y = ax^2 + bx + 1$, we get

$$y = ax^2 + bx + 1 = a\{x^2 + \tfrac{b}{a}x + (\tfrac{b}{2a})^2 - (\tfrac{b}{2a})^2\} + 1 = a(x + \tfrac{b}{2a})^2 - \tfrac{b^2}{4a} + 1$$

$\Rightarrow y = a(x + \frac{b}{2a})^2 - \frac{b^2-4a}{4a} \Rightarrow y + \frac{b^2-4a}{4a} = a(x + \frac{b}{2a})^2$. Thus, the vertex is $(-\frac{b}{2a}, -\frac{b^2-4a}{4a})$.

At the vertex, the y-coordinate is 2, so we get $-\frac{b^2-4a}{4a} = 2 \Rightarrow b^2 - 4a = -8a \Rightarrow b^2 + 4a = 0$.

Therefore, $a = -(1 + b) \Rightarrow b^2 + 4a = b^2 - 4(1 + b) = b^2 - 4b - 4 = (b - 2)^2 - 8 = 0$

$\Rightarrow b - 2 = \pm\sqrt{8} \Rightarrow b = 2 \pm \sqrt{8} \Rightarrow a = -(1 + b) = -(1 + 2 \pm \sqrt{8}) = -3 \mp \sqrt{8}$.

Therefore, $a = -3 - \sqrt{8}$, and $b = 2 + \sqrt{8}$, or $a = -3 + \sqrt{8}$, and $b = 2 - \sqrt{8}$.

Next, at the vertex, the x-coordinate is $-\frac{b}{2a} > 0$, so we get $-\frac{b}{2a} > 0 \Rightarrow \frac{b}{2a} < 0 \Rightarrow \frac{b}{a} < 0$.

If $\boldsymbol{a = -3 - \sqrt{8}}$, and $\boldsymbol{b = 2 + \sqrt{8}}$, we get $\frac{b}{a} < 0 \Rightarrow$ OK.

If $\boldsymbol{a = -3 + \sqrt{8}}$, and $\boldsymbol{b = 2 - \sqrt{8}}$, we get $\frac{b}{a} > 0 \Rightarrow$ not OK.

Next, moving on to the parabola with the steepness positive, and putting into the equation the point **(1, 2)**, we get $\boldsymbol{2 = a + b + 1 \Rightarrow a = 1 - b}$.

Next, at the vertex, the $\boldsymbol{y}$-coordinate = **0**. So we get $-\frac{b^2-4a}{4a} = 0 \Rightarrow b^2 - 4a = 0$.

Thus, $\boldsymbol{a = 1 - b \Rightarrow b^2 - 4a = b^2 - 4(1 - b) = b^2 + 4b - 4 = (b + 2)^2 - 8 = 0}$

$\boldsymbol{\Rightarrow b + 2 = \pm\sqrt{8} \Rightarrow b = \text{-}2 \pm \sqrt{8} \Rightarrow a = 1 - b = 1 - (\text{-}2 \pm \sqrt{8}) = 3 \mp \sqrt{8}}$.

Therefore, we get $\boldsymbol{a = 3 \mp \sqrt{8}}$, and $\boldsymbol{b = -2 \pm \sqrt{8}}$.

Next, at the vertex, the $\boldsymbol{x}$-coordinate is $-\frac{b}{2a} > 0$. So we get $-\frac{b}{2a} > 0 \Rightarrow \frac{b}{2a} < 0 \Rightarrow \frac{b}{a} < 0$.

If $\boldsymbol{a = 3 - \sqrt{8}}$, and $\boldsymbol{b = -2 + \sqrt{8}}$, we get $\frac{b}{a} > 0 \Rightarrow$ not OK.

If $\boldsymbol{a = 3 + \sqrt{8}}$, and $\boldsymbol{b = \text{-}2 - \sqrt{8}}$, we get $\frac{b}{a} < 0 \Rightarrow$ OK.

Therefore, $\boldsymbol{a = \text{-}(3 + \sqrt{8})}$ and $\boldsymbol{b = 2 + \sqrt{8}}$, or $\boldsymbol{a = 3 + \sqrt{8}}$ and $\boldsymbol{b = \text{-}(2 + \sqrt{8})}$.

If not quite sure of the idea behind the processes above, follow the steps below.

We have $\boldsymbol{y = f(x) = ax^2 + bx + 1}$, and $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$, and want to find $\boldsymbol{a}$ and $\boldsymbol{b}$.

Then first, what do we mean by "$\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$."?

We have a function $\boldsymbol{y = f(x)}$, so $\boldsymbol{f}$ has to produce values from 0 to 2 when $\boldsymbol{0 \le x \le 1}$.

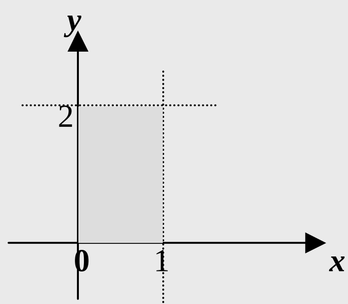
Choosing a pair of values for the constants $\boldsymbol{b}$ and $\boldsymbol{c}$, we get a different function.

So we want to determine the values of $\boldsymbol{a}$ and $\boldsymbol{b}$ so that $\boldsymbol{f}$ produces values from 0 to 2 when $\boldsymbol{x}$ gets its value from 0 to 1. How then, can we get the values of $\boldsymbol{a}$ and $\boldsymbol{b}$?

As stated in the example 0, if we know how ***f*** should behave in a region, it can be easier to determine the values of the constants. We can see how ***f*** behaves looking at the curve of ***f***, which is this time, a parabola, too.

And the region is a rectangular area specified by $\mathbf{0 \le \boldsymbol{x} \le 1}$ and $\mathbf{0 \le \boldsymbol{y} \le 2}$ in the ***x*-*y*** plane.
And the region is the entire rectangle including the boundary.
So the region is the gray rectangle in the ***x*-*y*** plane as shown below.

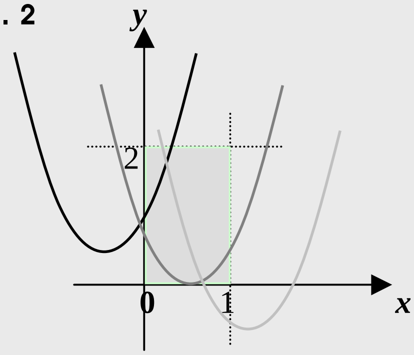
Fig. 1



Try putting a parabola in the region so that it covers <u>all the values from 0 to 1 along the ***x***-axis and all the values from 0 to 2 along the ***y***-axis</u>.

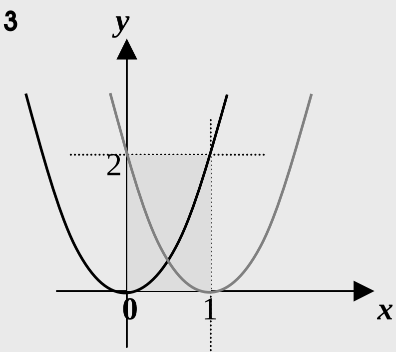
Even if you find however, a parabola that fits, try several others, too.
That's because there can be some others that can fit in the area, too.
Particularly this time, the steepness (the coefficient of $\boldsymbol{x}^2$) is not just a number, but is a constant. So keep that in mind putting parabolas in the region.
Suppose now, we put some parabolas as shown in the graph below.

Fig. 2



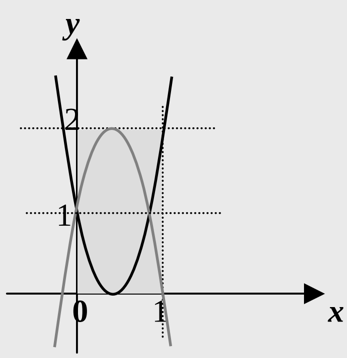
None of the above fits. None of the parabolas above covers all the values in both the intervals, **$0 \le y \le 2$** and **$0 \le x \le 1$**. So we want to try again, and can get the ones below.

Fig. 3



None again fits.
Each of the two covers all such values, yet the ***y***-intercept is not correct.
In the equation $\boldsymbol{y = ax^2 + bx + 1}$, the ***y***-intercept is 1. So we want to give another try.

Fig. 4



This time, we've got the ones fit. Each of the two above not only covers all the values but has the correct ***y***-intercept, too.

So we can now, begin finding the values of the constants ***a*** and ***b***.

Examining the graph above, we can see each parabola seems to have two critical points at the border of the region. Only one of the two is though, useful for finding the values of the constants. What are the two points, which of the two is not useful, and why not?

The point **(1, 0)** is useful, but the point **(0, 1)** is not. That's because, for instance, putting the point **(0, 1)** into $\boldsymbol{y = ax^2 + bx + 1}$, we get $\mathbf{1 = 0 + 0 + 1}$, which can therefore, leave us nothing significant.

Let's now, begin with the parabola concave-down in the graph above.

We have $\boldsymbol{y = f(x) = ax^2 + bx + 1}$, and $\mathbf{0} \le \boldsymbol{y} \le \mathbf{2}$ for $\mathbf{0} \le \boldsymbol{x} \le \mathbf{1}$.

So putting into the equation the point **(1, 0)**, we get $\boldsymbol{0 = a + b + 1 \Rightarrow a = -(1 + b)}$.

We have to find the two unknowns ***a*** and ***b***, so we need a system of two equations.
So we need to get another equation, which is for ***a*** and ***b***, of course.
Getting the other, we need something to work with. We used the point **(1, 0)** already.
What else then, can we use to get the other equation?

Usually, points in the boundary of a region are critical.
And at the boundary of the rectangle, we can see another point. What point then, is it?

It is the vertex. So we can use the vertex of the parabola, concave-down, of course.
At the vertex, we can see the ***x***-coordinate > **0**, and the ***y***-coordinate = **2**. So let's first, get the vertex of the parabola now. Then, we can put the equation the way below.

$$\boldsymbol{y = ax^2 + bx + 1 = a\{x^2 + \tfrac{b}{a}x + (\tfrac{b}{2a})^2 - (\tfrac{b}{2a})^2\} + 1 = a(x + \tfrac{b}{2a})^2 - \tfrac{b^2}{4a} + 1}$$

$\boldsymbol{\Rightarrow\ y = a(x + \tfrac{b}{2a})^2 - \tfrac{b^2-4a}{4a}}$ or $\boldsymbol{y + \tfrac{b^2-4a}{4a} = a(x + \tfrac{b}{2a})^2}$, which is said to be in the vertex form.

So the vertex is $\boldsymbol{(-\tfrac{b}{2a}, -\tfrac{b^2-4a}{4a})}$. And we know that the ***y***-coordinate is 2 at the vertex.
So the other equation in the system is as follows.

$$\boldsymbol{-\tfrac{b^2-4a}{4a} = 2 \Rightarrow b^2 - 4a = -8a \Rightarrow b^2 + 4a = 0}.$$

Thus, we now have a system of two equations for ***a*** and ***b***, $\boldsymbol{a = -(1 + b)}$, and $\boldsymbol{b^2 + 4a = 0}$.
So next, solving the system above, we get

$$\boldsymbol{a = -(1 + b) \Rightarrow b^2 + 4a = b^2 - 4(1 + b) = b^2 - 4b - 4 = (b - 2)^2 - 8 = 0}$$

$$\boldsymbol{\Rightarrow b - 2 = \pm\sqrt{8} \Rightarrow b = 2 \pm \sqrt{8} \Rightarrow a = -(1 + b) = -(1 + 2 \pm \sqrt{8}) = -3 \mp \sqrt{8}}.$$

Therefore, we have $\boldsymbol{a} = \textbf{-3} \mp \sqrt{\mathbf{8}}$, and $\boldsymbol{b} = \mathbf{2} \pm \sqrt{\mathbf{8}}$.

In other words, we have $\boldsymbol{a} = \textbf{-3} - \sqrt{\mathbf{8}}$, and $\boldsymbol{b} = \mathbf{2} + \sqrt{\mathbf{8}}$, or $\boldsymbol{a} = \textbf{-3} + \sqrt{\mathbf{8}}$, and $\boldsymbol{b} = \mathbf{2} - \sqrt{\mathbf{8}}$.

So we get two pairs of values of ***a*** and ***b***. So do we get two parabolas concave-down?

No, it is not the case, so we now get to choose one from the two values of ***a***. How?

Examining the graph, we can notice that at the vertex, the ***x***-coordinate > **0**.

