Examples 8 in Parabolas

Click this to start.

Examples 8

Note: this set of examples is for those students who want to study parabolas in depth.

This set of examples is again, about graphing.
Graphing matters. And it does a lot when we work with functions as well as when we do word problems. Using functions, we can keep track of some change. Keeping track of it, we can put it in a graph so that we can actually see it. Then, we call it a curve. Of all kinds in curves, one of the most often used is a parabola. In this set of examples, we are going to do some practices on putting a parabola in a graph.

Suppose $\boldsymbol{c}$, $\boldsymbol{m}$, and $\boldsymbol{n}$ are constant. Then,

0. Assuming $\boldsymbol{y = f(x) = \frac{1}{7}(2x^2 + 3)}$, and $\boldsymbol{m} \le \boldsymbol{y} \le \boldsymbol{n}$ for $\boldsymbol{m} \le \boldsymbol{x} \le \boldsymbol{n}$, find $\boldsymbol{m}$ and $\boldsymbol{n}$.

1. Assuming $\boldsymbol{y = f(x) = \frac{1}{5}(2x^2 - 3)}$, and $\boldsymbol{m} \le \boldsymbol{y} \le \boldsymbol{n}$ when $\boldsymbol{m} \le \boldsymbol{x} \le \boldsymbol{n}$, find $\boldsymbol{m}$ and $\boldsymbol{n}$.

2. Assuming $\boldsymbol{y = f(x) = \frac{1}{5}(2x^2 + c)}$, and $\boldsymbol{m} \le \boldsymbol{y} \le \boldsymbol{n}$ when $\boldsymbol{m} \le \boldsymbol{x} \le \boldsymbol{n}$, find $\boldsymbol{c}$, $\boldsymbol{m}$, and $\boldsymbol{n}$.

Suggestions or Solutions
To the Problem in the Example 0

Assuming *m* and *n* are constant, $y = f(x) = \frac{1}{7}(2x^2 + 3)$, and $m \le y \le n$ for $m \le x \le n$, find *m* and *n*.

$\boldsymbol{x = \frac{1}{7}(2x^2 + 3) \Rightarrow 7x = 2x^2 + 3 \Rightarrow 2x^2 - 7x + 3 = 0 \Rightarrow (2x - 1)(x - 3) = 0 \Rightarrow x = \frac{1}{2}}$ or $\mathbf{3}$.

And the axis of symmetry is $\boldsymbol{x = 0}$. So we get $\boldsymbol{m = \frac{1}{2}}$, and $\boldsymbol{n = 3}$.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, what do we mean by "$\boldsymbol{m \le y \le n}$ for $\boldsymbol{m \le x \le n}$."?

We have a function $\boldsymbol{y = f(x)}$, so the function $\boldsymbol{f}$ produces values from $\boldsymbol{m}$ to $\boldsymbol{n}$ for $\boldsymbol{m \le x \le n}$. So what?

Of the function $\boldsymbol{f}$, the range is the same as the domain, and both are of course, not infinite but finite.

In other words, the extent of the output is the same as that of the input, and is limited. So we need to find the extent. How then, can we find such an extent?

Suppose for instance, an input equals the output for the input, and another input equals the output for the other input.

Then, can we say that the two inputs are the values of $\boldsymbol{m}$ and $\boldsymbol{n}$?

No, we can't. It can be the case though, the two inputs are the solution.
So we can just say that the two inputs <u>could be</u> the values of $\boldsymbol{m}$ and $\boldsymbol{n}$.

Not quite sure? Let's this time, put it the way below.

Assuming for instance, $a < b$, $f(a) = a$, and $f(b) = b$, can we say that $m = a$, and $n = b$?

For instance, if $f(1) = 1$, and $f(3) = 3$, can we say that $m = 1$, and $n = 3$?

We can say yes if the domain is the same as the range.

So even if we can find two values, for each of which the output is the same, we cannot just say that the two values are the values of m and n. Why not?

Even though we have found such two values, the domain can be different from the range. So even finding such two values, we cannot just say that the domain equals the range.

So for the function f, it <u>can be</u> the case the values of m and n do not exist.
Anyway, how can we find the two values if they do exist?

We have a function where every input is the same as its corresponding output.
That is, if the function gets an input value, the output value is the same as the input value.
What then, is the function?

It is called an identity function. And the curve of the function is a line $y = x$.

So the identity function is $y = g(x) = x$ for x real.

(If not quite sure of functions, refer to **BASIC FUNCTIONS**.)

Now, the curve of the function f is a parabola.
So finding the two points where the parabola meets the line $y = x$, we <u>could</u> find the values of m and n.
Specifically, if of the two points, one is (u, u), and the other is (v, v), u and v <u>could be</u> the values of m and n.

That is, solving the equation $x = f(x)$, and getting two roots, we could take the two roots as the values of m and n. One root could be m, and the other could be n, of course.

So let’s find the points where the parabola meets the line $y = x$.

