Examples 9 in Parabolas

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Examples 9

Note: this set of examples is for those students who want to study conics in depth.

Suppose $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are constant. Then,

0. Show that a parabola $\mathbf{2}\boldsymbol{x}^{\mathbf{2}} + (\mathbf{2}\boldsymbol{c} + \mathbf{1})\boldsymbol{x} - (\boldsymbol{c} + \mathbf{1})\boldsymbol{y} + \boldsymbol{c} + \mathbf{1} = \mathbf{0}$ includes two particular points for any value of $\boldsymbol{c} \neq \mathbf{-1}$, and find the two particular points.

1. Show that a parabola $(\boldsymbol{c} + \mathbf{2})\boldsymbol{x}^{\mathbf{2}} + (\mathbf{2}\boldsymbol{c} + \mathbf{1})\boldsymbol{x} - (\boldsymbol{c} + \mathbf{1})\boldsymbol{y} + \mathbf{3}\boldsymbol{c} + \mathbf{1} = \mathbf{0}$ passes through two particular points for any value of $\boldsymbol{c} \neq \mathbf{-2}$ or $\mathbf{-1}$, and find the two particular points.

2. Show that for any value of $\boldsymbol{c} \neq \mathbf{0}$ or $\mathbf{-1}$, a parabola $\boldsymbol{cx}^{\mathbf{2}} + \mathbf{2}\boldsymbol{cx} - (\boldsymbol{c} + \mathbf{1})\boldsymbol{y} + \mathbf{3}\boldsymbol{c} + \mathbf{3} = \mathbf{0}$ passes through two particular points, and find the two particular points.

3. Show that a parabola $\boldsymbol{cx}^{\mathbf{2}} + (\mathbf{2}\boldsymbol{c} + \mathbf{1})\boldsymbol{x} - \boldsymbol{cy} + \mathbf{3}\boldsymbol{c} + \mathbf{1} = \mathbf{0}$ passes through a particular point for any value of $\boldsymbol{c} \neq \mathbf{0}$, and find the particular point.

4. Assuming this time, a parabola $\boldsymbol{y} = \boldsymbol{cx}^{\mathbf{2}} - (\boldsymbol{a} - \mathbf{2}\boldsymbol{c})\boldsymbol{x} - \mathbf{2}\boldsymbol{b}(\boldsymbol{c} - \mathbf{1})$ passes through two points $(\mathbf{1}, \mathbf{2})$ and $(\boldsymbol{s}, \boldsymbol{t})$ for any value of $\boldsymbol{c} \neq \mathbf{0}$, find the point $(\boldsymbol{s}, \boldsymbol{t})$.

Suggestions or Solutions
To the Problem in the Example 0

Show that a parabola $2x^2 + (2c + 1)x - (c + 1)y + c + 1 = 0$ includes two particular points for any value of $c \neq -1$, and find the two particular points.

To begin with, we can put the equation given this way: $\mathbf{2x^2 + (2c + 1)x - (c + 1)y + c + 1 = 2cx - cy + c + 2x^2 + x - y + 1 = c(2x - y + 1) + 2x^2 + x - y + 1 = 0}$, which indicates a parabola for each value of $\mathbf{c \neq -1}$.

Next, suppose a line $\mathbf{2x - y + 1 = 0}$ meets a parabola $\mathbf{2x^2 + x - y + 1 = 0}$ at $\mathbf{(s, t)}$.

Then, we get $\mathbf{2s - t + 1 = 0}$, and $\mathbf{2s^2 + s - t + 1 = 0}$, so $\mathbf{c(2s - t + 1) + 2s^2 + s - t + 1 = 0}$.

So we can say that for any value of $\mathbf{c}$ if $\mathbf{c \neq -1, c(2x - y + 1) + 2x^2 + x - y + 1 = 0}$, that is, $\mathbf{2x^2 + (2c + 1)x - (c + 1)y + c + 1 = 0}$ indicates a parabola passing through the point $\mathbf{(s, t)}$, which is thus, the particular point.

And let's next, find the point $\mathbf{(s, t)}$.

To begin with, $\mathbf{2x - y + 1 = 0 \Rightarrow y = 2x + 1}$, and $\mathbf{2x^2 + x - y + 1 = 0 \Rightarrow y = 2x^2 + x + 1}$.

Thus next, $\mathbf{2x + 1 = 2x^2 + x + 1 \Rightarrow 2x^2 - x = 0 \Rightarrow x(2x - 1) = 0 \Rightarrow x = 0}$ or $\frac{1}{2}$.

So $\mathbf{x = 0 \Rightarrow y = 2x + 1 = 0 + 1 \Rightarrow y = 1}$, and $\mathbf{x = \frac{1}{2} \Rightarrow y = 2x + 1 = 1 + 1 = 2 \Rightarrow y = 2}$.

Thus, two points can be $\mathbf{(s, t)}$, and are $\mathbf{(0, 1)}$ and $\mathbf{(\frac{1}{2}, 2)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Suppose $\boldsymbol{A}$ is the parabola given. Then, since $\boldsymbol{A}$ is in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, $\boldsymbol{x}$ and $\boldsymbol{y}$ are variables, and $\boldsymbol{c}$ is constant, of course.

A parabola gets determined as the value of the constant $\boldsymbol{c}$ gets determined.
So infinitely many parabolas can be represented by $\boldsymbol{A}$ since $\boldsymbol{c}$ can be any real number.

In nature, this example is the same as some of the examples in **CONICS 1**, which is a book on equations for lines, and covers a fact that if two lines meet at a particular point, a linear combination of the two passes through the particular point, too.

A linear combination of two objects means a sum of any multiples of the two.

For instance, if $\boldsymbol{Z = mX + nY}$ where $\boldsymbol{m}$ and $\boldsymbol{n}$ are constants, $\boldsymbol{Z}$ can be called a linear combination of $\boldsymbol{X}$ and $\boldsymbol{Y}$. So we can have $\boldsymbol{Z = X}$, $\boldsymbol{3Y}$, $\boldsymbol{X + 2Y}$, $\boldsymbol{X - 0.3Y}$, $\boldsymbol{-0.1X + Y}$, etc.

• Suppose now, $\boldsymbol{m}$ and $\boldsymbol{n}$ are constant, $\boldsymbol{m + n \neq 0}$, and the curves of equations $\boldsymbol{f(x, y) = 0}$ and $\boldsymbol{g(x, y) = 0}$ meet each other at particular points, so both share the particular points.