And we know the ***x***-coordinate $= -\frac{b}{2a}$, so we get $-\frac{b}{2a} > \mathbf{0} \Rightarrow \frac{b}{2a} < \mathbf{0} \Rightarrow \frac{b}{a} < \mathbf{0}$. So what?

We have $\boldsymbol{a} = \textbf{-3} \mp \sqrt{\mathbf{8}}$, and $\boldsymbol{b} = \mathbf{2} \pm \sqrt{\mathbf{8}}$. That is to say that

$\boldsymbol{a} = \textbf{-3} - \sqrt{\mathbf{8}}$ has to be with $\boldsymbol{b} = \mathbf{2} + \sqrt{\mathbf{8}}$, and $\boldsymbol{a} = \textbf{-3} + \sqrt{\mathbf{8}}$ has to be with $\boldsymbol{b} = \mathbf{2} - \sqrt{\mathbf{8}}$.

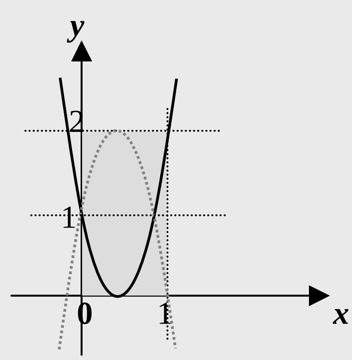
So first, we have $\boldsymbol{a} = \textbf{-3} - \sqrt{\mathbf{8}} < \mathbf{0}$, and $\boldsymbol{b} = \mathbf{2} + \sqrt{\mathbf{8}} > \mathbf{0}$. Thus, we get $\frac{b}{a} < \mathbf{0} \Rightarrow$ OK.

Next, we have $\boldsymbol{a} = -\mathbf{3} + \sqrt{\mathbf{8}} < \mathbf{0}$, and $\boldsymbol{b} = \mathbf{2} - \sqrt{\mathbf{8}} < \mathbf{0}$. So we get $\frac{b}{a} > \mathbf{0} \Rightarrow$ not OK.

Therefore, the parabola we want is $\boldsymbol{y} = \textbf{-}(\mathbf{3} + \sqrt{\mathbf{8}})\boldsymbol{x}^2 + (\mathbf{2} + \sqrt{\mathbf{8}})\boldsymbol{x} + \mathbf{1}$.

Now, we have another parabola that fits, and in the other parabola, which is concave-up. So moving on to the other parabola, we have

Fig. 5



Looking at closely though, the two parabolas, we can get the other parabola with no calculation.

Why?

Examining closely the two parabolas above, we can notice that the two are symmetric to each other about a line, ***y* = 1**. So we can see that the magnitudes of the steepness, that is, the coefficients of $\boldsymbol{x}^2$ are the same. Changing thus, the sign of the steepness of the parabola we've just found above, we get the steepness of the parabola we are after now.

So we can see that $\boldsymbol{a} = \mathbf{3} + \sqrt{\mathbf{8}}$ in the other parabola. What then, about the value of $\boldsymbol{b}$?

We can get the value of $\boldsymbol{b}$ using the point **(1, 2)**, since it passes through the point.
So putting the point **(1, 2)** into the equation $\boldsymbol{y = ax^2 + bx + 1}$, we get

$$\mathbf{2} = \boldsymbol{a} + \boldsymbol{b} + \mathbf{1} \Rightarrow \boldsymbol{b} = \mathbf{1} - \boldsymbol{a} = \mathbf{1} - (\mathbf{3} + \sqrt{\mathbf{8}}) = -\mathbf{2} - \sqrt{\mathbf{8}}.$$

Therefore, the other parabola is $\boldsymbol{y} = (\mathbf{3} + \sqrt{\mathbf{8}})\boldsymbol{x}^2 - (\mathbf{2} + \sqrt{\mathbf{8}})\boldsymbol{x} + \mathbf{1}$.

For practice though, let's find again, the values of $\boldsymbol{a}$ and $\boldsymbol{b}$ by the method we used for the parabola with the steepness negative (that is, concave-down).

So going back to the equation, we have $\boldsymbol{y = ax^2 + bx + 1}$, and $\mathbf{0} \le \boldsymbol{y} \le \mathbf{2}$ for $\mathbf{0} \le \boldsymbol{x} \le \mathbf{1}$.

Then first, putting **(1, 2)** into the equation above, we get $\mathbf{2} = \boldsymbol{a} + \boldsymbol{b} + \mathbf{1} \Rightarrow \boldsymbol{a} = \mathbf{1} - \boldsymbol{b}$.
What then, is the next?

The next is the vertex, and at the vertex, the $\boldsymbol{x}$-coordinate > **0**, and the $\boldsymbol{y}$-coordinate = **0**.
We know the vertex is at $(-\frac{\boldsymbol{b}}{\mathbf{2}\boldsymbol{a}}, -\frac{\boldsymbol{b}^2 - \mathbf{4}\boldsymbol{a}}{\mathbf{4}\boldsymbol{a}})$. So we get $-\frac{\boldsymbol{b}^2 - \mathbf{4}\boldsymbol{a}}{\mathbf{4}\boldsymbol{a}} = \mathbf{0} \Rightarrow \boldsymbol{b}^2 - \mathbf{4}\boldsymbol{a} = \mathbf{0}$.

Thus, we now have a system of two equations for $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a} = \mathbf{1} - \boldsymbol{b}$, and $\boldsymbol{b}^2 - \mathbf{4}\boldsymbol{a} = \mathbf{0}$.
So solving the system, we get

$$\boldsymbol{a} = \mathbf{1} - \boldsymbol{b} \Rightarrow \boldsymbol{b}^2 - \mathbf{4}\boldsymbol{a} = \boldsymbol{b}^2 - \mathbf{4}(\mathbf{1} - \boldsymbol{b}) = \boldsymbol{b}^2 + \mathbf{4}\boldsymbol{b} - \mathbf{4} = (\boldsymbol{b} + \mathbf{2})^2 - \mathbf{8} = \mathbf{0}$$
$$\Rightarrow \boldsymbol{b} + \mathbf{2} = \pm\sqrt{\mathbf{8}} \Rightarrow \boldsymbol{b} = \text{-}\mathbf{2} \pm \sqrt{\mathbf{8}} \Rightarrow \boldsymbol{a} = \mathbf{1} - \boldsymbol{b} = \mathbf{1} - (\text{-}\mathbf{2} \pm \sqrt{\mathbf{8}}) = \mathbf{3} \mp \sqrt{\mathbf{8}}.$$

Therefore, we have $\boldsymbol{a} = \mathbf{3} \mp \sqrt{\mathbf{8}}$, and $\boldsymbol{b} = \text{-}\mathbf{2} \pm \sqrt{\mathbf{8}}$, for now.
That is, we have $\boldsymbol{a} = \mathbf{3} - \sqrt{\mathbf{8}}$ and $\boldsymbol{b} = \text{-}\mathbf{2} + \sqrt{\mathbf{8}}$, or $\boldsymbol{a} = \mathbf{3} + \sqrt{\mathbf{8}}$ and $\boldsymbol{b} = \text{-}\mathbf{2} - \sqrt{\mathbf{8}}$.

We know we have only one parabola concave-up in the graph.
So from the two values of $\boldsymbol{a}$ above, we get to choose the true value of $\boldsymbol{a}$.
That is, we get to choose one from $\mathbf{3} - \sqrt{\mathbf{8}}$ and $\mathbf{3} + \sqrt{\mathbf{8}}$. How though?

Examining the parabola, we can notice that at the vertex, the $\boldsymbol{x}$-coordinate > **0**.

We know the x-coordinate is $-\frac{b}{2a}$, so we get $-\frac{b}{2a} > 0 \Rightarrow \frac{b}{2a} < 0 \Rightarrow \frac{b}{a} < 0$.

We have $a = 3 - \sqrt{8} > 0$, and $b = -2 + \sqrt{8} > 0$. So we get $\frac{b}{a} > 0 \Rightarrow$ not OK.

We have $a = 3 + \sqrt{8} > 0$, and $b = -2 - \sqrt{8} < 0$. So we get $\frac{b}{a} < 0 \Rightarrow$ OK.

Now, during the calculations we have done for a and b, even though not the right values for a and b, the following values have been found.

For the parabola with steepness negative, we have found $a = -3 + \sqrt{8}$, and $b = 2 - \sqrt{8}$.

For the parabola with steepness positive, We have hound $a = 3 - \sqrt{8}$, and $b = -2 + \sqrt{8}$.

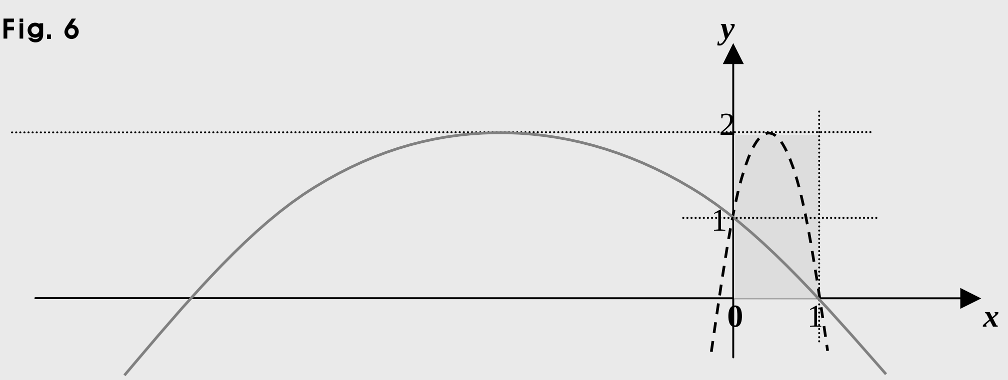
So using the values above, we can get two parabolas below.

One is $y = (-3 + \sqrt{8})x^2 + (2 - \sqrt{8})x + 1$.

And the other parabola is $y = (3 - \sqrt{8})x^2 - (2 - \sqrt{8})x + 1$.

The two ones are not the right ones, of course.
For reference purposes however, we may want to take a look at what parabolas the two equations indicate.

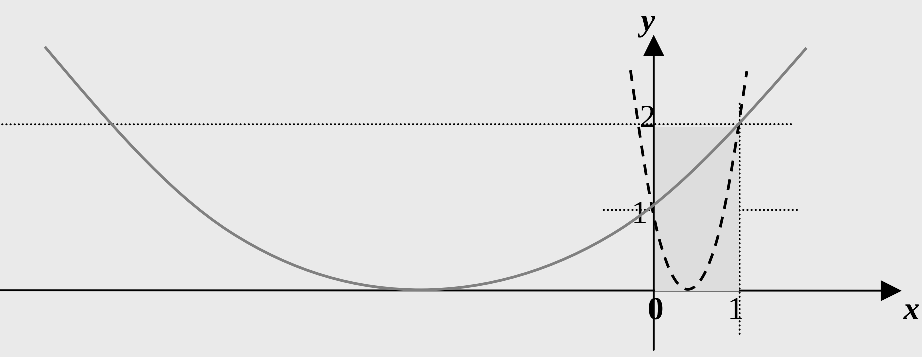
Fig. 6



The parabola solid is $y = (-3 + \sqrt{8})x^2 + (2 - \sqrt{8})x + 1$.

And the parabola dashed is $y = -(3 + \sqrt{8})x^2 + (2 + \sqrt{8})x + 1$.

Fig. 7



The parabola solid is $\boldsymbol{y=(3-\sqrt{8})x^2-(2-\sqrt{8})x+1}$.

And the parabola dashed is $\boldsymbol{y=(3-\sqrt{8})x^2-(2-\sqrt{8})x+1}$.

Suggestions or Solutions
To the Problem in the Example 2

Assuming a, b, and c are constant, $y = f(x) = ax^2 + bx + c$, and $0 \le y \le 2$ for $0 \le x \le 1$, find a, b, and c.

When x changes its value from 0 to 1, y gets every value from 0 to 2, so the function f produces all values from 0 to 2 since $y = f(x)$.

So it's *not* necessarily true that we get $y = f(x) = 0$ when $x = 0$.
That is, it's *not* necessarily the case where $y = f(0) = 0$.

If fact, we have $y = f(0) = c$ because $y = f(0) = a{\cdot}0^2 + b{\cdot}0 + c = c$. So if $c = 0$, $y = f(0) = 0$.

And by the same token, it's *not* necessarily true that we get $y = f(1) = 2$.