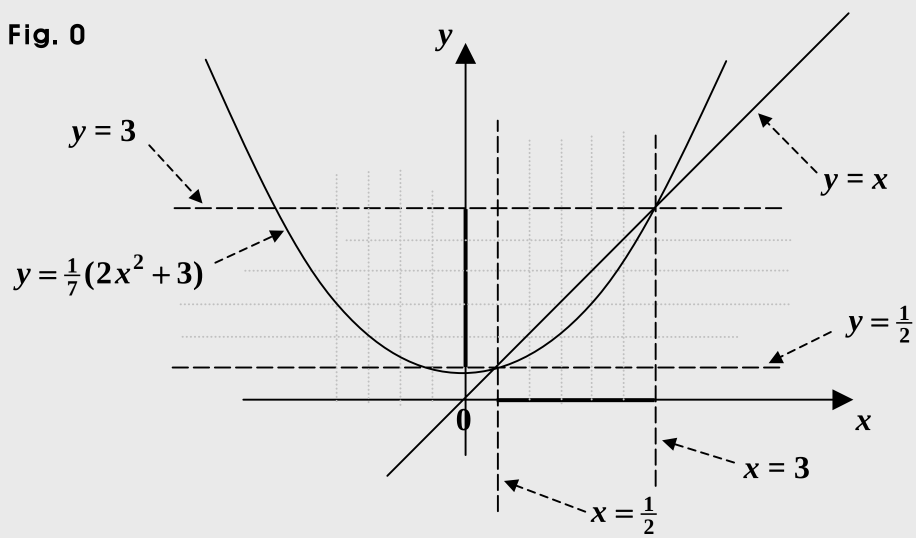
Now, the parabola is $y = \frac{1}{7}(2x^2 + 3)$, and the line is $y = x$. So we get

$$x = \frac{1}{7}(2x^2+3) \Rightarrow 7x = 2x^2 + 3 \Rightarrow 2x^2 - 7x + 3 = 0 \Rightarrow (2x-1)(x-3) = 0 \Rightarrow x = \frac{1}{2} \text{ or } 3.$$

So the parabola meets the line when x is $\frac{1}{2}$ and 3, which could be thus, the solution, that is, the values of m and n. What do we mean by though, could be?

So instead of just thinking, put the idea in a graph so that you can actually see it. So let's now, put the curves in a graph, and see if the x-values above are the values of m and n. The curves are the line $y = x$ and the parabola above, of course.

Fig. 0



So we can see that $m = \frac{1}{2}$, and $n = 3$. That is, we get $\frac{1}{2} \le y \le 3$ for $\frac{1}{2} \le x \le 3$.

And thus, the domain is $\frac{1}{2} \le x \le 3$, and the range is $\frac{1}{2} \le y \le 3$.

Do we have to do then, construct such a graph finding such a domain?

Doing problems with equations or functions, construct their graphs as much as you can. Then, you can see better the solution's whereabouts so that you can get access to the solution faster and safer as well as easier.

Suggestions or Solutions
To the Problem in the Example 1

Assuming *m* and *n* are constant, $y = f(x) = \frac{1}{5}(2x^2 - 3)$, and $m \le y \le n$ for $m \le x \le n$, find *m* and *n*.

$x = \frac{1}{5}(2x^2 - 3) \Rightarrow 5x = 2x^2 - 3 \Rightarrow (2x+1)(x-3) = 0 \Rightarrow x = -\frac{1}{2}$ or 3.

However, the axis of symmetry is $x = 0$, which is between the two roots.
So there is no solution to this problem.

If not quite sure of the idea behind the processes above, follow the steps below.

As in the case of the previous example, we want the set of ***x***-values to be the same as the set of ***y***-values. That is, we want the domain of the function ***f*** to be the same as the range.

And as stated in the previous example, <u>if such a domain does exist</u>, we can find the domain, that is, we can find the values of ***m*** and ***n***.   How then, can we find the values?

Suppose now, that we have found two particular inputs, for each of which the output is the same. Then, can we say that the two inputs are the values of ***m*** and ***n***?

Assuming for instance, $u < v$, $u = f(u)$, and $v = f(v)$, can we say that $m = u$ and $n = v$?

No, we can't. It's not necessarily case where $m = u$ and $n = v$. It will be the case though, if some condition is met.   And you will see shortly what the condition is.

Now, the curve of ***f*** is a parabola $y = \frac{1}{5}(2x^2 - 3)$.

If the values of ***m*** and ***n*** exist, we can find them using an identity function, of which the curve is a line $y = x$.   What do we mean by though, using an identity function?

Finding two points where the parabola meets the line $y = x$, we could get the value of ***m*** and ***n***. That is, if we find the ***x***-coordinates at the two points, the coordinate values could be the value of ***m*** and ***n***.