Then, the curve of $\boldsymbol{mf(x, y) + ng(x, y) = 0}$ meets the curves of $\boldsymbol{f}$ and $\boldsymbol{g}$ at the particular points. In other words, the curve of $\boldsymbol{mf(x, y) + ng(x, y) = 0}$, too, includes the particular points, so all the three curves share the particular points.

What do we mean by though, $\boldsymbol{f(x, y) = 0}$?

The equation $\boldsymbol{f}$ is made of two variables $\boldsymbol{x}$ and $\boldsymbol{y}$.
For instance, it can be $\boldsymbol{f(x, y) = x + 2y = 0}$ and $\boldsymbol{f(x, y) = 3x^2 - y + 2y^3 - xy + 4 = 0}$. It can be $\boldsymbol{f(x, y) = 2x + 5 = 0}$, too, because we can put it this way: $\boldsymbol{2x + 0 \cdot y + 5 = 0}$, which is an equation in terms of $\boldsymbol{x}$ and $\boldsymbol{y}$.

Now, the same idea applies to parabolas, too. The same is true, too, for lines, circles, etc. So if two parabolas meet each other at particular points, all the linear combinations of the two meet each other at the particular points, too. How though?

Suppose now, $\boldsymbol{A}$ and $\boldsymbol{B}$ are two parabolas, and meet each other at particular points, and their equations are as follows.

$\boldsymbol{A}$ is $\boldsymbol{x^2 + ax + by + c = 0}$, and $\boldsymbol{B}$ is $\boldsymbol{x^2 + dx + ey + f = 0}$.

Then, we can see that if $\boldsymbol{m + n \neq 0}$, $\boldsymbol{m(x^2 + ax + by + c) + n(x^2 + dx + ey + f) = 0}$ can be a parabola, too. Why?

Rearranging first, the equation above, we can get

$\boldsymbol{m(x^2 + ax + by + c) + n(x^2 + dx + ey + f)}$
$\boldsymbol{= (m + n)x^2 + (ma + nd)x + (mb + ne)y + mc + nf = 0}$, which is a parabola, since all the letters other than $\boldsymbol{x}$ and $\boldsymbol{y}$ are constant, and in turn, their products and sums are constant.

So suppose now, that $\boldsymbol{m + n \neq 0}$, and that

the equation $\boldsymbol{m(x^2 + ax + by + c) + n(x^2 + dx + ey + f) = 0}$ indicates a parabola $\boldsymbol{P}$,

and the parabolas $\boldsymbol{A}$ and $\boldsymbol{B}$ meet each other at two particular points $\boldsymbol{(s, t)}$ and $\boldsymbol{(u, v)}$.

Then, the parabola $\boldsymbol{P}$ passes through the two particular points, too.   Why?

First, we get $\boldsymbol{s^2 + as + bt + c = 0}$ and $\boldsymbol{s^2 + ds + et + f = 0}$, because the two parabolas $\boldsymbol{A}$ and $\boldsymbol{B}$ share the point $\boldsymbol{(s, t)}$.

So we get $\boldsymbol{m(s^2 + as + bt + c) + n(s^2 + ds + et + f) = 0}$.

Thus, the parabola $\boldsymbol{P}$ passes through the point $\boldsymbol{(s, t)}$, too.

So the three parabolas $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{P}$ share the point $\boldsymbol{(s, t)}$.

Next, we get $\boldsymbol{u^2 + au + bv + c = 0}$ and $\boldsymbol{u^2 + du + ev + f = 0}$, too, since $\boldsymbol{A}$ and $\boldsymbol{B}$ share the point $\boldsymbol{(u, v)}$, also.

So we get $\boldsymbol{m(u^2 + au + bv + c) + n(u^2 + du + ev + f) = 0}$.

Thus, the parabola $\boldsymbol{P}$ passes through the point $\boldsymbol{(u, v)}$, too. So $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{P}$ share the point.

Thus, the curve of $\boldsymbol{m(x^2 + ax + by + c) + n(x^2 + dx + ey + f) = 0}$, that is, the parabola $\boldsymbol{P}$ passes through the two points $\boldsymbol{(s, t)}$ and $\boldsymbol{(u, v)}$.

So the parabola $\boldsymbol{P}$ and the two parabolas $\boldsymbol{x^2 + ax + by + c = 0}$ and $\boldsymbol{x^2 + dx + ey + f = 0}$ meet each other at the two points $\boldsymbol{(s, t)}$ and $\boldsymbol{(u, v)}$. In other words, the three parabolas $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{P}$ share the two points $\boldsymbol{(s, t)}$ and $\boldsymbol{(u, v)}$.

What if however, $\boldsymbol{m + n = 0}$?

Then, $m(x^2 + ax + by + c) + n(x^2 + dx + ey + f) = 0$ is an equation of a line passing through the two points (s, t) and (u, v). Why?

First, we can put the equation above the way below.

$m(x^2 + ax + by + c) + n(x^2 + dx + ey + f)$

$= (m + n)x^2 + (am + dn)x + (bm + en)y + cm + fn$

$= (am + dn)x + (bm + en)y + cm + fn = 0$, since $m + n = 0$.

We know $am + dn$, $bm + en$, and $cm + fn$ are constant since a, b, c, d, e, f, m, and n are constant. So $(am + dn)x + (bm + en)y + cm + fn = 0$ is an equation of a line.

Next, since the two parabolas $x^2 + ax + by + c = 0$ and $x^2 + dx + ey + f = 0$ share the point (s, t), we get $s^2 + as + bt + c = 0$ and $s^2 + ds + et + f = 0$.

So we get $m(s^2 + as + bt + c) + n(s^2 + ds + et + f) = 0$.

Thus, since $m + n = 0$, we get

$m(s^2 + as + bt + c) + n(s^2 + ds + et + f)$

$= (m + n)s^2 + (am + dn)s + (bm + en)t + cm + fn$

$= (am + dn)s + (bm + en)t + cm + fn =0$.

Also, since the two parabolas $x^2 + ax + by + c = 0$ and $x^2 + dx + ey + f = 0$ share the point (u, v), we get $u^2 + au + bv + c = 0$ and $u^2 + du + ev + f = 0$.

So we get $m(u^2 + au + bv + c) + n(u^2 + du + ev + f) = 0$.