Now, we want to find a, b, and c, for which $0 \le y \le 2$ when $0 \le x \le 1$.
That is, we want to determine the values of a, b, and c so that $0 \le y \le 2$ when $0 \le x \le 1$.

So looking at how the function f behaves in a region specified by $0 \le x \le 1$ and $0 \le y \le 2$ in the x-y plane, we can see better what we can do to find the constants a, b, and c.

That is, by means of the curve of f, we can see how the function behaves in the region, and see better the solution's whereabouts. And the curve is a parabola, of course.

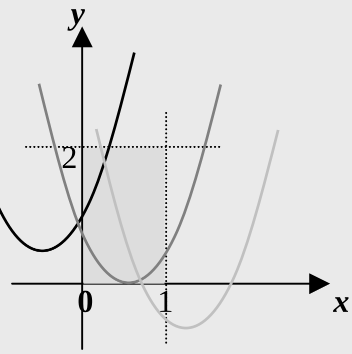
So let's begin with putting some parabolas in an area in the x-y plane, and see how we can place them there so that they fit in the area, which is the rectangle in gray in the x-y plane below.

Try putting a parabola in the graph so that a parabola covers all the values from 0 to 1 along the x-axis and all the values from 0 to 2 along the y-axis.

And even if you find a parabola that fits, try several others, too. There may be some others that can fit in the rectangle. So try more until you are sure that there is no more.

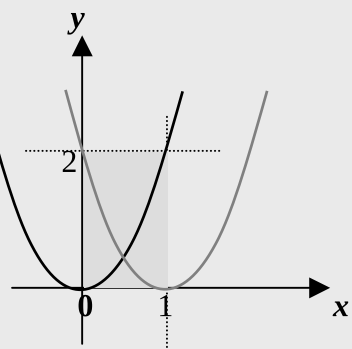
Suppose now, we put some parabolas as shown in the graph below.

Fig. 0



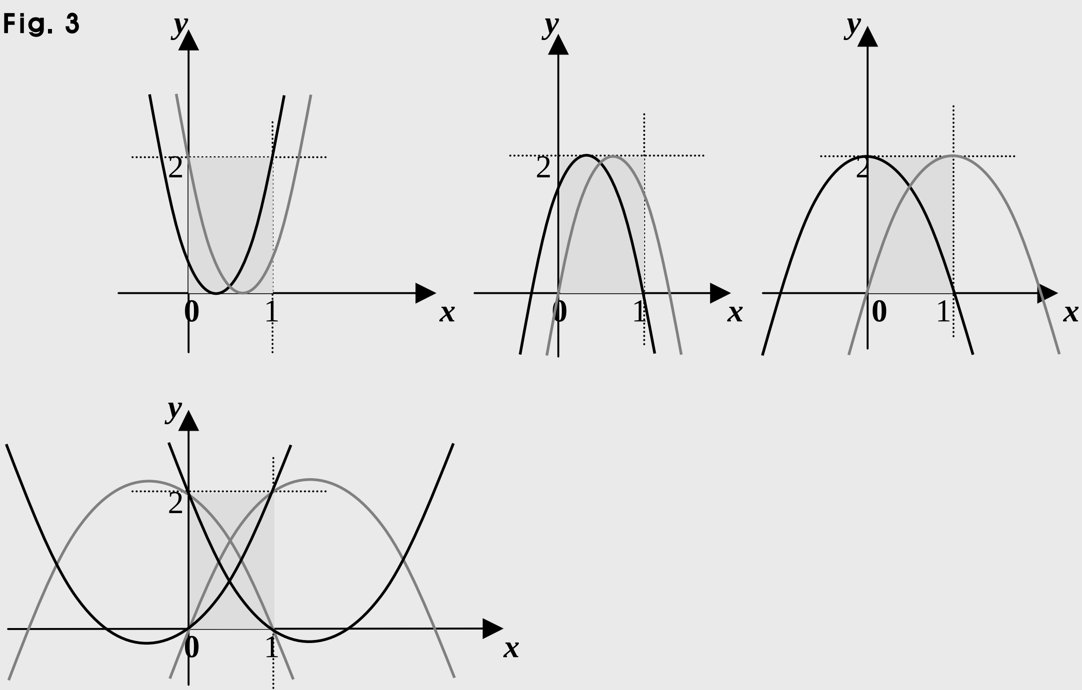
None of them fits. That's because no one covers all the values in the two intervals, $\mathbf{0 \leq y \leq 2}$ and $\mathbf{0 \leq x \leq 1}$. So we want to give another try.

Fig. 1



All the ones fit, but are not the only ones. So we want to try some more.

Fig. 3



We can see now, that there are quite a few cases to consider.
We can put all the cases in two groups.

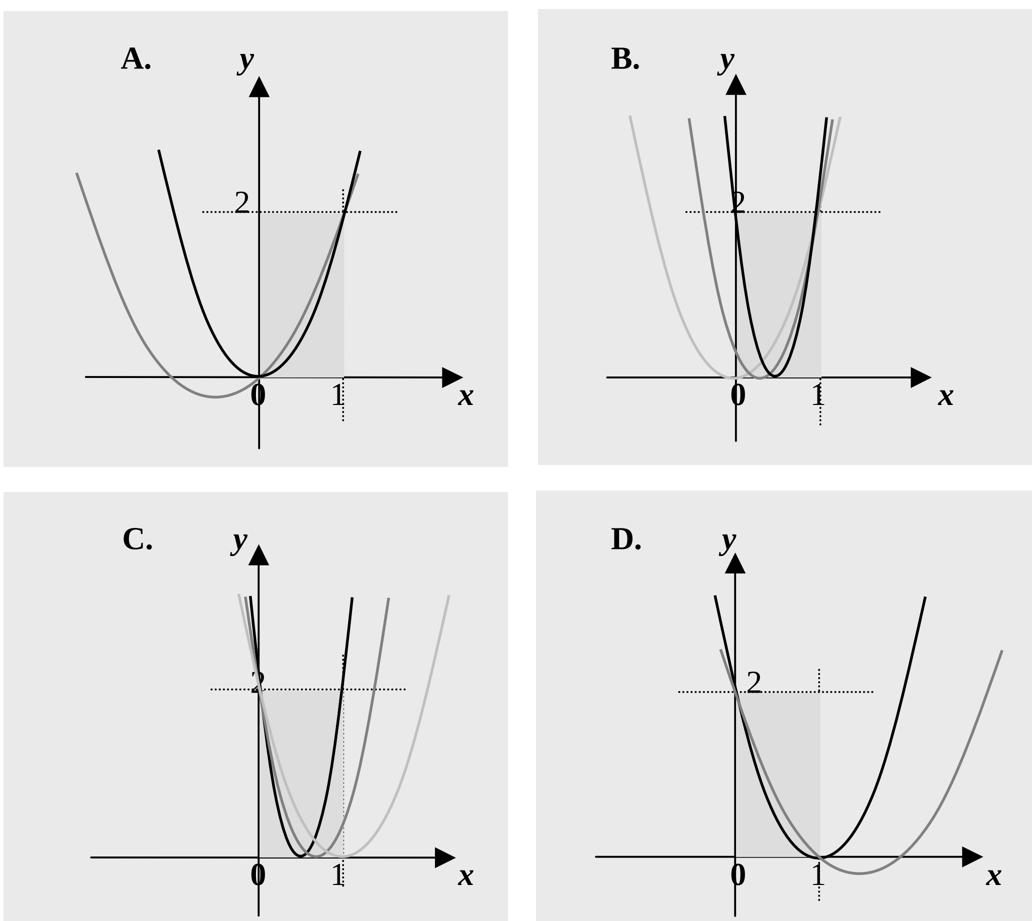
First, the equation is $\boldsymbol{y = ax^2 + bx + c}$, and $\mathbf{0 \le y \le 2}$ for $\mathbf{0 \le x \le 1}$.

So in one group, all the parabolas with the ***a***-value positive, that is, all the parabolas concave-up, and in the other, all the ones with the ***a***-value negative.

In each group, there are many parabolas that can fit.   How many, though?

Indefinitely many.
So instead of several sets of values of ***a***, ***b***, and ***c***, we will get to find the *extents* of or connective expressions between those constants.
Now, let's begin with the group where the parabolas concave-up.

Fig. 4



So as shown above in this group, we can put the parabolas in four cases.

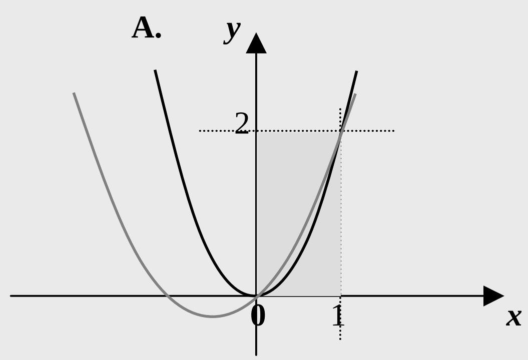
In the case **A**, we have $\boldsymbol{f(0) = 0}$, $\boldsymbol{f(1) = 2}$, and also at the vertex, **the *x*-coordinate** $\leq$ **0**.

In the case **B**, we have $\mathbf{0} \leq \boldsymbol{f(0)} \leq \mathbf{2}$, and $\boldsymbol{f(1) = 2}$, and also at the vertex, we have
$\mathbf{0} \leq$ **the *x*-coordinate** $\leq \frac{1}{2}$, and **the *y*-coordinate = 0**.
However, we don't have to consider $\mathbf{0} \leq \boldsymbol{f(0)} \leq \mathbf{2}$ as a condition. We'll cover it shortly.

In the case **C**, we have $\boldsymbol{f(0) = 2}$, and $\mathbf{0} \leq \boldsymbol{f(1)} \leq \mathbf{2}$, and also at the vertex, we have
$\frac{1}{2} \leq$ **the *x*-coordinate** $\leq \mathbf{1}$, and **the *y*-coordinate = 0**.

In the case **D**, we have $\boldsymbol{f(0) = 2}$, $\boldsymbol{f(1) = 0}$, and also at the vertex, **the *x*-coordinate** $\geq$ **1**.
Now, let's begin with the case **A**.

Fig. 5



We need to find the values of the constants ***a***, ***b***, and ***c*** satisfying the function below.
$\boldsymbol{y = f(x) = ax^2 + bx + c}$, and $\mathbf{0} \leq \boldsymbol{y} \leq \mathbf{2}$ for $\mathbf{0} \leq \boldsymbol{x} \leq \mathbf{1}$.

Then, we need to meet the conditions that
$\boldsymbol{f(0) = 0}$, $\boldsymbol{f(1) = 2}$, and also, **the *x*-coordinate** $\leq$ **0** at the vertex.

Why not consider **the *y*-coordinate** at the vertex, though? It has to be $\leq$ **0**, doesn't it?

The *y*-coordinate has to be $\leq$ **0**, of course.
However, we don't have to set **the *y*-coordinate** that way. If we do, it will be redundant. All the conditions specified above are exhaustive now, and will force **the *y*-coordinate** to be $\leq$ **0**. In other words, if all the conditions are met, **the *y*-coordinate** at the vertex cannot be positive.

Now, let's satisfy the conditions.

To begin with, we get $\boldsymbol{f(0) = 0 \Rightarrow a\cdot 0^2 + b\cdot 0 + c = 0 \Rightarrow c = 0 \Rightarrow f(x) = ax^2 + bx}$.

So next, we get $\boldsymbol{f(1) = 2 \Rightarrow a\cdot 1^2 + b\cdot 1 = 2 \Rightarrow a + b = 2}$.

Next, getting the vertex, we can begin with

$$\boldsymbol{y = ax^2 + bx + c = a\{x^2 + \tfrac{b}{a}x + (\tfrac{b}{2a})^2 - (\tfrac{b}{2a})^2\} + c = a(x + \tfrac{b}{2a}) - \tfrac{b^2-4ac}{4a}}.$$

So the vertex is $\boldsymbol{(-\tfrac{b}{2a}, -\tfrac{b^2-4ac}{4a})}$.

Then, at the vertex, **the x-coordinate** $\boldsymbol{\le 0 \Rightarrow -\tfrac{b}{2a} \le 0 \Rightarrow \tfrac{b}{a} \ge 0 \Rightarrow b \ge 0}$, since $\boldsymbol{a > 0}$.

Now, as an interim result, we have $\boldsymbol{c = 0}$, $\boldsymbol{a + b = 2}$, and $\boldsymbol{b \ge 0}$.