So finding now, the $\boldsymbol{x}$-coordinates at the two points, we get

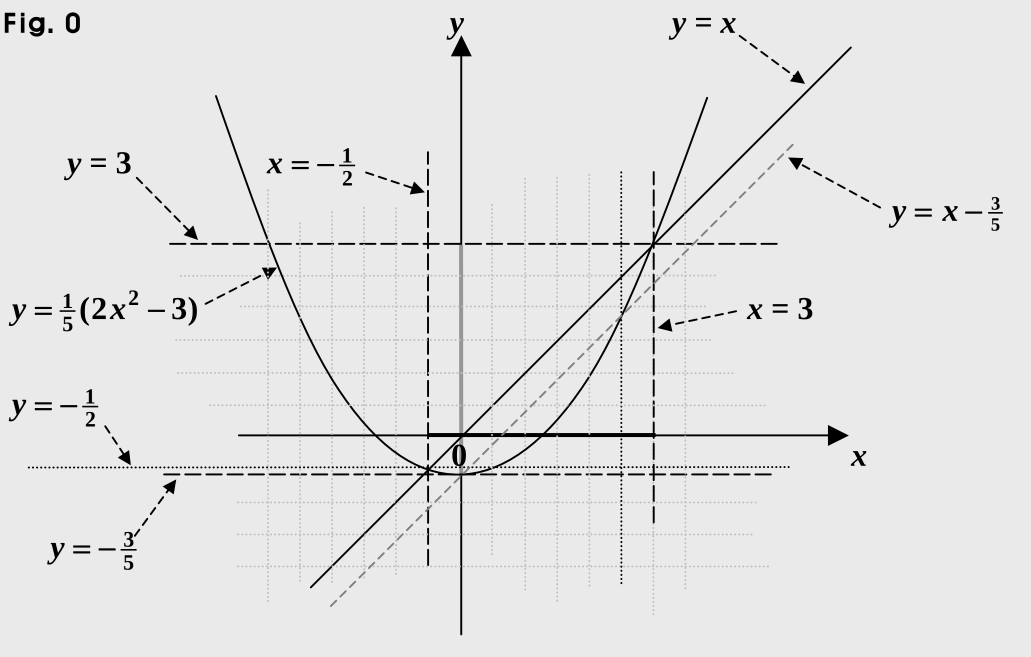
$$x = \tfrac{1}{5}(2x^2 - 3) \Rightarrow 5x = 2x^2 - 3 \Rightarrow (2x+1)(x-3) = 0 \Rightarrow x = -\tfrac{1}{2} \text{ or } 3.$$

So it looks like $m = -\frac{1}{2}$, and $n = 3$. However, the parabola has its vertex at $(0, -\frac{3}{5})$.

So at the vertex, the x-coordinate is bigger than $-\frac{1}{2}$, but the y-coordinate is less than $-\frac{1}{2}$.

That is, the domain cannot be the same as the range.

Fig. 0



What if we use a line passing through the vertex? That is, what if the line is $y = x - \frac{3}{5}$?

Then, the magnitudes of the domain and the range are the same, but the domain and the range are still different.
That is because we get $0 \le x \le$ some number, but $-\frac{3}{5} \le y \le$ some other number.

Therefore, there is no solution to this problem.

Suggestions or Solutions
To the Problem in the Example 2

Assuming this time, c, m, and n are constant, $y = f(x) = \frac{1}{5}(2x^2 + c)$, and $m \le y \le n$ for $m \le x \le n$, find c, m, and n.

To begin with, finding two points where the parabola $y = \frac{1}{5}(2x^2 + c)$ meets the line $y = x$, we get $x = \frac{1}{5}(2x^2 + c) \Rightarrow 2x^2 - 5x + c = 0 \Rightarrow x = \frac{5 \pm \sqrt{25-8c}}{4}$.

Next, the quadratic equation above needs to have two roots. So the discriminant is > 0.

Thus, we get $D = 5^2 - 4 \cdot 2c = 25 - 8c > 0 \Rightarrow c < \frac{25}{8}$.

Next, the x-coordinate at the vertex is $\le$ the smaller of the two roots above. So we get

$0 \le \frac{5-\sqrt{25-8c}}{4} \Rightarrow 0 \le 5 - \sqrt{25-8c} \Rightarrow \sqrt{25-8c} \le 5 \Rightarrow 25 - 8c \le 25 \Rightarrow c \ge 0$.

(Equivalently, "At the vertex, the y-coordinate $\ge$ the x-coordinate."

The vertex is $(0, \frac{c}{5})$, so we get $\frac{c}{5} \ge 0 \Rightarrow c \ge 0$.)

Now, putting threads together, we get $0 \le c < \frac{25}{8}$.

Therefore, the interval is $\frac{5-\sqrt{25-8c}}{4} \le x \le \frac{5+\sqrt{25-8c}}{4}$ where $0 \le c < \frac{25}{8}$.

That is, $m = \frac{5-\sqrt{25-8c}}{4}$, and $n = \frac{5+\sqrt{25-8c}}{4}$ where $0 \le c < \frac{25}{8}$.