Thus, since $m + n = 0$, we get

$m(u^2 + au + bv + c) + n(u^2 + du + ev + f)$

$= (m + n)u^2 + (am + dn)u + (bm + en)v + cm + fn$

$= (am + dn)u + (bm + en)v + cm + fn = 0$.

Thus, if $m + n = 0$, $m(x^2 + ax + by + c) + n(x^2 + dx + ey + f) = 0$ is an equation of a line, which is $(am + dn)x + (bm + en)y + cm + fn = 0$, and the line includes (s, t) and (u, v), at which the two parabolas $x^2 + ax + by + c = 0$ and $x^2 + dx + ey + f = 0$ meet each other.

So the line and the two parabolas share the two points (s, t) and (u, v).

• In addition, the same is true for a parabola and a line, too. That is, if a parabola meets a line at particular points, all the linear combinations of the parabola and line meet each other at the particular points, too.

Suppose for instance,

$A(x, y) = a_1x^2 + b_1x + c_1 + d_1y = 0$ where a_1, b_1, c_1, and d_1 are constant.

$B(x, y) = a_2x + b_2y + c_2 = 0$ where a_2, b_2, and c_2 are constant.

$C(x, y) = m(a_1x^2 + b_1x + c_1 + d_1y) + n(a_2x + b_2y + c_2) = 0$ where m and n are constant.

Suppose also, that the parabola A meets the line B at particular points.

Then, the parabola C, too, passes through the particular points, and thus, meets both of A and B at the particular points.

Suppose for another instance,

$A(x, y) = a_1x^2 + b_1x + c_1 + d_1y = 0$.

$B(x, y) = a_2x^2 + b_2x + c_2 + d_2y = 0$.

$C(x, y) = m(a_1x^2 + b_1x + c_1 + d_1y) + n(a_2x^2 + b_2x + c_2 + d_2y) = 0$.

Suppose also, that the parabolas A and B meet each other at particular points.

Then, all the three parabolas, A, B, and C share the particular points.

In fact, the same idea applies to curves in other kinds, also.

Suppose for instance, m and n are constants, and a curve $f(x, y) = 0$ meets another curve $g(x, y) = 0$ at particular points.

Then, $\boldsymbol{mf(x, y) + ng(x, y) = 0}$ represents a group of curves, each of which passes through the particular points.

That is, if a curve $\boldsymbol{f(x, y) = 0}$ meets a curve $\boldsymbol{g(x, y) = 0}$ at particular points, all the curves represented by $\boldsymbol{mf(x, y) + ng(x, y) = 0}$ meet each other altogether at the particular points.

Let's now, start doing the problem in this example. The problem goes as follows.

Show that a parabola $2x^2 + (2c + 1)x - (c + 1)y + c + 1 = 0$ includes two particular points for any value of c if $c \neq -1$, and find the two particular points.

To begin with, we can put the parabola given this way: $\boldsymbol{2x^2 + (2c + 1)x - (c + 1)y + c + 1 = 2cx - cy + c + 2x^2 + x - y + 1 = c(2x - y + 1) + 2x^2 + x - y + 1 = 0}$, which is a linear combination of a line $\boldsymbol{2x - y + 1 = 0}$ and a parabola $\boldsymbol{2x^2 + x - y + 1 = 0}$ for $\boldsymbol{c \neq -1}$.

Next, suppose the line $\boldsymbol{2x - y + 1 = 0}$ meets the parabola $\boldsymbol{2x^2 + x - y + 1 = 0}$ at $\boldsymbol{(s, t)}$.

Then, we get $\boldsymbol{2s - t + 1 = 0}$, and $\boldsymbol{2s^2 + s - t + 1 = 0}$, so $\boldsymbol{c(2s - t + 1) + 2s^2 + s - t + 1 = 0}$.

It means that the parabola given passes through the point $\boldsymbol{(s, t)}$, too.

So we can say that for any value of $\boldsymbol{c}$ if $\boldsymbol{c \neq -1}$, $\boldsymbol{c(2x - y + 1) + 2x^2 + x - y + 1 = 0}$, that is, $\boldsymbol{2x^2 + (2c + 1)x - (c + 1)y + c + 1 = 0}$ indicates a parabola passing through the point $\boldsymbol{(s, t)}$, which is thus, the particular point. And finding $\boldsymbol{(s, t)}$, we can get it the way below.

To begin with, $\boldsymbol{2x - y + 1 = 0 \Rightarrow y = 2x + 1}$, and $\boldsymbol{2x^2 + x - y + 1 = 0 \Rightarrow y = 2x^2 + x + 1}$.

Thus next, $\boldsymbol{2x + 1 = 2x^2 + x + 1 \Rightarrow 2x^2 - x = 0 \Rightarrow x(2x - 1) = 0 \Rightarrow x = 0}$ or $\boldsymbol{\frac{1}{2}}$.

So $\boldsymbol{x = 0 \Rightarrow y = 2x + 1 = 0 + 1 \Rightarrow y = 1}$, and $\boldsymbol{x = \frac{1}{2} \Rightarrow y = 2x + 1 = 1 + 1 = 2 \Rightarrow y = 2}$.

Thus, two points can be $\boldsymbol{(s, t)}$, and are $\boldsymbol{(0, 1)}$ and $\boldsymbol{(\frac{1}{2}, 2)}$.

So all the parabolas represented by $\boldsymbol{2x^2 + (2c + 1)x - (c + 1)y + c + 1 = 0}$ meet altogether at $\boldsymbol{(0, 1)}$ and $\boldsymbol{(\frac{1}{2}, 2)}$.

And if finding the particular points only, we can get the points the way below, too.

Just solving a system of two equations we can get putting into c any two values other than -1, in the equation $2x^2 + (2c + 1)x - (c + 1)y + c + 1 = 0$, we can find the two particular points, too. That is because for any $c \neq -1$, the equation above indicates a parabola passing through the two particular points.

So for instance, putting 0 into c, we get a parabola $2x^2 + x - y + 1 = 0$, and putting 1 into c, we get another parabola $2x^2 + 3x - 2y + 2 = 0$.

Then, solving the system of $2x^2 + x - y + 1 = 0$ and $2x^2 + 3x - 2y + 2 = 0$, we can get the two points.

So solving the system, we can get first,

$2x^2 + x - y + 1 = 0 \Rightarrow y = 2x^2 + x + 1$, and $2x^2 + 3x - 2y + 2 = 0 \Rightarrow y = x^2 + 3x/2 + 1$.

