Examples in the Non-Vertex Form

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Examples in the Non–Vertex Form

Find the vertex, focus, and directrix of each of parabolas as follows.

0. $y = x^2 - x - \frac{1}{2}$.

1. $y = 2x^2 - 4x + 2$, and $2x^2 - 3x + 2y - 1 = 0$.

2. $x = 3y^2 - 6y + 3$, and $2y^2 + x - y - 1 = 0$.

Suggestions or Solutions
To the Problem in the Example 0

Find the vertex, focus, and directrix of a parabola as follows. $y = x^2 - x - \frac{1}{2}$**.**

$(x - m)^2 = 4p(y - n)$

$\Rightarrow$ the vertex = (m, n), the focus = $(m, p + n)$, and the directrix is $y = n - p$.

$y = x^2 - x - \frac{1}{2} = x^2 - x + (\frac{1}{2})^2 - (\frac{1}{2})^2 - \frac{1}{2} = (x - \frac{1}{2})^2 - \frac{3}{4}$

$\Rightarrow (x - \frac{1}{2})^2 = y + \frac{3}{4} = 4 \cdot \frac{1}{4}\{y - (-\frac{3}{4})\}$.

Thus, $p = \frac{1}{4}$, $m = \frac{1}{2}$, and $n = -\frac{3}{4}$. So we get

The vertex = (m, n) = $(\frac{1}{2}, -\frac{3}{4})$.

The focus = $(m, p + n)$ = $(\frac{1}{2}, \frac{1}{4} - \frac{3}{4}) = (\frac{1}{2}, -\frac{1}{2})$.

The directrix is $y = n - p = -\frac{3}{4} - \frac{1}{4} = -1$.

If not quite sure of the idea behind the processes above, follow the steps below.

The equation given is in the non-vertex form. Putting the equation in the vertex form, we can readily see the vertex, focus, and directrix.

If the axis of symmetry is perpendicular to the x-axis, the vertex form often used is as follows. $x^2 = 4py$, or $(x - m)^2 = 4p(y - n)$.

In a parabola $x^2 = 4py$, the vertex is $(0, 0)$, the focus is $(0, p)$, and the directrix is $y = -p$.

Moving the parabola above by m in the direction of the x-axis, and by n in the direction of the y-axis, we get a new parabola $(x - m)^2 = 4p(y - n)$.

Then, the vertex, focus, and directrix, etc. get moved in the same manner, too.
So we get

The vertex = $(0 + m, 0 + n) = (m, n)$.

The focus = $(0 + m, p + n) = (m, p + n)$, and the directrix is $y = -p + n = n - p$.

Let's now, put in the vertex form, the right hand side in the equation given.

$$x^2 - x - \tfrac{1}{2} = x^2 - x + (\tfrac{1}{2})^2 - (\tfrac{1}{2})^2 - \tfrac{1}{2} = (x - \tfrac{1}{2})^2 - \tfrac{1}{4} - \tfrac{1}{2} = (x - \tfrac{1}{2})^2 - \tfrac{3}{4}.$$

Next, putting the given equation in the vertex form, we get

$$y = x^2 - x - \tfrac{1}{2} \Rightarrow y = (x - \tfrac{1}{2})^2 - \tfrac{3}{4} \Rightarrow y - (-\tfrac{3}{4}) = (x - \tfrac{1}{2})^2 \Rightarrow (x - \tfrac{1}{2})^2 = 4 \cdot \tfrac{1}{4}\{y - (-\tfrac{3}{4})\}.$$

Then, comparing the equation above with $(x - m)^2 = 4p(y - n)$, we can see that

$p = \frac{1}{4}$, $m = \frac{1}{2}$, and $n = -\frac{3}{4}$.

So we get

The vertex = $(m, n) = (\frac{1}{2}, -\frac{3}{4})$.

The focus = $(m, p + n) = (\frac{1}{2}, \frac{1}{4} - \frac{3}{4}) = (\frac{1}{2}, -\frac{1}{2})$.

The directrix is $y = n - p = -\frac{3}{4} - \frac{1}{4} = -1$.

In short:

$(x - m)^2 = 4p(y - n)$

$\Rightarrow$ the vertex = (m, n), the focus = $(m, p + n)$, and the directrix is $y = n - p$.

$$y = x^2 - x - \tfrac{1}{2} = x^2 - x + (\tfrac{1}{2})^2 - (\tfrac{1}{2})^2 - \tfrac{1}{2} = (x - \tfrac{1}{2})^2 - \tfrac{3}{4}$$

$$\Rightarrow (x - \tfrac{1}{2})^2 = y + \tfrac{3}{4} = 4 \cdot \tfrac{1}{4}\{y - (-\tfrac{3}{4})\}.$$

Thus, $p = \frac{1}{4}$, $m = \frac{1}{2}$, and $n = -\frac{3}{4}$. So we get

The vertex = $(m, n) = (\frac{1}{2}, -\frac{3}{4})$.