If not quite sure of the idea behind the processes above, follow the steps below.

As in the case of the previous example, we want the set of x-values to be the same as the set of y-values. That is, we want the domain of the function f to be the same as the range.

And as stated in the previous example, if such a domain does exist, we can find the domain, that is, we can find the values of m and n. How then, can we find the values?

The curve of $\boldsymbol{f}$ is a parabola $\boldsymbol{y = \frac{1}{5}(2x^2 + c)}$.

If such a domain exists, we can find it using the line $\boldsymbol{y = x}$.

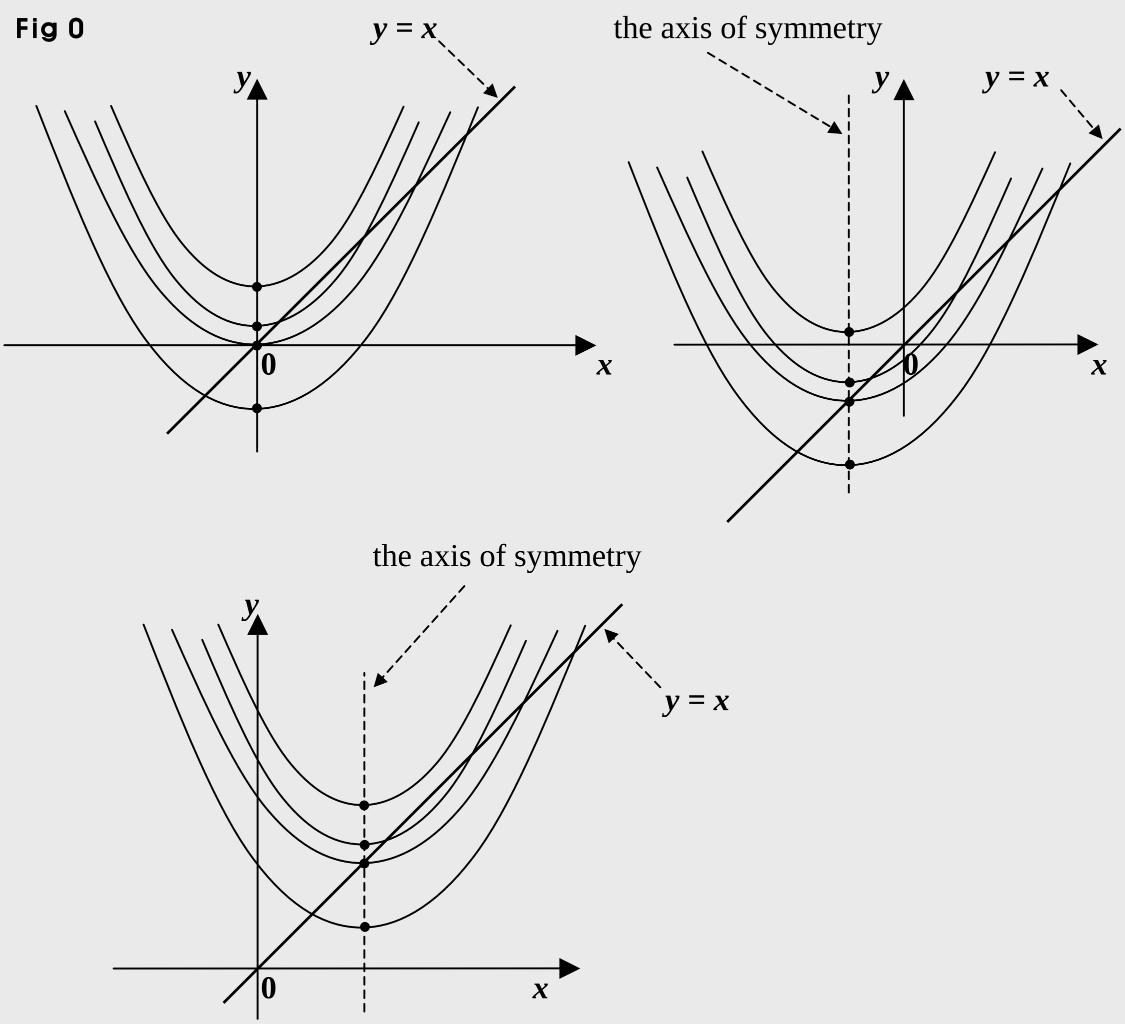
So first, find two points where the parabola meets the line $\boldsymbol{y = x}$.

Next, check to see if a domain can be the same as a range. Then, specify the domain.

In this case though, we may want to begin with putting some parabolas in a graph, and see in what circumstances solutions can exist. We are going to consider cases where parabolas are concave up, since the equation given indicates such a parabola.

Now, constructing some graphs schematic, we can have

Fig 0



Now, examining the graphs above, we can notice that if solutions exist, two conditions have to be met at the same time, and the two are as below.