Next, let's find the extent of $\boldsymbol{a}$. We can get the extent using $\boldsymbol{a + b = 2}$, and $\boldsymbol{b \ge 0}$.

So to begin with, $\boldsymbol{a + b = 2 \Rightarrow b = 2 - a}$. Thus, we get $\boldsymbol{b \ge 0 \Rightarrow 2 - a \ge 0 \Rightarrow a \le 2}$.

Besides, we have $\boldsymbol{a > 0}$, too. Thus, we get $\boldsymbol{0 < a \le 2}$.

So we now have $\boldsymbol{c = 0}$, $\boldsymbol{a + b = 2}$, and $\boldsymbol{0 < a \le 2}$.

Therefore in the case **A**, the parabola is $\boldsymbol{y = ax^2 + bx}$, where $\boldsymbol{a + b = 2}$ for $\boldsymbol{0 < a \le 2}$.

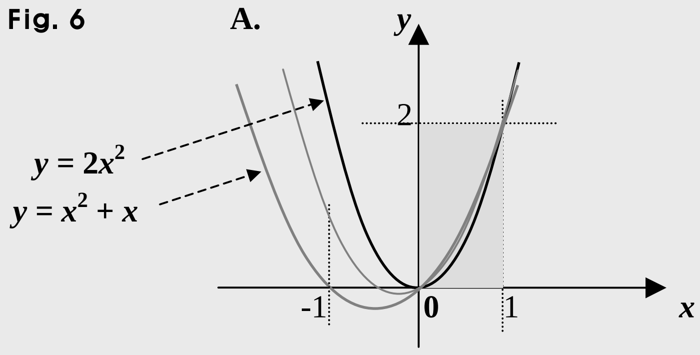
That is, choosing any pair of values for $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying $\boldsymbol{0 < a \le 2}$ and $\boldsymbol{a + b = 2}$, we get a parabola $\boldsymbol{y = ax^2 + bx}$, and then, we can get $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$.

So for instance, we can set $\boldsymbol{a = 1}$, and $\boldsymbol{b = 1}$, and then, we get a parabola $\boldsymbol{y = x^2 + x}$.
Then, we can get $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$.

For another instance, we can set $\boldsymbol{a = 2}$, and $\boldsymbol{b = 0}$, and then, we get a parabola $\boldsymbol{y = 2x^2}$.
Then, we can get $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$.

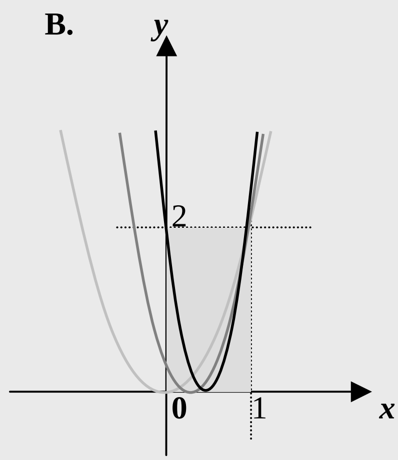
Putting now in a graph the two parabolas above, along with another in between, we get

Fig. 6 A.



Now, let's begin the case **B**.

Fig. 7 B.



We want to find the values of the constants $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ satisfying the function below.
$\boldsymbol{y = f(x) = ax^2 + bx + c}$, and $\mathbf{0 \le y \le 2}$ for $\mathbf{0 \le x \le 1}$.

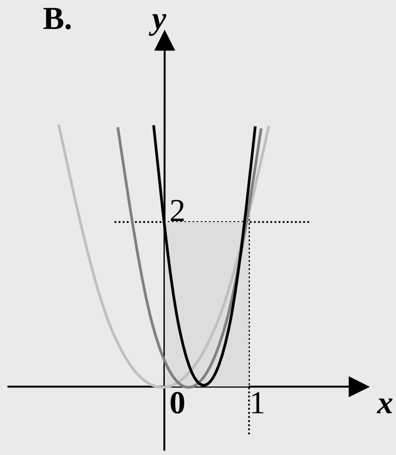
Then, we need to meet the conditions that
$\boldsymbol{f(1) = 2}$, and at the vertex, $\mathbf{0} \le$ **the *x*-coordinate** $\le \frac{1}{2}$, and **the *y*-coordinate = 0**.

Why not consider the value of $\boldsymbol{f(0)}$, though? That is, don't we have $\mathbf{0 \le \boldsymbol{f}(0) \le 2}$?

Of course, we do. However, we don't have to consider the case. If we do so, it will be redundant. The conditions specified above are exhaustive now, and will force $\boldsymbol{f(0)}$ to be so. That is, if all the conditions are met, we will get $\mathbf{0 \le \boldsymbol{f}(0) \le 2}$ automatically.

Now, let's satisfy the conditions. The parabolas in the group **B** are as follows.

Fig. 8 B.



To begin with, we have $\boldsymbol{y = f(x) = ax^2 + bx + c}$, so we get $\boldsymbol{f(1) = 2 \Rightarrow 2 = a + b + c}$.

Next, we know that the vertex is $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$. So we get

First, the $\boldsymbol{y}$**-coordinate** $= 0 \Rightarrow -\frac{b^2-4ac}{4a} = 0 \Rightarrow b^2 - 4ac = 0$.

Next, $0 \le$ **the** $\boldsymbol{x}$**-coordinate** $\le \frac{1}{2} \Rightarrow 0 \le -\frac{b}{2a} \le \frac{1}{2} \Rightarrow -\frac{1}{2} \le \frac{b}{2a} \le 0 \Rightarrow -1 \le \frac{b}{a} \le 0$.

Now, as an interim result, we have $a + b + c = 2$, $b^2 - 4ac = 0$, and $-1 \le \frac{b}{a} \le 0$.

Next, let's find the extent of $\boldsymbol{a}$.
Looking at closely the graph above, we can see that $2 \le a \le 8$. Why?

The steepness of the parabola in light gray is 2. Why?

We can put the parabola in this equation $\boldsymbol{y = ax^2}$, since the vertex is at the origin.
Then, since the parabola includes $(1, 2)$, we get $2 = a \cdot 1^2 = a \Rightarrow a = 2$.

Next, the $\boldsymbol{a}$-value of the parabola in black is 8. Why?

We can put the parabola this way: $y = a\left(x - \frac{1}{2}\right)^2$, because the vertex is at $\left(\frac{1}{2}, 0\right)$.

Then, since the parabola includes **(1, 2)**, we get $\mathbf{2 = \boldsymbol{a}(1 - \frac{1}{2})^2 = \boldsymbol{a}(\frac{1}{2})^2 = \frac{\boldsymbol{a}}{4} \Rightarrow \boldsymbol{a} = 8}$.

Therefore, as another interim result, we have

$\mathbf{2 \le \boldsymbol{a} \le 8}$, $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \mathbf{2}$, $\boldsymbol{b}^2 - \mathbf{4}\boldsymbol{ac} = \mathbf{0}$, and $\mathbf{-1 \le \frac{\boldsymbol{b}}{\boldsymbol{a}} \le 0}$.

Now, let's move on to $\boldsymbol{b}$.

We have $\mathbf{-1 \le \frac{\boldsymbol{b}}{\boldsymbol{a}} \le 0}$. Thus, we get $\mathbf{-\boldsymbol{a} \le \boldsymbol{b} \le 0}$.

So next, since $\mathbf{2 \le \boldsymbol{a} \le 8}$, we get $\mathbf{-\boldsymbol{a} \le \boldsymbol{b} \le 0 \Rightarrow -8 \le \boldsymbol{b} \le 0}$. Why?

"$\mathbf{2 \le \boldsymbol{a} \le 8}$**"** means that $\boldsymbol{a}$ can have any value from 2 to 8.
And also, we have the facts below.

When $\boldsymbol{a} = \mathbf{2}$, we get $\mathbf{-\boldsymbol{a} \le \boldsymbol{b} \le 0 \Rightarrow -2 \le \boldsymbol{b} \le 0}$.

When $\boldsymbol{a} = \mathbf{8}$, we get $\mathbf{-\boldsymbol{a} \le \boldsymbol{b} \le 0 \Rightarrow -8 \le \boldsymbol{b} \le 0}$.

So we can see that $\mathbf{-8 \le \boldsymbol{b} \le 0}$.

Thus, we now, have $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \mathbf{2}$, $\boldsymbol{b}^2 - \mathbf{4}\boldsymbol{ac} = \mathbf{0}$, $\mathbf{2 \le \boldsymbol{a} \le 8}$, and $\mathbf{-8 \le \boldsymbol{b} \le 0}$.

What about the extent of $\boldsymbol{c}$, though?

Examining the graph again, we can see that $\mathbf{0 \le \boldsymbol{c} \le 2}$. Why?

In the graph, we can see that $\mathbf{0 \le}$ ***y*-intercept** $\mathbf{\le 2}$.

And in $\boldsymbol{y} = \boldsymbol{ax}^2 + \boldsymbol{bx} + \boldsymbol{c}$, $\boldsymbol{c}$ is the ***y***-intercept.

Let's check to see if ($\mathbf{0 \le \boldsymbol{c} \le 2}$) and ($\mathbf{2 \le \boldsymbol{a} \le 8}$) satisfy ($\mathbf{-8 \le \boldsymbol{b} \le 0}$).

To begin with, we get $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \mathbf{2} \Rightarrow \boldsymbol{a} + \boldsymbol{c} = \mathbf{2} - \boldsymbol{b}$. Next, we get

$\mathbf{(0 \le \boldsymbol{c} \le 2) + (2 \le \boldsymbol{a} \le 8) \Rightarrow 2 \le \boldsymbol{a} + \boldsymbol{c} \le 10 \Rightarrow 2 \le 2 - \boldsymbol{b} \le 10 \Rightarrow 0 \le -\boldsymbol{b} \le 8 \Rightarrow -8 \le \boldsymbol{b} \le 0}$.

Also, we can get rid of $\boldsymbol{c}$ in the expression, $\boldsymbol{ax}^2 + \boldsymbol{bx} + \boldsymbol{c}$.

Since we have $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \mathbf{2}$, we get $\boldsymbol{c} = \mathbf{2} - (\boldsymbol{a} + \boldsymbol{b})$.

Thus, we get $ax^2 + bx + c \Rightarrow ax^2 + bx + 2 - (a + b)$.

Besides, we can combine the two connective expressions, $a + b + c = 2$ and $b^2 - 4ac = 0$.

We have $c = 2 - (a + b)$. So we get

$-4ac = \{2 - (a + b)\}(-4a) = -8a + 4a^2 + 4ab = 4a^2 + 4ab + b^2 - b^2 - 8a$
$= (2a + b)^2 - b^2 - 8a \Rightarrow -4ac = (2a + b)^2 - b^2 - 8a$.

Thus, we get $b^2 - 4ac = b^2 + (2a + b)^2 - b^2 - 8a = (2a + b)^2 - 8a = 0$.

So we get $(2a + b)^2 = 8a \Rightarrow 2a + b = \pm\sqrt{8a} \Rightarrow b = -2a \pm \sqrt{8a}$.

Now, we need to choose for b one from $-2a + \sqrt{8a}$ and $-2a - \sqrt{8a}$.

When a changes from 2 to 8, the value of $(-2a + \sqrt{8a})$ changes from 0 to -8, but that of $(-2a - \sqrt{8a})$ changes from -8 to -24.
Besides, we have $-8 \le b \le 0$. Therefore, we can see that $b = -2a + \sqrt{8a}$.

Thus, we now have $b = -2a + \sqrt{8a}$, $2 \le a \le 8$, and $c = 2 - (a + b)$.

Therefore, in the case of **B**, the parabola is as follows.
$y = ax^2 + bx + 2 - (a + b)$ where $b = -2a + \sqrt{8a}$ for $2 \le a \le 8$.

That is, choosing any pair of values for a and b satisfying $2 \le a \le 8$ and $b = -2a + \sqrt{8a}$, we get a parabola $y = ax^2 + bx + 2 - (a + b)$, and then, we can get $0 \le y \le 2$ for $0 \le x \le 1$.