So next, we can get

$2x^2 + x + 1 = x^2 + 3x/2 + 1 \Rightarrow x^2 - x/2 = 0 \Rightarrow x(x - 1/2) = 0 \Rightarrow x = 0$ or $1/2$.

Thus next, we get

$x = 0 \Rightarrow 2x^2 + x - y + 1 = -y + 1 = 0 \Rightarrow y = 1$.

$x = 1/2 \Rightarrow 2x^2 + x - y + 1 = 1/2 + 1/2 - y + 1 = 0 \Rightarrow y = 2$.

Suggestions or Solutions
To the Problem in the Example 1

Show that a parabola $(c + 2)x^2 + (2c + 1)x - (c + 1)y + 3c + 1 = 0$ includes two particular points for any value of $c \neq -2$ or -1, and find the two particular points.

To begin with, we can put the parabola given the way below.

$(c + 2)x^2 + (2c + 1)x - (c + 1)y + 3c + 1 = cx^2 + 2cx - cy + 3c + 2x^2 + x - y + 1$
$= c(x^2 + 2x - y + 3) + 2x^2 + x - y + 1 = 0$, which is a linear combination of two parabolas $x^2 + 2x - y + 3 = 0$ and $2x^2 + x - y + 1 = 0$ for any $c \neq -2$ or -1.

Next, suppose two parabolas $x^2 + 2x - y + 3 = 0$ and $2x^2 + x - y + 1 = 0$ meet at (s, t).

Then, we get $s^2 + 2s - t + 3 = 0$, and $2s^2 + s - t + 1 = 0$, so we get
$c(s^2 + 2s - t + 3) + 2s^2 + s - t + 1 = 0$, which means that the parabola given passes through the point (s, t), too.

So if the two parabolas above meet at a particular point, every parabola indicated by the equation given includes the particular point, too.
And the same is true for another particular point, too, if the two curves share the point.

So for any $c \neq -2$ or -1, $(c + 2)x^2 + (2c + 1)x - (c + 1)y + 3c + 1 = 0$ indicates a parabola passing through the points stated above. And we can get the points the way below.

Finding first, the x-coordinates at the points, we get first,

$x^2 + 2x - y + 3 = 0 \Rightarrow y = x^2 + 2x + 3$, and $2x^2 + x - y + 1 = 0 \Rightarrow y = 2x^2 + x + 1$.

So next, $x^2 + 2x + 3 = 2x^2 + x + 1 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x + 1)(x - 2) = 0 \Rightarrow x = -1$ or 2.

Thus next, finding the y-coordinates, we get

$x = -1 \Rightarrow y = x^2 + 2x + 3 = 1 - 2 + 3 = 2$.

$x = 2 \Rightarrow y = x^2 + 2x + 3 = 4 + 4 + 3 = 11$.

Therefore, the points are $(-1, 2)$ and $(2, 11)$.

And if finding the particular points only, we can get the points the way below, too.

Just solving a system of two equations we can get putting into $\boldsymbol{c}$ any two values other than -2 and -1, in the equation $\mathbf{(c + 2)x^2 + (2c + 1)x - (c + 1)y + 3c + 1 = 0}$, we can find the two particular points, too. That is because for any $\boldsymbol{c \neq -2}$ or $\mathbf{-1}$, the equation above indicates a parabola passing through the two particular points.

So for instance, putting 0 into $\boldsymbol{c}$, we get a parabola $\mathbf{2x^2 + x - y + 1 = 0}$, and putting 1 into $\boldsymbol{c}$, we get another parabola $\mathbf{3x^2 + 3x - 2y + 4 = 0}$.

Then, solving the system of $\mathbf{2x^2 + x - y + 1 = 0}$ and $\mathbf{3x^2 + 3x - 2y + 4 = 0}$, we can get the two points.

So solving the system, we can get first,

$\mathbf{2x^2 + x - y + 1 = 0 \Rightarrow y = 2x^2 + x + 1}$, and $\mathbf{3x^2 + 3x - 2y + 4 = 0 \Rightarrow y = 3x^2/2 + 3x/2 + 2}$.

So next, we can get

$\mathbf{2x^2 + x + 1 = 3x^2/2 + 3x/2 + 2 \Rightarrow x^2/2 - x/2 - 1 = 0 \Rightarrow x^2 - x - 2 = 0}$
$\mathbf{\Rightarrow (x + 1)(x - 2) = 0 \Rightarrow x = -1}$ or $\mathbf{2}$.

Thus next, we get

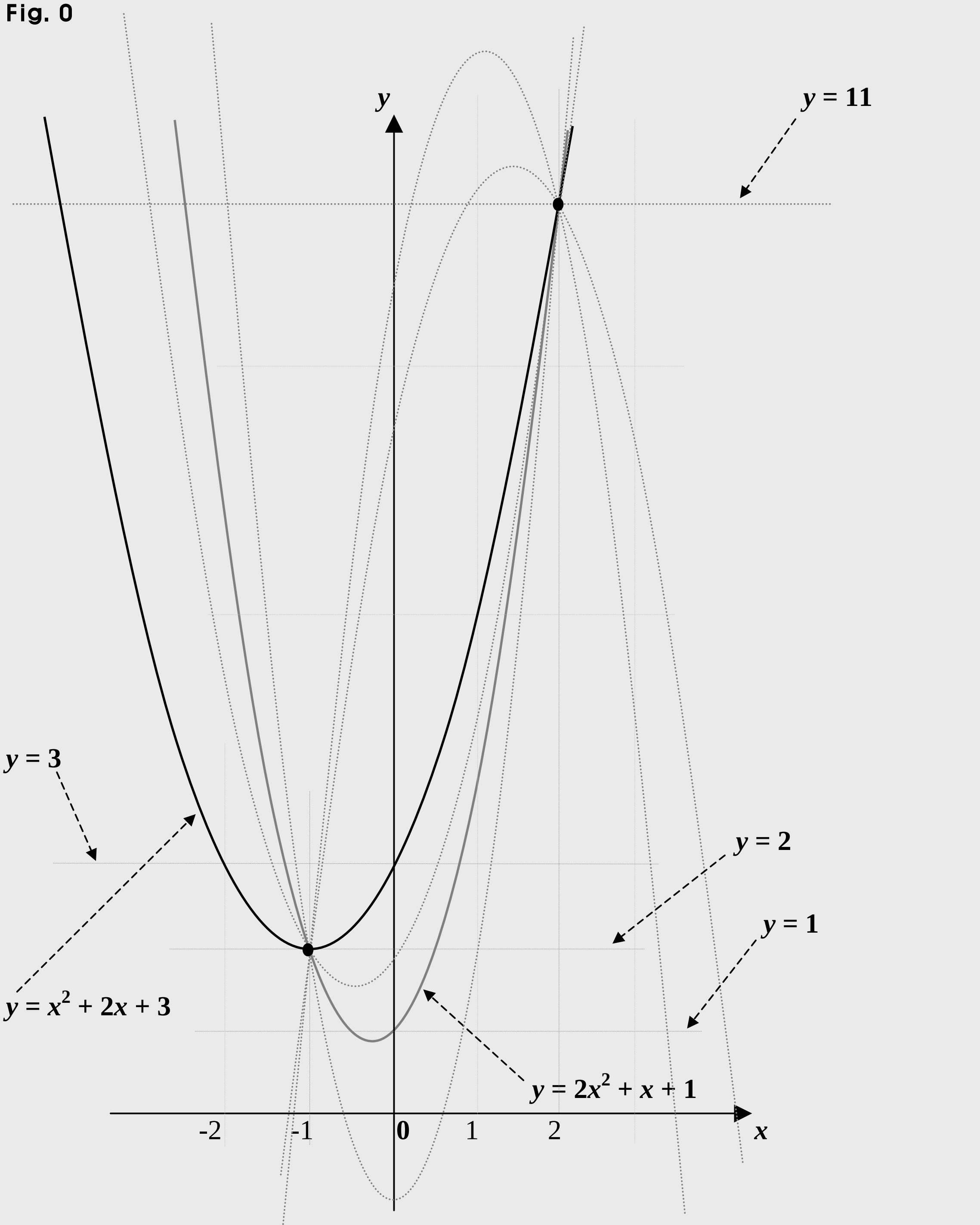
$\mathbf{x = -1 \Rightarrow 2x^2 + x - y + 1 = 2 - 1 - y + 1 = 0 \Rightarrow y = 2}$.