• First, the parabola meets the line at two points, of course.

• Next, the vertex is not below the line $\boldsymbol{y = x}$.

What do we mean by the vertex not below the line, though?

Now, looking at the vertex more closely, we can notice that the second condition above is equivalent to any of the two statements below.

• At the vertex, the $\boldsymbol{y}$-coordinate ≥ the $\boldsymbol{x}$-coordinate.

• The $\boldsymbol{x}$-coordinate at the vertex ≤ the smaller of the two roots.

What roots, though?

The roots are the solution to the equation where $\boldsymbol{x = f(x)}$.
How do we get the $\boldsymbol{x}$-coordinate at the vertex ≤ the smaller of the two roots, though?

Examining closely the graphs above, we can see it is the case if the vertex is not between the two points where the parabola meets the line. That is, the vertex is on the left of the two points or is the same as the point that is the left one of the two. So the $\boldsymbol{x}$-coordinate at the vertex ≤ the smaller of the two roots.

• Now, the procedure where we can get the interval can go as follows.

Suppose first, that the parabola is $\boldsymbol{y = ax^2 + bx + c}$.

Then first, we need to get two points where the parabola meets the line, $\boldsymbol{y = x}$.

So we need to solve a quadratic equation as follows. $\boldsymbol{x = ax^2 + bx + c}$.

That is, we want to solve $\boldsymbol{ax^2 + (b - 1)x + c = 0}$.

Then, the quadratic equation has to have two roots.

How then, do we know if such an equation can get two different roots without actually solving it, of course?

The discriminant for the equation has to be > **0**.

So suppose next, that ***D*** is the discriminant.

Then, we need to have $\boldsymbol{D = (b-1)^2 - 4ac > 0}$.

So we get $\boldsymbol{(b-1)^2 - 4ac = b^2 - 2b + 1 - 4ac > 0 \Rightarrow b^2 - 2b - 4ac > -1}$.

Now up to this point, we have covered the idea where we need to get two points where the parabola meets the line $\boldsymbol{y = x}$, which is the curve of the identity function, of course. What then, is the next?

The next is the condition that the vertex has to be *in* or *above* the line, $\boldsymbol{y = x}$.

The condition has been substantiated by either of the two statements stated above.

So let's see first, what the first of the two says.

The statement says, "At the vertex, the ***y***-coordinate ≥ the ***x***-coordinate."

We have $\boldsymbol{y = ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{b^2-4ac}{4a}}$. So the vertex is $\boldsymbol{(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})}$.

Thus, we need to get $\boldsymbol{-\frac{b}{2a} \le -\frac{b^2-4ac}{4a}}$.

Multiplying by both sides by **-4*a***, we get $\boldsymbol{2b \ge b^2 - 4ac \Rightarrow b^2 - 2b - 4ac \le 0}$.

Now, that's what the first statement says.

Let's see next, what the second statement says. It should be the same as above, of course.

The statement says, "The ***x***-coordinate at the vertex ≤ the smaller of the two roots."

To begin with, we know that the vertex is $\boldsymbol{(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})}$.

Next, by the quadratic formula, the two roots are $x = \frac{-(b-1) \pm \sqrt{(b-1)^2 - 4ac}}{2a}$.

The smaller of the two roots is $x = \frac{-(b-1) - \sqrt{(b-1)^2 - 4ac}}{2a}$.

So we get $-\frac{b}{2a} \le \frac{-(b-1) - \sqrt{(b-1)^2 - 4ac}}{2a}$. Thus, we get

$$-\frac{b}{2a} \le \frac{-(b-1) - \sqrt{(b-1)^2 - 4ac}}{2a} \Rightarrow b \ge b - 1 + \sqrt{(b-1)^2 - 4ac} \Rightarrow \sqrt{(b-1)^2 - 4ac} \le 1.$$

So we get $(b-1)^2 - 4ac \le 1 \Rightarrow b^2 - 2b + 1 - 4ac \le 1 \Rightarrow b^2 - 2b - 4ac \le 0$, which is the same as the one we have got from the first statement.

Now, putting threads together, we have $b^2 - 2b - 4ac > -1$, and $b^2 - 2b - 4ac \le 0$.

So we get $-1 < b^2 - 2b - 4ac \le 0$.

Therefore, if we have $-1 < b^2 - 2b - 4ac \le 0$, the interval is as follows.

$$\frac{-(b-1) - \sqrt{(b-1)^2 - 4ac}}{2a} \le x \le \frac{-(b-1) + \sqrt{(b-1)^2 - 4ac}}{2a}.$$

That is, in the case where $y = f(x) = ax^2 + bx + c$, if we have $-1 < b^2 - 2b - 4ac \le 0$, we can get $m \le y \le n$ for $m \le x \le n$, where

$$m = \frac{-(b-1) - \sqrt{(b-1)^2 - 4ac}}{2a}, \text{ and } n = \frac{-(b-1) + \sqrt{(b-1)^2 - 4ac}}{2a}.$$

Now, in this example, we have $y = f(x) = \frac{1}{5}(2x^2 + c)$ where c is constant.