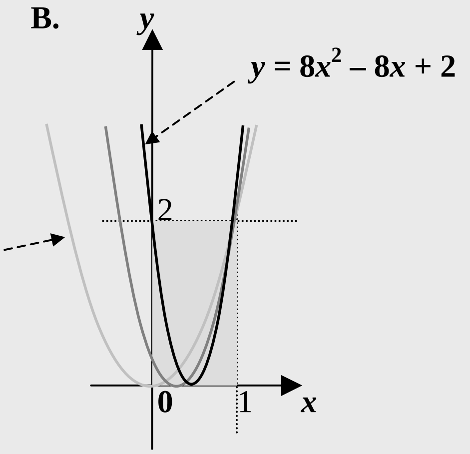
So for instance, we can set $a = 2$, so we get $b = -2a + \sqrt{8a} = -4 + \sqrt{16} = 0$, and then, we get a parabola $y = 2x^2$. Then, we can get $0 \le y \le 2$ for $0 \le x \le 1$.

For another instance, we can set $a = 8$, so we get $b = -2a + \sqrt{8a} = -16 + 8 = -8$, and then, we get a parabola $y = 8x^2 - 8x + 2$. Then, we can get $0 \le y \le 2$ for $0 \le x \le 1$.

And putting the parabola in the vertex form, we get
$y = 8x^2 - 8x + 2 = 8(x^2 - x + \frac{1}{4} - \frac{1}{4}) + 2 = 8(x - \frac{1}{2})^2 - 2 + 2 = 8(x - \frac{1}{2})^2 \Rightarrow y = 8(x - \frac{1}{2})^2$.

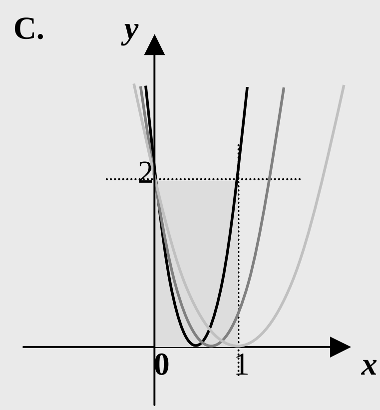
Let's now, put in a graph the two parabolas above, together with another one in between.

Fig. 9



Now, let's move on to the case **C**.

Fig. A



We want to find the values of the constants $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ satisfying the function below.

$\boldsymbol{y = f(x) = ax^2 + bx + c}$, and $\mathbf{0 \le y \le 2}$ for $\mathbf{0 \le x \le 1}$.

Then, we need to meet the conditions as follows.

$\boldsymbol{f(0) = 2}$, and also, at the vertex, $\frac{1}{2} \le$ **the *x*-coordinate** $\le \mathbf{1}$, and **the *y*-coordinate = 0**.

To begin with, we get $\boldsymbol{f(0) = 2 \Rightarrow 2 = a \cdot 0^2 + b \cdot 0 + c \Rightarrow c = 2 \Rightarrow f(x) = ax^2 + bx + 2}$.

Next, we know that the vertex is $\boldsymbol{(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})}$. So we get

First, the ***y*-coordinate = 0** $\boldsymbol{\Rightarrow -\frac{b^2-4ac}{4a} = 0 \Rightarrow b^2 - 4ac = 0 \Rightarrow b^2 = 4ac = 8a}$, since $\boldsymbol{c = 2}$.

Next, $\boldsymbol{\frac{1}{2} \le}$ **the *x*-coordinate** $\boldsymbol{\le 1 \Rightarrow \frac{1}{2} \le -\frac{b}{2a} \le 1 \Rightarrow -1 \le \frac{b}{2a} \le -\frac{1}{2} \Rightarrow -2 \le \frac{b}{a} \le -1}$.

Then, as an interim result, we have $\boldsymbol{c = 2}$, $\boldsymbol{b^2 = 8a}$, and $\boldsymbol{-2 \le \frac{b}{a} \le -1}$.

Next, finding the extent of $\boldsymbol{b}$, we get

$b^2 = 8a \Rightarrow \boldsymbol{\frac{b}{a} = \frac{8}{b}}$, and thus, $\boldsymbol{-2 \le \frac{b}{a} \le -1 \Rightarrow -2 \le \frac{8}{b} \le -1 \Rightarrow -1 \le \frac{b}{8} \le -\frac{1}{2} \Rightarrow -8 \le b \le -4}$.

Now, we have $\boldsymbol{c = 2}$, $\boldsymbol{-8 \le b \le -4}$, and $\boldsymbol{b^2 = 8a}$.

Thus, in the case **C**, the parabola is $\boldsymbol{y = ax^2 + bx + 2}$ where $\boldsymbol{b^2 = 8a}$ for $\boldsymbol{-8 \le b \le -4}$.

As in the case **B** above, looking at the graph only, we can see that $\boldsymbol{2 \le a \le 8}$.

First, we can see that the $\boldsymbol{a}$-value of the parabola in light gray is 2.
And we can put the parabola this way: $\boldsymbol{y = a(x - 1)^2}$, because the vertex is $\boldsymbol{(1, 0)}$.
Then, since the parabola includes $\boldsymbol{(0, 2)}$, we get $\boldsymbol{2 = a(0 - 1)^2 = a \Rightarrow a = 2}$.

Next, we can see that the $\boldsymbol{a}$-value of the parabola in black is 8.
And we can put the parabola this way: $\boldsymbol{y = a(x - \frac{1}{2})^2}$, because the vertex is $\boldsymbol{(\frac{1}{2}, 0)}$.
Then, since the parabola includes $\boldsymbol{(1, 2)}$, we get $\boldsymbol{2 = a(1 - \frac{1}{2})^2 = \frac{a}{4} \Rightarrow a = 8}$.

Let's check to see if the above is correct.

To begin with, we get $\boldsymbol{-8 \le b \le -4 \Rightarrow 16 \le b^2 \le 64 \Rightarrow 2 \le \frac{b^2}{8} \le 8}$.

Next, we get $\boldsymbol{b^2 = 8a \Rightarrow a = \frac{b^2}{8}}$. So we get $\boldsymbol{2 \le \frac{b^2}{8} \le 8 \Rightarrow 2 \le a \le 8}$.

So in the case **C**, we can the parabola the way below, too.

$\boldsymbol{y = ax^2 + bx + 2}$ where $\boldsymbol{b^2 = 8a}$ for $\boldsymbol{2 \le a \le 8}$.

And since $a = \frac{b^2}{8}$, we can put the parabola the way below, too.

$y = \frac{b^2}{8}x^2 + bx + 2$ where $-8 \le b \le -4$.

So choosing any value for b satisfying $-8 \le b \le -4$, we get a parabola $y = \frac{b^2}{8}x^2 + bx + 2$,

and then, we can get $0 \le y \le 2$ for $0 \le x \le 1$.

So for instance, we can set $b = -8$, and then, we get a parabola $y = 8x^2 – 8x + 2$.

Then, we can get $0 \le y \le 2$ for $0 \le x \le 1$.

For another instance, we can set $b = -4$, and then, we get a parabola $y = 2x^2 – 4x + 2$.

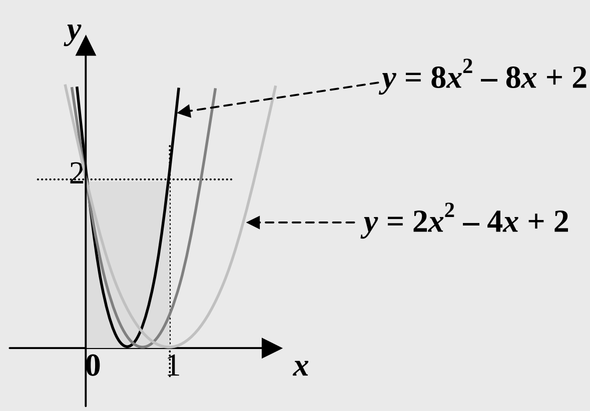
Then, we can get $0 \le y \le 2$ for $0 \le x \le 1$.

And putting the parabola in the vertex form, we get

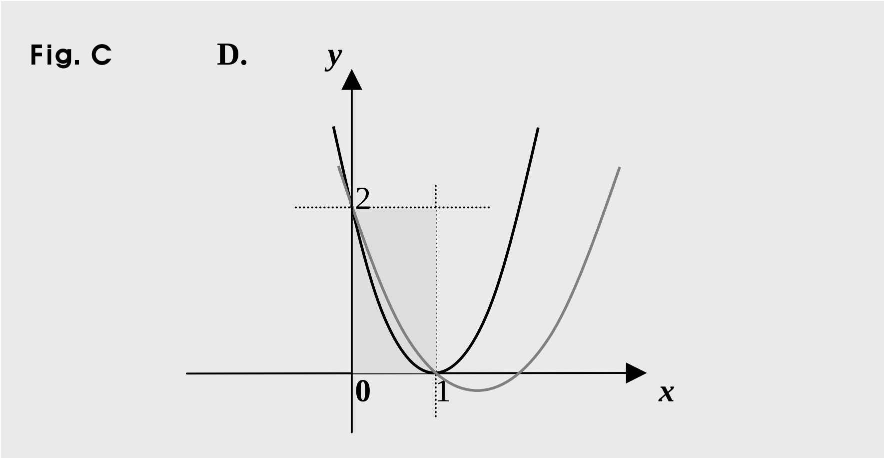
$y = 2x^2 – 4x + 2 = 2(x^2 – 2x + 1) = 2(x – 1)^2$.

Now, putting in a graph the two parabolas above, along with another in between, we get

Fig. B C.



Now, let’s move on to the case **D**.



We want to find the values of the constants $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ satisfying the function below.

$\boldsymbol{y = f(x) = ax^2 + bx + c}$, and $\mathbf{0 \le y \le 2}$ for $\mathbf{0 \le x \le 1}$.

Then, we need to meet the conditions as follows.

$\boldsymbol{f(0) = 2}$, $\boldsymbol{f(1) = 0}$, and also, at the vertex, **the *x*-coordinate** $\ge \mathbf{1}$.

To begin with, we get $\boldsymbol{f(0) = a \cdot 0^2 + b \cdot 0 + c = 2 \Rightarrow c = 2 \Rightarrow f(x) = ax^2 + bx + 2}$.

So next, we get $\boldsymbol{f(1) = a \cdot 1^2 + b \cdot 1 + 2 = 0}$. Therefore, we get $\boldsymbol{a + b = -2}$.

Next, we know that the vertex is $\boldsymbol{(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})}$. So we get

The *x*-coordinate $\boldsymbol{\ge 1 \Rightarrow -\frac{b}{2a} \ge 1 \Rightarrow \frac{b}{a} \le -2 \Rightarrow b \le -2a}$ since $\boldsymbol{a > 0}$.

Meanwhile, $\boldsymbol{a + b = -2 \Rightarrow b = -2 - a}$. So we get $\boldsymbol{b \le -2a \Rightarrow -2 - a \le -2a \Rightarrow a \le 2}$.

Besides, we have $\boldsymbol{a > 0}$, too. Therefore, we can see that $\boldsymbol{0 < a \le 2}$.

Thus, in the case **D**, the parabola is $\boldsymbol{y = ax^2 + bx + 2}$ where $\boldsymbol{a + b = -2}$ for $\boldsymbol{0 < a \le 2}$.

So choosing any pair of values for $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying $\boldsymbol{0 < a \le 2}$ and $\boldsymbol{a + b = -2}$, we get a parabola $\boldsymbol{y = ax^2 + bx + 2}$, and then, we can get $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$.

So for instance, we can set $\boldsymbol{a = 1}$, so we get $\boldsymbol{b = -3}$ since $\boldsymbol{a + b = -2}$, and then, we get a parabola $\boldsymbol{y = x^2 - 3x + 2 = (x - 2)(x - 1)}$. Then, we can get $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$.

And putting the parabola in the vertex form, we get

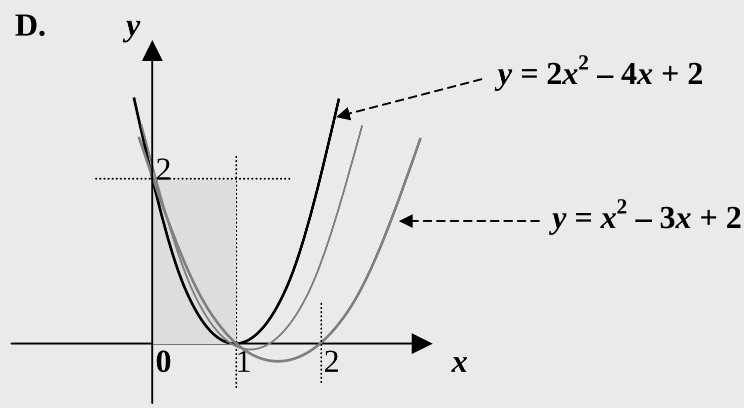
$$\boldsymbol{y = x^2 - 3x + 2 = x^2 - 3x + \tfrac{9}{4} - \tfrac{9}{4} + 2 = (x - \tfrac{3}{2})^2 - \tfrac{1}{4}}.$$

For another instance, we can set $\boldsymbol{a = 2}$, so we get $\boldsymbol{b = -4}$ since $\boldsymbol{a + b = -2}$, and then, we get a parabola $\boldsymbol{y = 2x^2 - 4x + 2}$. Then, we can get $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$.