$\mathbf{x = 2 \Rightarrow 2x^2 + x - y + 1 = 8 + 2 - y + 1 = 0 \Rightarrow y = 11}$.

So every parabola indicated by the equation $\mathbf{(c + 2)x^2 + (2c + 1)x - (c + 1)y + 3c + 1 = 0}$ passes through the two points (-1, 2) and (2, 11).

And putting in one graph, some of those parabolas, we can put them the way below.

Fig. 0



Except the parabola in black, all the parabolas in gray are some of the parabolas that can be represented by the equation $(c + 2)x^2 + (2c + 1)x - (c + 1)y + 3c + 1 = 0$ if $c \neq -2$ or -1.

Suggestions or Solutions
To the Problem in the Example 2

Show that a parabola $cx^2 + 2cx - (c + 1)y + 3c + 3 = 0$ includes two particular points for any value of $c \neq 0$ or -1, and find the two particular points.

To begin with, we can put the parabola given the way below.

$cx^2 + 2cx - (c + 1)y + 3c + 3 = cx^2 + 2cx - cy + 3c - y + 3 = c(x^2 + 2x - y + 3) - y + 3$
$= c(x^2 + 2x - y + 3) - 1(y - 3) = 0$, which is a linear combination of a line $y - 3 = 0$ and a parabola $x^2 + 2x - y + 3 = 0$ for any $c \neq 0$ or -1.

And of course, we can put the line this way, too: $y = 3$.

Next, suppose the parabola $x^2 + 2x - y + 3 = 0$ and the line $y - 3 = 0$ meet at (s, t).

Then, we get $s^2 + 2s - t + 3 = 0$, and $t - 3 = 0$, so we get $c(s^2 + 2s - t + 3) + t - 3 = 0$, which means that the parabola given passes through the point (s, t), too.

So if the parabola above meets the line at a particular point, every parabola indicated by the equation given includes the particular point, too.
And the same is true for another particular point, too, if the two curves share the point.

So for any $c \neq 0$ or -1, $cx^2 + 2cx - (c + 1)y + 3c + 3 = 0$ indicates a parabola passing through the points stated above. And we can get the points the way below.

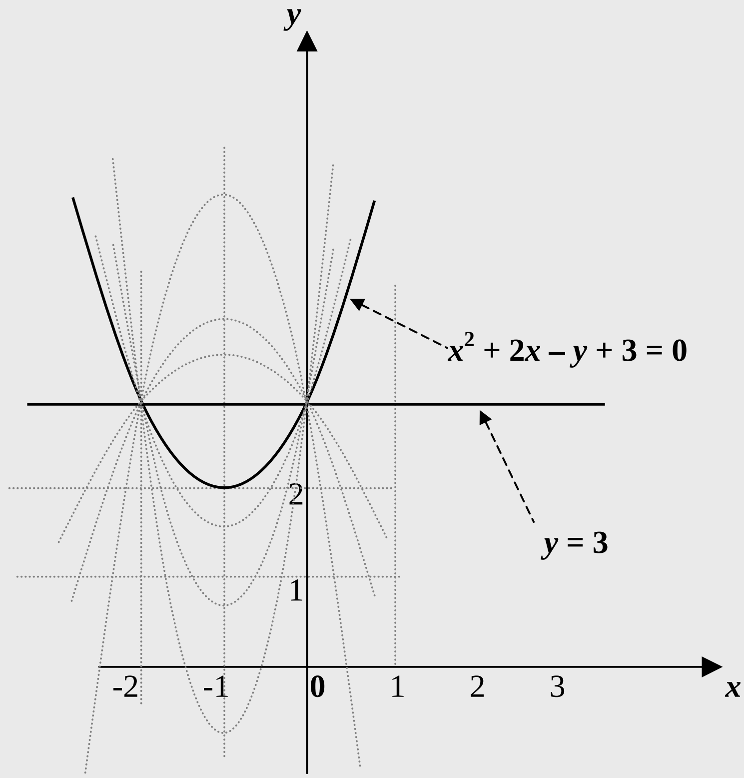
Since the parabola meets the line $y = 3$, the y-coordinates at the points are 3.
So we have only to get the x-coordinates at the points.
$x^2 + 2x + 3 = 3 \Rightarrow x^2 + 2x = 0 \Rightarrow x(x + 2) = 0 \Rightarrow x = 0$ or -2.

So the parabola and the line meet at **(0, 3)** and **(-2, 3)**.

Therefore, **(0, 3)** and **(-2, 3)** are the two particular points.