To begin with, we need to find two points where the curve of the function f, that is, the parabola, $y = \frac{1}{5}(2x^2 + c)$ meets the line, $y = x$.

So we want to solve $x = \frac{1}{5}(2x^2 + c)$, which is $2x^2 - 5x + c = 0$.

Next, the quadratic equation above needs to have two roots. So the discriminant is > $\mathbf{0}$.

Thus, we get $\boldsymbol{D} = \mathbf{5^2 - 4 \cdot 2}\boldsymbol{c} = \mathbf{25 - 8}\boldsymbol{c} > \mathbf{0} \Rightarrow \boldsymbol{c} < \frac{\mathbf{25}}{\mathbf{8}}$.

Then, we get two roots, which are $\boldsymbol{x} = \frac{5 \pm \sqrt{25-8c}}{4}$ by means of the quadratic formula.

Next, the $\boldsymbol{x}$-coordinate at the vertex is ≤ the smaller of the two roots above.

The $\boldsymbol{x}$-coordinate at the vertex is 0. So we get $\mathbf{0} \le \frac{5-\sqrt{25-8c}}{4}$. Thus, we get

$$\mathbf{0} \le \frac{5-\sqrt{25-8c}}{4} \Rightarrow \mathbf{0} \le \mathbf{5} - \sqrt{\mathbf{25}-\mathbf{8}\boldsymbol{c}} \Rightarrow \sqrt{\mathbf{25}-\mathbf{8}\boldsymbol{c}} \le \mathbf{5} \Rightarrow \mathbf{25}-\mathbf{8}\boldsymbol{c} \le \mathbf{25} \Rightarrow \boldsymbol{c} \ge \mathbf{0}.$$

What then, is the next?

This time, let's satisfy the statement as follows.

"At the vertex, the $\boldsymbol{y}$-coordinate ≥ the $\boldsymbol{x}$-coordinate."

Then, since the vertex is $(\mathbf{0}, \frac{c}{5})$, we get $\frac{c}{5} \ge \mathbf{0} \Rightarrow \boldsymbol{c} \ge \mathbf{0}$.

Now, putting threads together, we have $\boldsymbol{c} < \frac{25}{8}$, and also, $\boldsymbol{c} \ge \mathbf{0}$.

So we get $\mathbf{0} \le \boldsymbol{c} < \frac{25}{8}$.

Therefore, the domain is $\frac{5-\sqrt{25-8c}}{4} \le \boldsymbol{x} \le \frac{5+\sqrt{25-8c}}{4}$ where $\mathbf{0} \le \boldsymbol{c} < \frac{25}{8}$.

That is, we get $\boldsymbol{m} = \frac{5-\sqrt{25-8c}}{4}$, and $\boldsymbol{n} = \frac{5+\sqrt{25-8c}}{4}$ where $\mathbf{0} \le \boldsymbol{c} < \frac{25}{8}$.

Now, what if we have $\boldsymbol{m} \le \boldsymbol{y}$ for $\boldsymbol{m} \le \boldsymbol{x}$?

Then, the domain is not finite but infinite.

And if it is infinite as above, we need to consider several cases. By the way, an infinite domain can be open or closed. For instance,

($\mathbf{1} \le \boldsymbol{x} \le \mathbf{2}$) is finite and closed, and ($\mathbf{1} \le \boldsymbol{x}$) is infinite and closed.

($\mathbf{1} < \boldsymbol{x} < \mathbf{2}$), ($\mathbf{1} \le \boldsymbol{x} < \mathbf{2}$), or ($\mathbf{1} < \boldsymbol{x} \le \mathbf{2}$) is finite and open, and ($\mathbf{1} < \boldsymbol{x}$) is infinite and open.

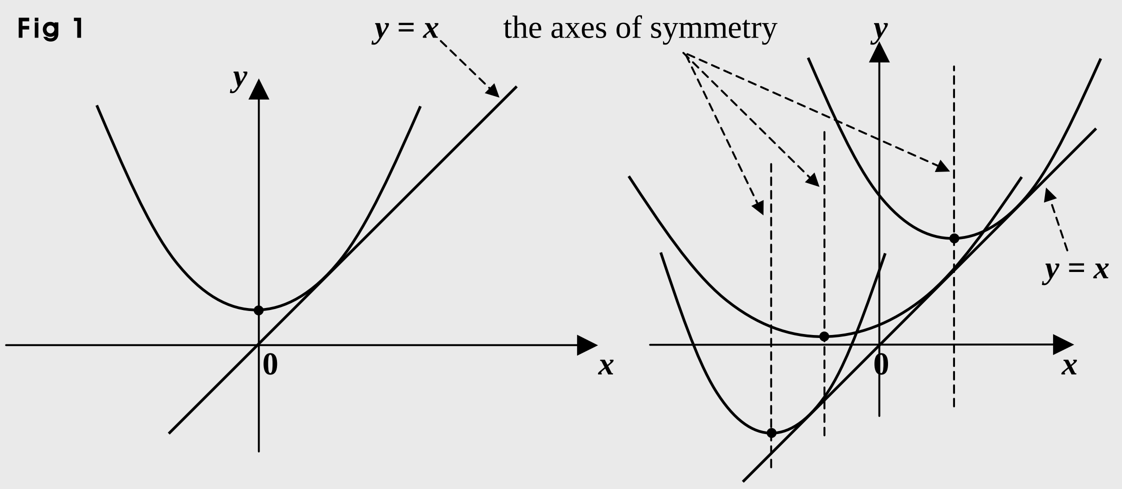
And of course, "$\boldsymbol{x}$ is real." means the entire number line, so it is infinite and open, too.