And putting the parabola in the vertex form, we get
$\boldsymbol{y = 2x^2 - 4x + 2 = 2(x^2 - 2x + 1) = 2(x - 1)^2}$.

Now, putting in a graph the two parabolas above, along with another in between, we get

Fig. D



Now, putting threads together, we have

In the case **A**, the parabola is $\boldsymbol{y = ax^2 + bx}$ where $\boldsymbol{a + b = 2}$ for $\boldsymbol{0 < a \le 2}$.

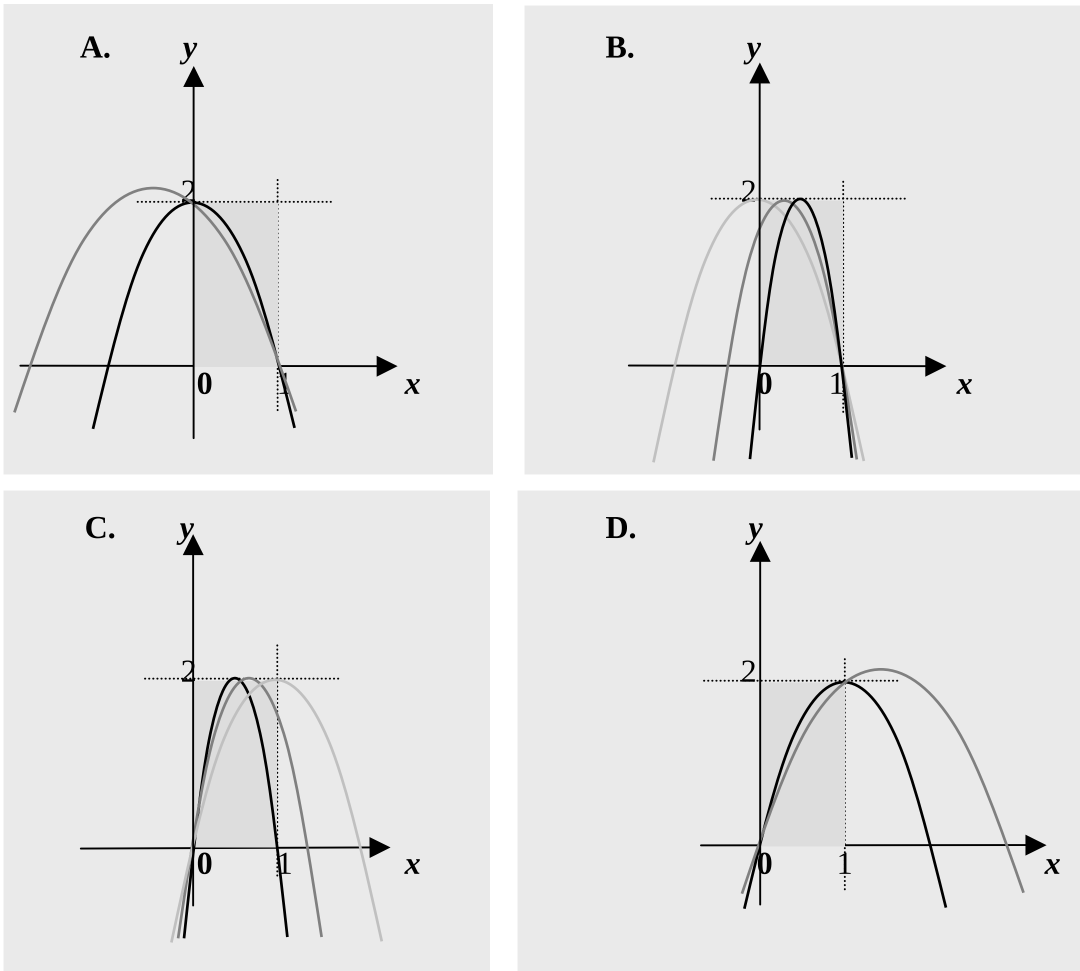
In the case of **B**, it is $\boldsymbol{y = ax^2 + bx + 2 - (a + b)}$ where $\boldsymbol{b = -2a + \sqrt{8a}}$ for $\boldsymbol{2 \le a \le 8}$.

In the case **C**, it is $\boldsymbol{y = ax^2 + bx + 2}$ where $\boldsymbol{b^2 = 8a}$ for $\boldsymbol{2 \le a \le 8}$.

And in the case **D**, the parabola is $\boldsymbol{y = ax^2 + bx + 2}$ where $\boldsymbol{a + b = -2}$ for $\boldsymbol{0 < a \le 2}$.

Now, let's move on to the category where the parabolas are concave-down.
We can put them in four cases as follows.

Fig. E



In the case **A**, we have ***f*(0) = 2**, ***f*(1) = 0**, and also at the vertex, **the *x*-coordinate ≤ 0**.

In the case **B**, we have **0 ≤ *f*(0) ≤ 2**, ***f*(1) = 0**, and also at the vertex, we have
0 ≤ the *x*-coordinate ≤ $\frac{1}{2}$, and **the *y*-coordinate = 2**.

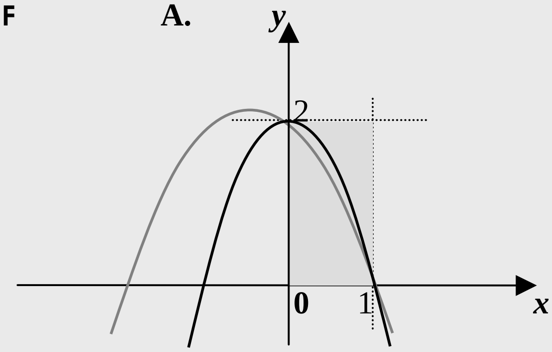
However, we don't have to consider **0 ≤ *f*(0) ≤ 2** as a condition. We will cover it shortly.

In the case **C**, we have ***f*(0) = 0**, **0 ≤ *f*(1) ≤ 2**, and also at the vertex, we have
$\frac{1}{2}$ **≤ the *x*-coordinate ≤ 1**, and **the *y*-coordinate = 2**.

In the case **D**, we have ***f*(0) = 0**, ***f*(1) = 2**, and also at the vertex, **the *x*-coordinate ≥ 1**.

Now, let's begin with the case **A**.

Fig. F A.



We want to find the values of the constants $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ satisfying the function below.
$\boldsymbol{y = f(x) = ax^2 + bx + c}$, and $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$.

Then, we need to satisfy the conditions that
$\boldsymbol{f(0) = 2}$, $\boldsymbol{f(1) = 0}$, and also, **the *x*-coordinate** $\boldsymbol{\le 0}$ at the vertex.

To begin with, we get $\boldsymbol{f(0) = 2 \Rightarrow 2 = 0 + 0 + c \Rightarrow c = 2 \Rightarrow f(x) = ax^2 + bx + 2}$.

So next, we get $\boldsymbol{f(1) = 0 \Rightarrow 0 = a + b + 2 \Rightarrow a + b = -2}$.

Next, the vertex is $\boldsymbol{(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})}$.

So at the vertex, **the *x*-coordinate** $\boldsymbol{\le 0 \Rightarrow -\frac{b}{2a} \le 0 \Rightarrow \frac{b}{a} \ge 0 \Rightarrow b \le 0}$, because $\boldsymbol{a < 0}$.

Now, as an interim result, we have $\boldsymbol{c = 2}$, $\boldsymbol{a + b = -2}$, and $\boldsymbol{b \le 0}$.

Let's now, get the extent of $\boldsymbol{a}$.

To begin with, we get $\boldsymbol{a + b = -2 \Rightarrow b = -2 - a}$. So we get $\boldsymbol{b \le 0 \Rightarrow -2 - a \le 0 \Rightarrow a \ge -2}$.

Besides, we have $\boldsymbol{a < 0}$, too. Therefore, we can see that $\boldsymbol{-2 \le a < 0}$.

So we have $\boldsymbol{c = 2}$, $\boldsymbol{a + b = -2}$, and $\boldsymbol{-2 \le a < 0}$.

Thus, in the case **A**, the parabola is $\boldsymbol{y = ax^2 + bx + 2}$ where $\boldsymbol{a + b = -2}$ for $\boldsymbol{-2 \le a < 0}$.

So choosing any pair of values for $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying $\mathbf{-2 < a \le 0}$ and $\boldsymbol{a + b = -2}$, we get a parabola $\boldsymbol{y = ax^2 + bx + 2}$, and then, we can get $\mathbf{0 \le y \le 2}$ for $\mathbf{0 \le x \le 1}$.

So for instance, we can set $\boldsymbol{a = -1}$, so we get $\boldsymbol{b = -1}$ since $\boldsymbol{a + b = -2}$, and then, we get a parabola $\boldsymbol{y = -x^2 - x + 2}$. Then, we can get $\mathbf{0 \le y \le 2}$ for $\mathbf{0 \le x \le 1}$.

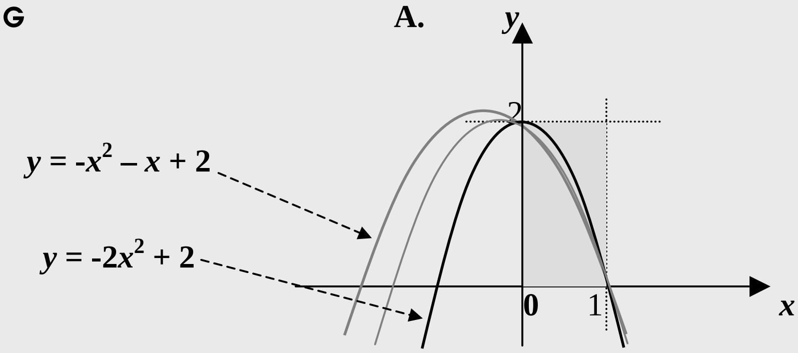
Putting it in the vertex form, we get

$$\boldsymbol{y = -x^2 - x + 2 = -(x^2 + x) + 2 = -(x^2 + x + \tfrac{1}{4} - \tfrac{1}{4}) + 2 = -(x + \tfrac{1}{2})^2 + \tfrac{9}{4}}.$$

For another instance, we can set $\boldsymbol{a = -2}$, and $\boldsymbol{b = 0}$, then we get a parabola $\boldsymbol{y = -2x^2 + 2}$. Then, we can get $\mathbf{0 \le y \le 2}$ for $\mathbf{0 \le x \le 1}$.

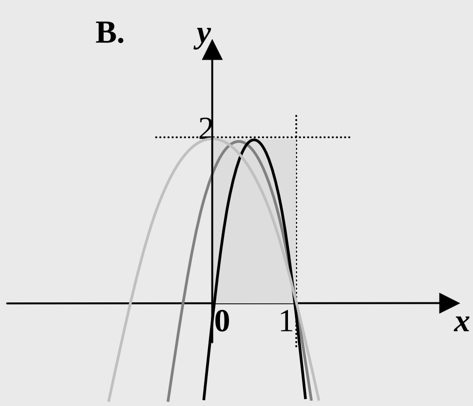
And putting in a graph the two parabolas above, along with another in between, we get

Fig. G **A.**



Now, let's begin the case **B.**

Fig. H **B.**



We want to find the values of the constants $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ satisfying the function below.
$\boldsymbol{y = f(x) = ax^2 + bx + c}$, and $\mathbf{0 \le} \boldsymbol{y} \mathbf{\le 2}$ for $\mathbf{0 \le} \boldsymbol{x} \mathbf{\le 1}$.

Then, we need to satisfy the conditions that
$\boldsymbol{f}\mathbf{(1) = 0}$, and also, at the vertex, **0 ≤ the *x*-coordinate** $\le \frac{1}{2}$, and **the *y*-coordinate = 2**.

Why not consider the value of $\boldsymbol{f}\mathbf{(0)}$, though? That is, don't we have $\mathbf{0 \le} \boldsymbol{f}\mathbf{(0) \le 2}$?