And putting in one graph, some of those parabolas, we can put them the way below.

Fig. 0



So if $c \neq 0$ or -1, the equation $cx^2 + 2cx - (c + 1)y + 3c + 3 = 0$ indicates all the parabolas dotted in gray above.

So note that the parabola in black above cannot be indicated by the equation given.

That is because for no value of c, we can get the parabola $x^2 + 2x - y + 3 = 0$.

Suggestions or Solutions
To the Problem in the Example 3

Show that a parabola $cx^2 + (2c + 1)x - cy + 3c + 1 = 0$ includes a particular point for any value of $c \neq 0$, and find the particular point.

To begin with, we can put the parabola given the way below.

$cx^2 + (2c + 1)x - cy + 3c + 1 = cx^2 + 2cx + x - cy + 3c + 1$
$= c(x^2 + 2x - y + 3) + x + 1 = 0$, which is a linear combination of a line $x + 1 = 0$ and a parabola $x^2 + 2x - y + 3 = 0$ for any $c \neq 0$.

And of course, we can put the line this way, too: $x = -1$.

Next, suppose the parabola $x^2 + 2x - y + 3 = 0$ and the line $x + 1 = 0$ meet at (s, t).

Then, we get $s^2 + 2s - t + 3 = 0$, and $s + 1 = 0$, so we get $c(s^2 + 2s - t + 3) + s + 1 = 0$, which means that the parabola given passes through the point (s, t), too.

So if the parabola above meets the line at a particular point, every parabola indicated by the equation given includes the particular point, too.

And the same is true for another particular point, too, if the two curves share the point.

So for any $c \neq 0$, $cx^2 + (2c + 1)x - cy + 3c + 1 = 0$ indicates a parabola passing through the points stated above. And we can get the points the way below.

First, the parabola $x^2 + 2x - y + 3 = 0$ meets the line $x = -1$, -1 is the x-coordinate at point where the parabola meets the line.

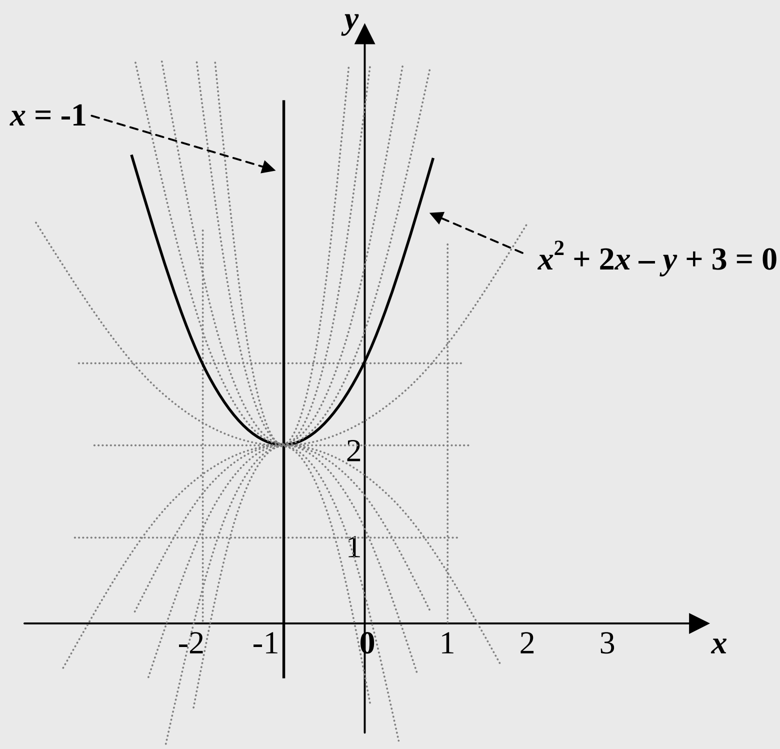
So next, getting the y-coordinate, we get

$x = -1 \Rightarrow y = x^2 + 2x + 3 = 1 - 2 + 3 = 2 \Rightarrow y = 2$.

So there is one particular point, which is $(-1, 2)$.

And putting in one graph, some of those parabolas, we can put them the way below.

Fig. 0



The equation given can indicate each parabola dotted in gray.
Note however, it cannot indicate the parabola in black.
That's because for no value of $\boldsymbol{c}$, we can get the equation $\boldsymbol{x^2 + 2x - y + 3 = 0}$.

Suggestions or Solutions
To the Problem in the Example 4

Assuming this time, a parabola $y = cx^2 - (a - 2c)x - 2b(c - 1)$ passes through two points (1, 2) and (s, t) for any value of $c \neq 0$, find the point (s, t).

To begin with, we can put the parabola given the way below.

$y = cx^2 - (a - 2c)x - 2b(c - 1) \Rightarrow cx^2 + 2cx - 2cb - ax + 2b - y = 0$
$\Rightarrow c(x^2 + 2x - 2b) - (ax - 2b + y) = 0$, which is a linear combination of a curve of an equation $x^2 + 2x - 2b = 0$ and a line $ax - 2b + y = 0$ for any $c \neq 0$.

Suppose now, A is the parabola $y = cx^2 - (a - 2c)x - 2b(c - 1)$.
Then, we know A passes through the two points **(1, 2)** and (s, t) for any value of c.
So the curve of $x^2 + 2x - 2b = 0$ includes **(1, 2)**, too, where the x-coordinate is 1.

Thus, we get $x = 1 \Rightarrow x^2 + 2x - 2b = 1 + 2 - 2b = 0 \Rightarrow b = \frac{3}{2}$.
So we get $x^2 + 2x - 2b = 0 \Rightarrow x^2 + 2x - 3 = 0$, and $ax - 2b + y = 0 \Rightarrow ax - 3 + y = 0$.

Therefore, $x^2 + 2x - 2b = 0 \Rightarrow x^2 + 2x - 3 = 0 \Rightarrow (x - 1)(x + 3) = 0 \Rightarrow x = 1$ or $x = -3$.
So the curve of $x^2 + 2x - 2b = 0$ is a pair of parallel lines $x = 1$ and $x = -3$.

Next, the curve above has to include the points **(1, 2)** and (s, t).
So the line $x = 1$ includes **(1, 2)**, and the other line $x = -3$ includes (s, t).
Thus, we get $s = -3$.   So we get $(s, t) = (-3, t)$.