Now, if the domain is infinite as $\boldsymbol{x} > \boldsymbol{m}$, we need to consider cases as below.

To begin with, we may want to note that there are infinitely many parabolas, of course, which the line $\boldsymbol{y} = \boldsymbol{x}$ does not meet.

Now in one case, the parabola meets the line $\boldsymbol{y} = \boldsymbol{x}$ at one point only. That is, the line is tangent to the parabola.

Fig 1



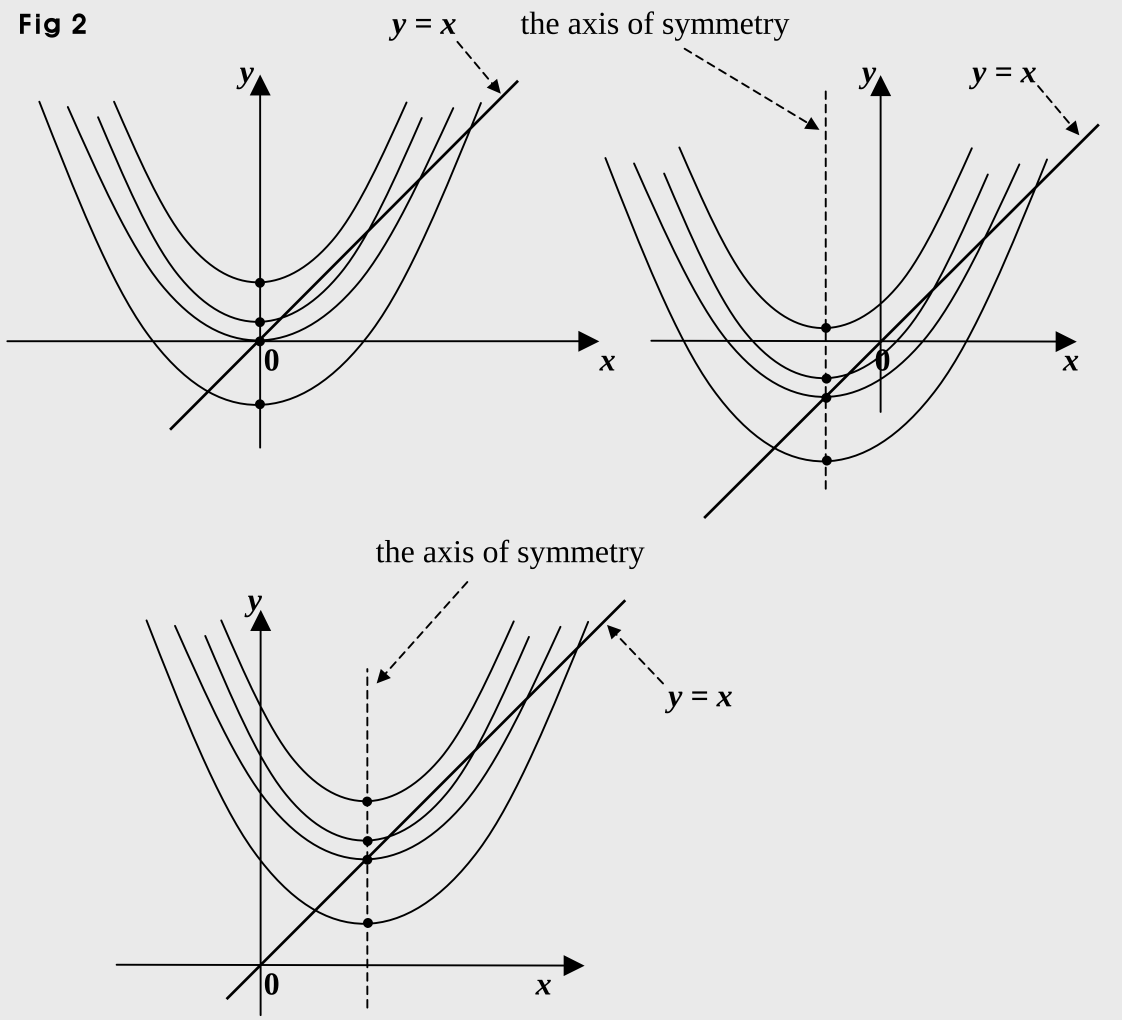
Then, all we have to do is to find the tangent point. So for instance, assuming the point is **(1, 1)**, we can see that the solution is simply $\boldsymbol{m} = \mathbf{1}$, that is, $\boldsymbol{y} > \mathbf{1}$ for $\boldsymbol{x} > \mathbf{1}$.