Of course, we do. However, we don't have to consider the case. If we do so, it will be redundant. The conditions specified above are exhaustive now, and will force $\boldsymbol{f}\mathbf{(0)}$ to be so. In other words, if all the conditions are met, we will get $\mathbf{0 \le} \boldsymbol{f}\mathbf{(0) \le 2}$ automatically.

Now, let's satisfy the conditions.

To begin with, we get $\boldsymbol{f}\mathbf{(1) = 0} \Rightarrow \mathbf{0} = \boldsymbol{a + b + c}$.

Next, we know that the vertex is $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$. So we get

First, the ***y*-coordinate = 2** $\Rightarrow -\frac{b^2-4ac}{4a} = 2 \Rightarrow b^2 - 4ac = -8a \Rightarrow b^2 - 4ac + 8a = 0$.

Next, **0 ≤ the *x*-coordinate** $\le \frac{1}{2} \Rightarrow 0 \le -\frac{b}{2a} \le \frac{1}{2} \Rightarrow -\frac{1}{2} \le \frac{b}{2a} \le 0 \Rightarrow -1 \le \frac{b}{a} \le 0$.

Now, as an interim result, we have $\boldsymbol{a + b + c} = \mathbf{0}$, $b^2 - 4ac + 8a = 0$, and $-1 \le \frac{b}{a} \le 0$.

Next, let's find the extent of $\boldsymbol{a}$.
Looking at closely the graph above, we can see that $\mathbf{-8 \le} \boldsymbol{a} \mathbf{\le -2}$. Why?

The $\boldsymbol{a}$-value of the parabola in light gray is -2. Why?

Since the vertex is **(0, 2)**, we can put the parabola this way: $\boldsymbol{y = ax^2} \mathbf{+ 2}$.
Then, since the parabola includes **(1, 0)**, we get $\mathbf{0} = \boldsymbol{a} \cdot \mathbf{1^2 + 2} \Rightarrow \boldsymbol{a} = \mathbf{-2}$.
Besides, the $\boldsymbol{a}$-value of the parabola in black is -8. Why?

Since the vertex is at $\left(\frac{1}{2}, 2\right)$, we can put the parabola this, way: $y = a\left(x - \frac{1}{2}\right)^2 + 2$.
And since the parabola includes **(1, 0)**, we get $0 = a\left(1 - \frac{1}{2}\right)^2 + 2 = \frac{a}{4} + 2 \Rightarrow a = -8$.

Therefore, as another interim result, we have

$\mathbf{-8 \le \boldsymbol{a} \le -2}$, $\boldsymbol{a + b + c} = \mathbf{0}$, $\boldsymbol{b}^2 - 4\boldsymbol{ac} + 8\boldsymbol{a} = \mathbf{0}$, and $-\mathbf{1} \le \frac{\boldsymbol{b}}{\boldsymbol{a}} \le \mathbf{0}$.

Next, let's find the extent of ***b***.

To begin with, we get $-\mathbf{1} \le \frac{\boldsymbol{b}}{\boldsymbol{a}} \le \mathbf{0} \Rightarrow \mathbf{0} \le \boldsymbol{b} \le -\boldsymbol{a}$ since $\boldsymbol{a} < \mathbf{0}$.

Then, since $\mathbf{-8} \le \boldsymbol{a} \le \mathbf{-2}$, we get $\mathbf{0} \le \boldsymbol{b} \le -\boldsymbol{a} \Rightarrow \mathbf{0 \le b \le 8}$. Why?

($\mathbf{-8} \le \boldsymbol{a} \le \mathbf{-2}$**)** means that ***a*** can have any value from -8 to -2.
And we have the facts below.

When $\boldsymbol{a} = \mathbf{-8}$, we get $\mathbf{0} \le \boldsymbol{b} \le -\boldsymbol{a} \Rightarrow \mathbf{0} \le \boldsymbol{b} \le -(-8) = 8 \Rightarrow \mathbf{0} \le \boldsymbol{b} \le \mathbf{8}$.

When, $\boldsymbol{a} = \mathbf{-2}$, we get $\mathbf{0} \le \boldsymbol{b} \le -\boldsymbol{a} \Rightarrow \mathbf{0 \le b} \le -(-2) = 2 \Rightarrow \mathbf{0} \le \boldsymbol{b} \le \mathbf{2}$.

Therefore, we can see that $\mathbf{0} \le \boldsymbol{b} \le \mathbf{8}$.

Now, we have $\boldsymbol{a + b + c} = \mathbf{0}$, $\boldsymbol{b}^2 - 4\boldsymbol{ac} + 8\boldsymbol{a} = \mathbf{0}$, $\mathbf{-8} \le \boldsymbol{a} \le \mathbf{-2}$, and $\mathbf{0} \le \boldsymbol{b} \le \mathbf{8}$.

Let's combine the two connective expressions, $\boldsymbol{a + b + c} = \mathbf{0}$ and $\boldsymbol{b}^2 - 4\boldsymbol{ac} + 8\boldsymbol{a} = \mathbf{0}$.

To begin with, we get $\boldsymbol{a + b + c} = \mathbf{0} \Rightarrow \boldsymbol{c} = -(\boldsymbol{a} + \boldsymbol{b})$

$\Rightarrow -4\boldsymbol{ac} = 4\boldsymbol{a}(\boldsymbol{a} + \boldsymbol{b}) = 4\boldsymbol{a}^2 + 4\boldsymbol{ab} = 4\boldsymbol{a}^2 + 4\boldsymbol{ab} + \boldsymbol{b}^2 - \boldsymbol{b}^2 = (2\boldsymbol{a} + \boldsymbol{b})^2 - \boldsymbol{b}^2$.

So we get $\boldsymbol{b}^2 - 4\boldsymbol{ac} + 8\boldsymbol{a} = \boldsymbol{b}^2 + (2\boldsymbol{a} + \boldsymbol{b})^2 - \boldsymbol{b}^2 + 8\boldsymbol{a} = (2\boldsymbol{a} + \boldsymbol{b})^2 + 8\boldsymbol{a} = \mathbf{0}$

$\Rightarrow (2\boldsymbol{a} + \boldsymbol{b})^2 = -8\boldsymbol{a}$, which is positive since ***a*** is negative.

Thus, we get $2\boldsymbol{a} + \boldsymbol{b} = \pm\sqrt{-8\boldsymbol{a}} \Rightarrow \boldsymbol{b} = -2\boldsymbol{a} \pm \sqrt{-8\boldsymbol{a}} \Rightarrow \boldsymbol{b} = -2\boldsymbol{a} + \sqrt{-8\boldsymbol{a}}$ or $-2\boldsymbol{a} - \sqrt{-8\boldsymbol{a}}$.

Now, we need to choose for ***b*** one from $(-2\boldsymbol{a} + \sqrt{-8\boldsymbol{a}})$ and $(-2\boldsymbol{a} - \sqrt{-8\boldsymbol{a}})$.

We have $\mathbf{-8} \le \boldsymbol{a} \le \mathbf{-2}$, and thus, when ***a*** changes from -8 to -2, we can see that
$(-2\boldsymbol{a} + \sqrt{-8\boldsymbol{a}})$ changes from 24 to 8, and $(-2\boldsymbol{a} - \sqrt{-8\boldsymbol{a}})$ changes from 8 to 0.

We know that $\mathbf{0 \le b \le 8}$. So we get $\boldsymbol{b = \text{-}2a - \sqrt{\text{-}8a}}$.

Now, we have $\boldsymbol{b = -2a - \sqrt{-8a}}$, $\boldsymbol{\text{-}8 \le a \le \text{-}2}$, and $\boldsymbol{c = -(a + b)}$.

Thus, in the case **B**, the parabola is $\boldsymbol{y = ax^2 + bx - (a + b)}$ where $\boldsymbol{b = \text{-}2a - \sqrt{\text{-}8a}}$ for $\boldsymbol{\text{-}8 \le a \le \text{-}2}$.

Examining the graph again, we can see that $\mathbf{0 \le c \le 2}$. Why?

In the graph, we can see that $\mathbf{0 \le}$ ***y*-intercept** $\mathbf{\le 2}$.
And in $\boldsymbol{y = ax^2 + bx + c}$, $\boldsymbol{c}$ is the *y*-intercept.

Let's check to see if ($\boldsymbol{0 \le c \le 2}$) and ($\boldsymbol{\text{-}8 \le a \le \text{-}2}$) satisfy ($\boldsymbol{0 \le b \le 8}$).

To begin with, we get $\boldsymbol{a + b + c = 0 \Rightarrow a + c = \text{-}b}$.

Next, we get $\boldsymbol{(0 \le c \le 2) + (\text{-}8 \le a \le \text{-}2) \Rightarrow \text{-}8 \le a + c \le 0 \Rightarrow \text{-}8 \le \text{-}b \le 0 \Rightarrow 0 \le b \le 8}$.

So we can put the parabola this way, too:
$\boldsymbol{y = ax^2 + bx - (a + b)}$ where $\boldsymbol{b = \text{-}2a - \sqrt{\text{-}8a}}$ for $\boldsymbol{0 \le b \le 8}$.

Now, choosing any pair of values for $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying $\boldsymbol{\text{-}8 \le a \le \text{-}2}$ and $\boldsymbol{b = \text{-}2a - \sqrt{\text{-}8a}}$, we get a parabola $\boldsymbol{y = ax^2 + bx - (a + b)}$, and then, we can get $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$.

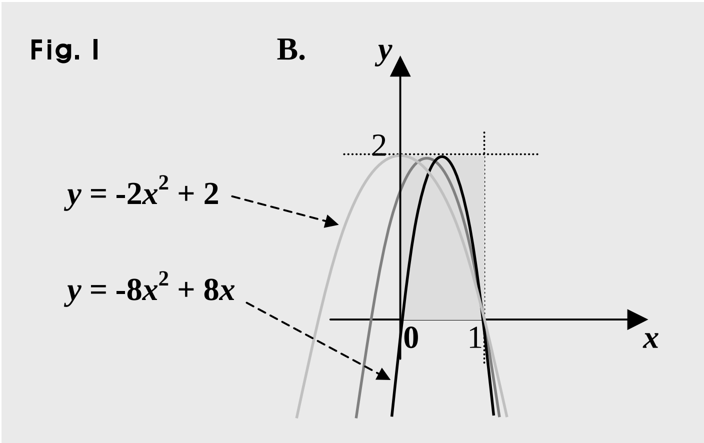
So for instance, we can set $\boldsymbol{a = \text{-}2}$, so we get $\boldsymbol{b = \text{-}2a - \sqrt{\text{-}8a} = 4 - \sqrt{16} = 0}$, and then, we get a parabola $\boldsymbol{y = \text{-}2x^2 + 2}$. Then, we can get $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$.

For another instance, we can set $\boldsymbol{a = \text{-}8}$, so we get $\boldsymbol{b = \text{-}2a - \sqrt{\text{-}8a} = 16 - 8 = 8}$, and then, we get a parabola $\boldsymbol{y = \text{-}8x^2 + 8x}$. And putting the parabola in the vertex form, we get

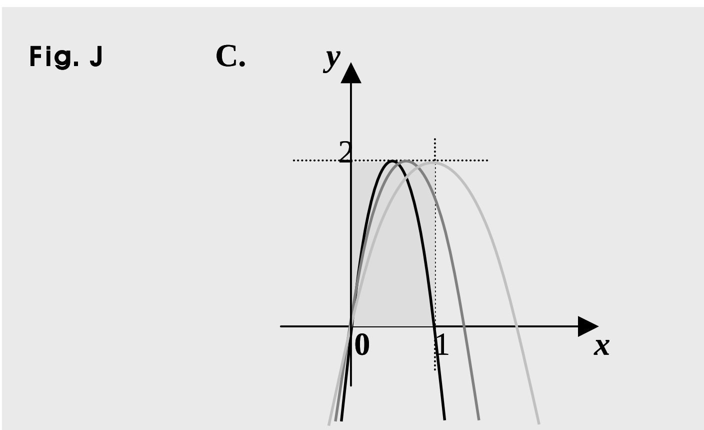
$$\boldsymbol{y = \text{-}8x^2 + 8x = \text{-}8(x^2 - x + \tfrac{1}{4} - \tfrac{1}{4}) = \text{-}8(x - \tfrac{1}{2})^2 + 2 = \text{-}8(x - \tfrac{1}{2})^2 + 2 \Rightarrow y = \text{-}8(x - \tfrac{1}{2})^2 + 2}.$$

Then, we can see that we can get $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$. Notice that $\boldsymbol{y = 2}$ when $\boldsymbol{x = \tfrac{1}{2}}$.