Next, the line $ax - 3 + y = 0$ includes **(1, 2)**, too, and thus, we get
$(1, 2) \Rightarrow ax - 3 + y = a - 3 + 2 = 0 \Rightarrow a = 1$. Therefore, the line is $x - 3 + y = 0$.

And also, the line $x - 3 + y = 0$ includes the point (s, t), too, which is, $(-3, t)$.
So we get $-3 - 3 + t = 0 \Rightarrow t = 6$.   Therefore, $(s, t) = (-3, 6)$.

If not quite sure of the idea behind the processes above, follow the steps below.

Putting curves together, we get another curve.

More specifically, adding together multiples of curves, we can get another curve.
Then, the other curve is called a linear combination of curves.

So the number of curves put together doesn't have to be two.
So we can put together as many curves as we want to get another curve. That is, a curve can be a linear combination of more than two curves.

Suppose now, $\boldsymbol{A}$ is the parabola given.
Then this time, unlike the previous example, $\boldsymbol{A}$ looks like a linear combination of more than two curves. Anyway, if $\boldsymbol{A}$ is a linear combination of curves, the curves meet at the two points $\mathbf{(1, 2)}$ and $(\boldsymbol{s}, \boldsymbol{t})$, so we may want to begin with checking to see if $\boldsymbol{A}$ is such a combination, and see what curves $\boldsymbol{A}$ is a linear combination of.

To begin with, we can put the parabola given the way below.

$\boldsymbol{y = cx^2 - (a - 2c)x - 2b(c - 1) \Rightarrow cx^2 + 2cx - 2cb - ax + 2b - y = 0}$

$\boldsymbol{\Rightarrow c(x^2 + 2x - 2b) - (ax - 2b + y) = c(x^2 + 2x - 2b) + (-1)(ax - 2b + y) = 0}$, which is a linear combination of a curve of $\boldsymbol{x^2 + 2x - 2b = 0}$ and a line $\boldsymbol{ax - 2b + y = 0}$ for any $\boldsymbol{c \neq 0}$.

So $\boldsymbol{A}$ is not a linear combination of a parabola and a line.   Why not?

The curve of $\boldsymbol{ax - 2b + y = 0}$ is a line, of course.
However, the curve of $\boldsymbol{x^2 + 2x - 2b = 0}$ is not a parabola.   What curve then, is it?

The equation $\boldsymbol{x^2 + 2x - 2b = 0}$ indicates two lines parallel to the $\boldsymbol{y}$-axis, so the curve is not a parabola but a pair of lines parallel to the $\boldsymbol{y}$-axis.
That is, the curve is made of two lines perpendicular to the $\boldsymbol{x}$-axis.   Why?

Suppose $\boldsymbol{B}$ is $\boldsymbol{x^2 + 2x - 2b = 0}$, and $\boldsymbol{C}$ is $\boldsymbol{ax - 2b + y = 0}$.

Then, $\boldsymbol{A}$ is a linear combination of $\boldsymbol{B}$ and $\boldsymbol{C}$. So $\boldsymbol{B}$ and $\boldsymbol{C}$ have to meet at $\mathbf{(1, 2)}$ and $(\boldsymbol{s}, \boldsymbol{t})$, because $\boldsymbol{A}$ includes the two points $\mathbf{(1, 2)}$ and $(\boldsymbol{s}, \boldsymbol{t})$ for any value of $\boldsymbol{c \neq 0}$.

So whatever the curve of $\boldsymbol{B}$ may be, it has to include the two points.

Thus, the curve of $\boldsymbol{B}$ has to pass through $\mathbf{(1, 2)}$ anyway.

At the point **(1, 2)**, the x-coordinate is 1, so the equation of B has to hold when $x = 1$.

That is, $x^2 + 2x - 2b = 0$ has to hold when $x = 1$.

So we get $x^2 + 2x - 2b = 1 + 2 - 2b = 0 \Rightarrow b = \frac{3}{2}$.

Thus, we can now see that the equation of B is now, $x^2 + 2x - 3 = 0$.

Why is the curve of B though, a pair of two lines parallel to the y-axis?

We can put it this way: $x^2 + 2x - 3 = 0 \Rightarrow (x - 1)(x + 3) = 0 \Rightarrow x = 1$ or $x = -3$.

And we know $x = 1$ is a line parallel to the y-axis, and so is $x = -3$.

So in this case, $x^2 + 2x - 3 = 0$ is not a quadratic equation for an unknown x but an equation of a curve.

And it is in fact, a part of an equation indicating another curve, which is the parabola A.

Those of you have read **CONICS 1**, which is about lines, should be able to notice that $(x - 1)(x + 3) = 0$ can indicate two lines, so it is composed of two equations, one is $x - 1 = 0$, which is $x = 1$, and the other is $x + 3 = 0$, which is $x = -3$.

So we can see that $x - 1 = 0$ indicates a line $x = 1$, and that $x + 3 = 0$ indicates another line $x = -3$. Both lines are perpendicular to the x-axis, and thus, are parallel to each other.

So two lines, $x = 1$ and $x = -3$ make the curve of B, which is $x^2 + 2x - 3 = 0$.

Let's now get the point (s, t).

To begin with, we know B and C have to meet at **(1, 2)** and (s, t), because A is a linear combination of B and C, and includes **(1, 2)** and (s, t) for any value of $c \neq 0$.

So whatever the curve of B may be, it has to include the two points .
Thus, the curve of B has to pass through **(1, 2)** and (s, t) anyway.

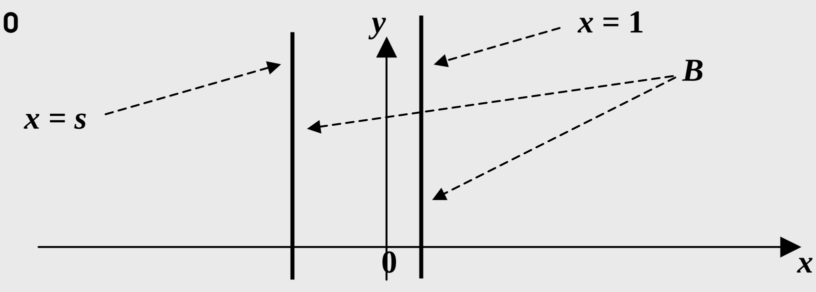
The curve of B is composed of two lines $x = 1$ and $x = -3$.

The line $x = 1$ clearly passes through the point **(1, 2)**.

What point then, does the line $x = -3$ have to pass through?

It is the other point (s, t), of course.
So we can see that $s = -3$, which is the x-coordinate at (s, t).