In another case, the parabola meets the line $\boldsymbol{y} = \boldsymbol{x}$ at two points.

What then, do we want to consider next?

We want to consider two cases, in one, the vertex is between the two points, and in the other, the vertex is between the two.

Fig 2



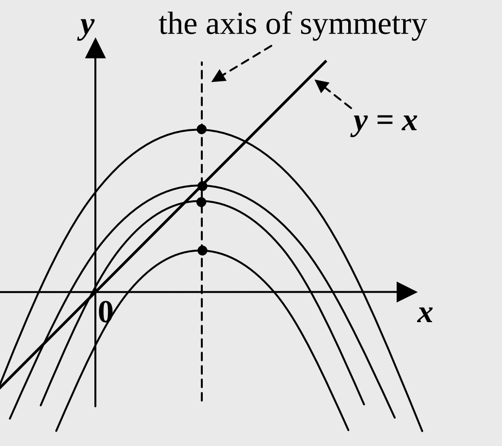
Then, examining the graphs above, we can notice that if the vertex is between the two points, we want to take for ***m*** the bigger of the ***x***-coordinates at the two points; otherwise, that is, if the vertex is not between the two points, we get two values for ***m***.

What are the two?

One is simply one of the two ***x***-coordinates at the two points, and the other is the other ***x***-coordinate.

And the similar idea applies to parabolas in other kinds, too.
That is to say that we can use the line ***y = x*** and the discriminant to find such an interval for a parabola concave down, right, or left.

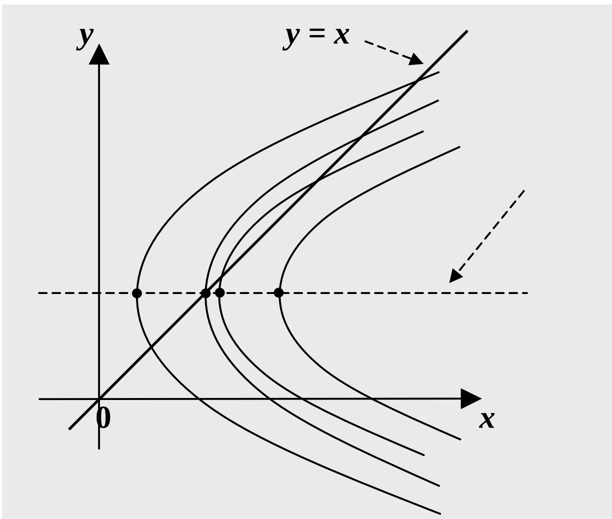
Fig. 3



If the parabola is concave down:
$y = ax^2 + bx + c$ for $a < 0$

The y-coordinate at the vertex is ≤ the smaller of the two roots.

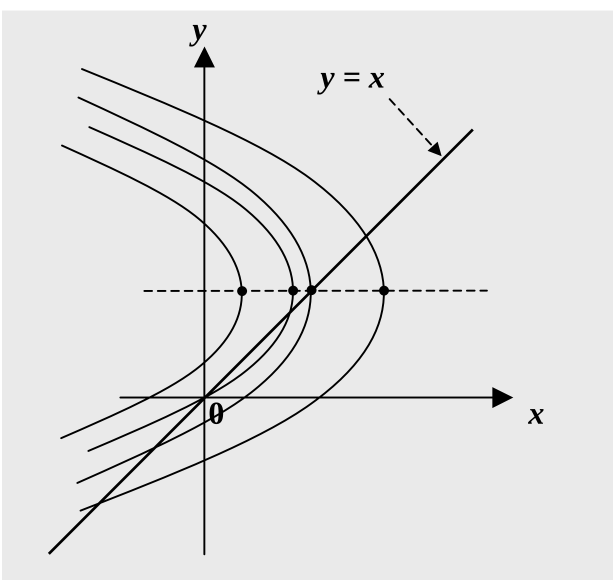
At the vertex, the y-coordinate ≤ the x-coordinate.



If the parabola is concave right:
$x = ay^2 + by + c$ for $a > 0$

The y-coordinate at the vertex is ≤ the smaller of the two roots.

At the vertex, the x-coordinate ≥ the y-coordinate.



If the parabola is concave right,
$x = ay^2 + by + c$ for $a < 0$

The y-coordinate at the vertex is ≥ the smaller of the two roots.

At the vertex, the x-coordinate ≤ the y-coordinate.