Let's now, put in a graph the two parabolas above, together with another one in between.



Now, let's move on to the case **C**.



We want to find the values of the constants $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ satisfying the function below.
$\boldsymbol{y = f(x) = ax^2 + bx + c}$, and $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$.

Then, we need to satisfy the conditions that
$\boldsymbol{f(0) = 0}$, and also, at the vertex, $\frac{1}{2} \le$ **the** $\boldsymbol{x}$**-coordinate** $\le$ **1**, and **the** $\boldsymbol{y}$**-coordinate = 2**.

To begin with, we get $\boldsymbol{f(0) = 0 \Rightarrow 0 = a \cdot 0^2 + b \cdot 0 + c \Rightarrow c = 0 \Rightarrow f(x) = ax^2 + bx}$.

Next, we know that the vertex is $\boldsymbol{(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})}$. So we get

First, the $\boldsymbol{y}$**-coordinate = 2** $\boldsymbol{\Rightarrow -\frac{b^2-4ac}{4a} = 2 \Rightarrow b^2 - 4ac = -8a \Rightarrow b^2 + 8a = 0}$ since $\boldsymbol{c = 0}$.

Next, $\frac{1}{2} \le$ **the** $\boldsymbol{x}$**-coordinate** $\boldsymbol{\le 1 \Rightarrow \frac{1}{2} \le -\frac{b}{2a} \le 1 \Rightarrow -1 \le \frac{b}{2a} \le -\frac{1}{2} \Rightarrow -2 \le \frac{b}{a} \le -1}$.

Then, as an interim result, we have $\boldsymbol{c = 0}$, $\boldsymbol{b^2 + 8a = 0}$, and $\boldsymbol{-2 \le \frac{b}{a} \le -1}$.

Let's next, get the extent of $\boldsymbol{b}$ using the two expressions, $\boldsymbol{b^2 + 8a = 0}$ and $\boldsymbol{-2 \le \frac{b}{a} \le -1}$.

To begin with, we get $\boldsymbol{b^2 + 8a = 0 \Rightarrow b^2 = -8a \Rightarrow a = -\frac{b^2}{8} \Rightarrow \frac{b}{a} = b \cdot \frac{1}{a} = b \cdot (-\frac{8}{b^2}) = -\frac{8}{b}}$.

So we can get $\boldsymbol{-2 \le \frac{b}{a} \le -1 \Rightarrow -2 \le -\frac{8}{b} \le -1 \Rightarrow 1 \le \frac{8}{b} \le 2 \Rightarrow \frac{1}{2} \le \frac{b}{8} \le 1 \Rightarrow 4 \le b \le 8}$.

Thus, we now have $\boldsymbol{c = 0}$, $\boldsymbol{b^2 + 8a = 0}$, and $\boldsymbol{4 \le b \le 8}$.

So in the case **C**, the parabola is $\boldsymbol{y = ax^2 + bx}$ where $\boldsymbol{b^2 + 8a = 0}$ for $\boldsymbol{4 \le b \le 8}$.

Also, as in the case **B** above, looking at the graph only, we can get $\boldsymbol{-8 \le a \le -2}$.

First, the $\boldsymbol{a}$-value of the parabola in light gray is -2.
Since the vertex is **(1, 2)**, we can put the equation this way: $\boldsymbol{y = a(x - 1)^2 + 2}$.
And since the parabola includes **(0, 0)**, we get $\boldsymbol{0 = a(0 - 1)^2 + 2 \Rightarrow 0 = a + 2 \Rightarrow a = -2}$.

Next, the $\boldsymbol{a}$-value of the parabola in black is -8.
Since the vertex is $\boldsymbol{(\frac{1}{2}, 2)}$, suppose that the equation is $\boldsymbol{y = a(x - \frac{1}{2})^2 + 2}$.
Then, since the parabola includes **(0, 0)**, we get $\boldsymbol{0 = a(0 - \frac{1}{2})^2 + 2 = \frac{a}{4} + 2 \Rightarrow a = -8}$.

Checking to see if the above is correct, we get first, $\boldsymbol{b^2 + 8a = 0 \Rightarrow a = -\frac{b^2}{8}}$.

So next, we get $\boldsymbol{4 \le b \le 8 \Rightarrow 16 \le b^2 \le 64 \Rightarrow 2 \le \frac{b^2}{8} \le 8 \Rightarrow -8 \le -\frac{b^2}{8} \le -2 \Rightarrow -8 \le a \le -2}$.

So we can put the parabola this way, too: $\boldsymbol{y = ax^2 + bx}$ where $\boldsymbol{b^2 + 8a = 0}$ for $\boldsymbol{-8 \le a \le -2}$.

So choosing any pair of values for $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying $\boldsymbol{4 \le b \le 8}$ and $\boldsymbol{b^2 + 8a = 0}$, we get a parabola $\boldsymbol{y = ax^2 + bx}$, and then, we can get $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$.

So for instance, we can set $\boldsymbol{b = 4}$, so we get $\boldsymbol{a = -2}$ since $\boldsymbol{b^2 + 8a = 0}$, and then, we get a parabola $\boldsymbol{y = -2x^2 + 4x = -2x(x - 2)}$. Then, we can get $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$.

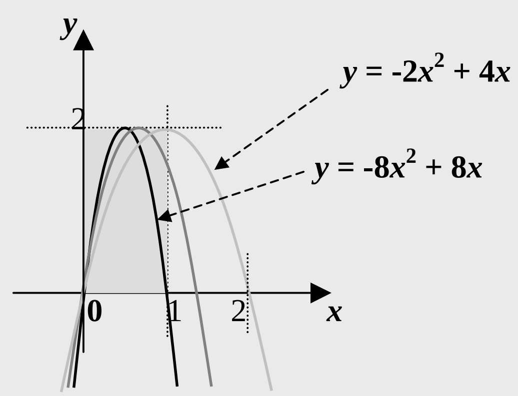
And putting the parabola in the vertex form, we get $\boldsymbol{y = -2x^2 + 4x = -2(x-1)^2 + 2}$.

For another, we can set $\boldsymbol{b = 8}$, so we get $\boldsymbol{a = -8}$ since $\boldsymbol{b^2 + 8a = 0}$, and then, we get a parabola $\boldsymbol{y = -8x^2 + 8x}$. And putting it in the vertex form, we get $\boldsymbol{y = -8(x - \frac{1}{2})^2 + 2}$.

So we can get $\boldsymbol{0 \le y \le 2}$ for $\boldsymbol{0 \le x \le 1}$. We know that we get $\boldsymbol{y = 2}$ when $\boldsymbol{x = \frac{1}{2}}$.

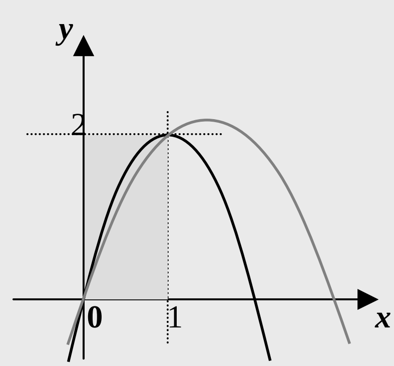
Now, putting in a graph the two parabolas above, along with another in between, we get

Fig. K C.



Now, let's move on to the case **D**.

Fig. L D.



We want to find the values of the constants $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ satisfying the function below.

$\boldsymbol{y = f(x) = ax^2 + bx + c}$ for $\boldsymbol{0 \le x \le 1}$ and $\boldsymbol{0 \le y \le 2}$.

Then, we need to satisfy the conditions as follows.

$\boldsymbol{f(0) = 0}$, and $\boldsymbol{f(1) = 2}$; also, the $\boldsymbol{x}$**-coordinate** $\boldsymbol{\ge 1}$ at the vertex.

To begin with, we get $f(0) = 0 \Rightarrow 0 = a\cdot 0^2 + b\cdot 0 + c \Rightarrow c = 0 \Rightarrow f(x) = ax^2 + bx$.

Next, we get $f(1) = 2 \Rightarrow 2 = a\cdot 1^2 + b\cdot 1$. Therefore, $a + b = 2$.

Next, we know that the vertex is $(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})$. So we get

The x-**coordinate** $\geq 1 \Rightarrow -\frac{b}{2a} \geq 1 \Rightarrow \frac{b}{a} \leq -2 \Rightarrow b \geq -2a$ since $a < 0$.

Then, as an interim result, we have $c = 0$, $a + b = 2$, and $b \geq -2a$.

Next, let's get the extent of a.

To begin with, we get $a + b = 2 \Rightarrow b = 2 - a$.

Then, we get $b \geq -2a \Rightarrow 2 - a \geq -2a \Rightarrow a \geq -2$.

Besides, we have $a < 0$, too. So we get $-2 \leq a < 0$.

Therefore, in the case **D**, the parabola is $y = ax^2 + bx$ where $a + b = 2$ for $-2 \leq a < 0$.

So choosing any pair of values for a and b satisfying $-2 \leq a < 0$ and $a + b = 2$, we get a parabola $y = ax^2 + bx$, and then, we can get $0 \leq y \leq 2$ for $0 \leq x \leq 1$.

So for instance, we can set $a = -2$, so we get $b = 4$ since $a + b = 2$, and then, we get a parabola $y = -2x^2 + 4x = -2x(x - 2)$. Then, we can get $0 \leq y \leq 2$ for $0 \leq x \leq 1$.

And putting the parabola in the vertex form, we get

$$y = -2x^2 + 4x = -2(x^2 - 2x + 1 - 1) = -2(x - 1)^2 + 2.$$

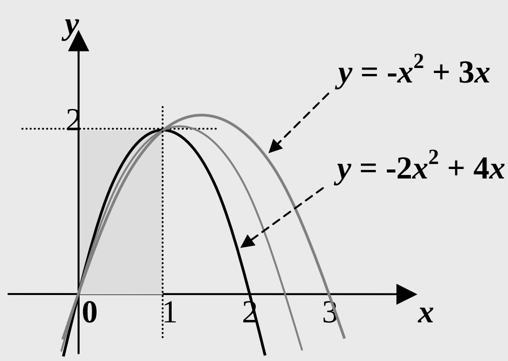
For another instance, we can set $a = -1$, so we get $b = 3$ since $a + b = 2$, and then, we get a parabola $y = -x^2 + 3x = -x(x - 3)$. Then, we can get $0 \leq y \leq 2$ for $0 \leq x \leq 1$.

And putting the parabola in the vertex form, we get

$$y = -x^2 + 3x = -(x^2 - 3x + \tfrac{9}{4} - \tfrac{9}{4}) = -(x - \tfrac{3}{2})^2 + \tfrac{9}{4}.$$

Now, putting in a graph the two parabolas above, along with another in between, we get

Fig. M D.



So putting threads together in the case where the parabolas are concave-down, we have

In the case **A**, the parabola is $\boldsymbol{y = ax^2 + bx + 2}$ where $\boldsymbol{a + b = -2}$ for $\boldsymbol{-2 \le a < 0}$.

In the case **B**, the parabola is $\boldsymbol{y = ax^2 + bx - (a + b)}$ where $\boldsymbol{b = -2a - \sqrt{-8a}}$ for $\boldsymbol{-8 \le a \le -2}$.

In the case **C**, the parabola is $\boldsymbol{y = ax^2 + bx}$ where $\boldsymbol{b^2 + 8a = 0}$ for $\boldsymbol{4 \le b \le 8}$.

And in the case **D**, the parabola is $\boldsymbol{y = ax^2 + bx}$ where $\boldsymbol{a + b = 2}$ for $\boldsymbol{-2 \le a < 0}$.

Besides, in the case where the parabolas are concave-up, we have

In the case **A**, the parabola is $\boldsymbol{y = ax^2 + bx}$ where $\boldsymbol{a + b = 2}$ for $\boldsymbol{0 < a \le 2}$.

In the case **B**, it is $\boldsymbol{y = ax^2 + bx + 2 - (a + b)}$ where $\boldsymbol{b = -2a + \sqrt{8a}}$ for $\boldsymbol{2 \le a \le 8}$.

In the case **C**, it is $\boldsymbol{y = ax^2 + bx + 2}$ where $\boldsymbol{b^2 = 8a}$ for $\boldsymbol{2 \le a \le 8}$.

And in the case **D**, the parabola is $\boldsymbol{y = ax^2 + bx + 2}$ where $\boldsymbol{a + b = -2}$ for $\boldsymbol{0 < a \le 2}$.