Fig. 0



So as shown in the graph above, B is composed of two lines parallel to the y-axis.
One of the two is the line $x = 1$, and the other is the line $x = -3$.

Although the two lines are not connected, B is still a curve, which is in two parts.

One part of B is the line $x = -3$, which passes through the point (s, t), so we can see that $(s, t) = (-3, t)$.

And the other part of B is the line $x = 1$, which clearly passes through the point $(1, 2)$.

Now, let's next, move on to the other curve C, which is the curve of $ax - 2b + y = 0$.

First, we know $b = \frac{3}{2}$. So the equation of C is $ax - 3 + y = 0$.

Next, since the line $ax - 3 + y = 0$, too, includes $(1, 2)$, we get
$(1, 2) \Rightarrow ax - 3 + y = a(1) - 3 + 2 = a - 3 + 2 = 0 \Rightarrow a = 1$.

So the other curve is a line $x - 3 + y = 0$.
Therefore, the curves used in the linear combination are three lines as follows.
$x = 1$, $x = -3$, and $x - 3 + y = 0$.

Next, the line $x - 3 + y = 0$, too, is one of the curves used in the linear combination, so it has to include the point (s, t), that is, $(-3, t)$.

So we get **(-3, *t*)** $\Rightarrow$ ***x* – 3 + *y* = -3 – 3 + *t* = 0** $\Rightarrow$ ***t* = 6**. Therefore, **(*s*, *t*) = (-3, 6)**.

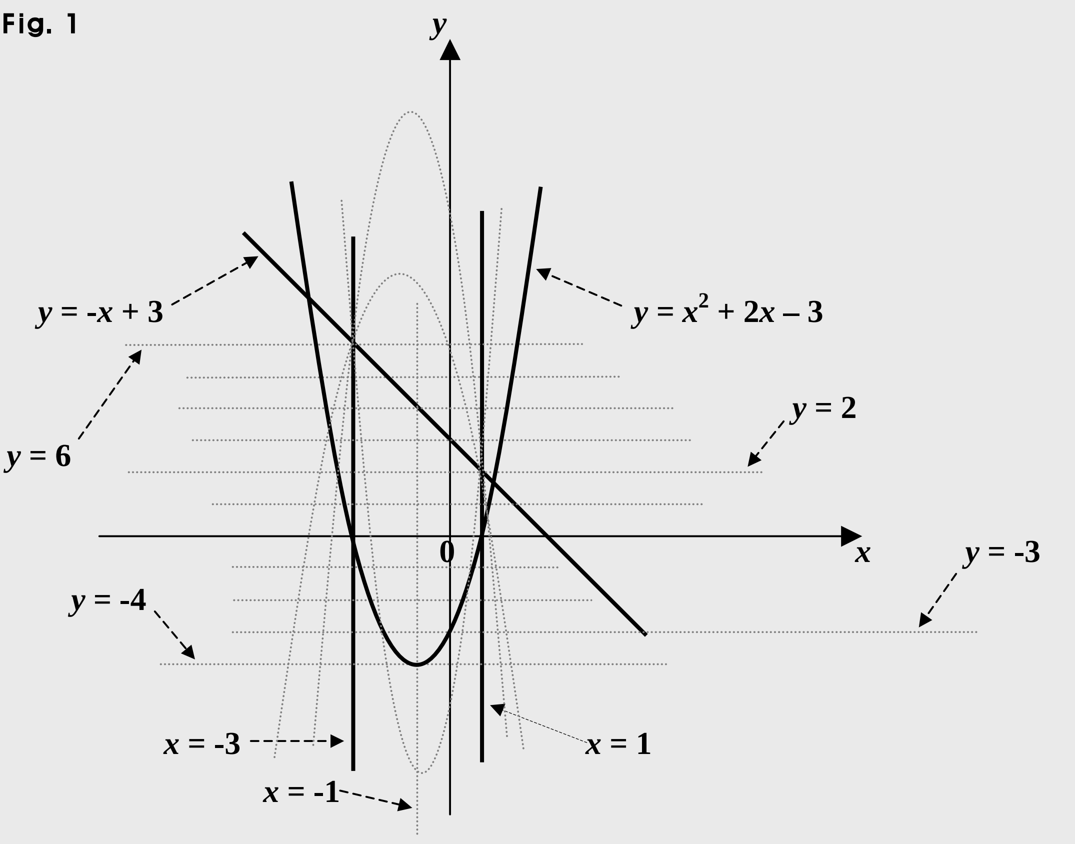
Let's now, put in a graph the curves (three lines) and some parabolas each of which is indicated by ***A*** for some ***c***. The equation ***A*** is $y = cx^2 - (a - 2c)x - 2b(c - 1)$ where $c \neq 0$.

So let's get first, the actual equation of ***A***. We know $a = 1$, and $b = \frac{3}{2}$. So we get

$y = cx^2 - (a - 2c)x - 2b(c - 1) \Rightarrow y = cx^2 - (1 - 2c)x - 3(c - 1)$

$\Rightarrow c(x^2 + 2x - 3) - x + 3 - y = 0$. So putting now, in a graph some parabolas, together with the curves of ***B*** and ***C***, we can put them the way below.

Fig. 1



The parabolas in gray only can be indicated by ***A*** for some ***c***.
So note that the parabola in black cannot be indicated by the equation ***A***.
Why then, there is no parabola among the curves used in the linear combination?

There are roots of parabolas, though. What in the world are those roots?

The two lines, $x^2 + 2x - 3 = (x - 1)(x + 3) = 0$, which indicates $x = 1$ and $x = -3$, can be said to be from a parabola, $y = x^2 + 2x - 3$.

The two numbers **1** and **-3** are the solution to a quadratic equation $x^2 + 2x - 3 = 0$, and are usually called the roots of the equation.

The roots are the x-intercepts of a parabola, $y = x^2 + 2x - 3$.

So the solution to $x^2 + 2x - 3 = 0$ can be called the roots of the curve of $y = x^2 + 2x - 3$, which is a parabola.

The solution can be taken for roots of a tree, too. How?

If we take the x-axis for surface of ground, and the curve for a tree, then the solution can be taken for the roots.

Now, the two lines, $x = 1$ and $x = -3$, which are the roots of a parabola, get combined by another line $y = -x + 3$, and then, the combination (compound) of the three lines forms a tree, $y = x^2 + 2x - 3$, which is the parabola stated above.