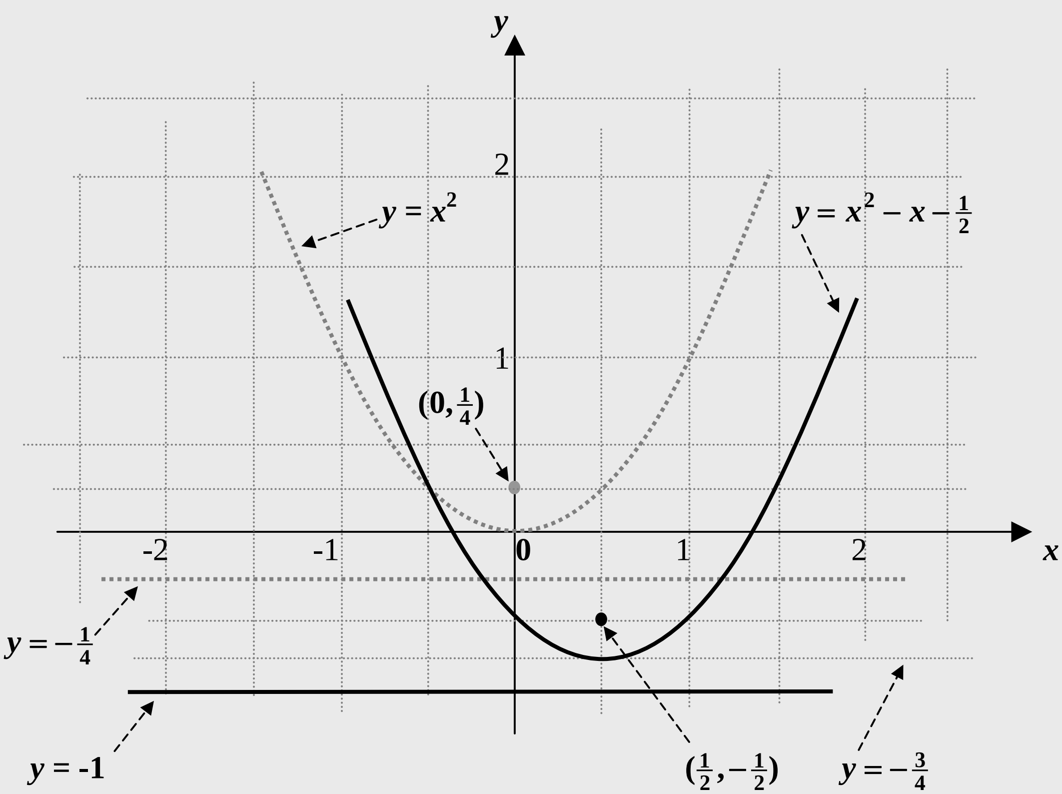
The focus = $(m, p + n) = (\frac{1}{2}, \frac{1}{4} - \frac{3}{4}) = (\frac{1}{2}, -\frac{1}{2})$.

The directrix is $y = n - p = -\frac{3}{4} - \frac{1}{4} = -1$.

And putting the equation in a graph, we get

Fig. 0

$$y = x^2 - x - \frac{1}{2} \Leftrightarrow (x - \frac{1}{2})^2 = 4 \cdot \frac{1}{4}\{y - (-\frac{3}{4})\}$$



Suggestions or Solutions
To the Problem in the Example 1

Find the vertices, the focuses, and the directrices of a parabolas as follows.
$y = 2x^2 - 4x + 2$ and $2x^2 - 3x + 2y - 1 = 0$.

$\boldsymbol{(x - m)^2 = 4p(y - n)}$

$\Rightarrow$ the vertex = $\boldsymbol{(m, n)}$, the focus = $\boldsymbol{(m, p + n)}$, and the directrix is $\boldsymbol{y = n - p}$.

So putting the first equation into the vertex form above, we get

$$y = 2x^2 - 4x + 2 = 2(x^2 - 2x + 1) = 2(x-1)^2 \Rightarrow y = 2(x-1)^2 \Rightarrow (x-1)^2 = 4\cdot\tfrac{1}{8}y.$$

So $\boldsymbol{p = \tfrac{1}{8}, m = 1}$, and $\boldsymbol{n = 0}$. Thus,

The vertex = $\boldsymbol{(m, n) = (1, 0)}$.

The focus = $\boldsymbol{(m, p + n) = (1, \tfrac{1}{8} + 0) = (1, \tfrac{1}{8})}$.

The directrix is $\boldsymbol{y = n - p = 0 - \tfrac{1}{8} = -\tfrac{1}{8}}$.

Next, putting the second equation into the vertex form, we get

$$2x^2 - 3x + 2y - 1 = 0 \Rightarrow 2y = -2x^2 + 3x + 1 \Rightarrow y = -x^2 + \tfrac{3}{2}x + \tfrac{1}{2} = -(x^2 - \tfrac{3}{2}x) + \tfrac{1}{2}$$
$$= -\{x^2 - \tfrac{3}{2}x + (\tfrac{3}{4})^2 - (\tfrac{3}{4})^2\} + \tfrac{1}{2} = -(x - \tfrac{3}{4})^2 + (\tfrac{3}{4})^2 + \tfrac{1}{2} = -(x - \tfrac{3}{4})^2 + \tfrac{9+8}{16}$$
$$\Rightarrow y = -(x - \tfrac{3}{4})^2 + \tfrac{17}{16} \Rightarrow (x - \tfrac{3}{4})^2 = -y + \tfrac{17}{16} = -4\cdot\tfrac{1}{4}(y - \tfrac{17}{16}) \Rightarrow (x - \tfrac{3}{4})^2 = 4\cdot(-\tfrac{1}{4})(y - \tfrac{17}{16}).$$

So $\boldsymbol{p = -\tfrac{1}{4}, m = \tfrac{3}{4}}$, and $\boldsymbol{n = \tfrac{17}{16}}$. Thus,

The vertex = $\boldsymbol{(m, n) = (\tfrac{3}{4}, \tfrac{17}{16})}$.

The focus = $\boldsymbol{(m, p + n) = (\tfrac{3}{4}, -\tfrac{1}{4} + \tfrac{17}{16}) = (\tfrac{3}{4}, \tfrac{13}{16})}$.

The directrix is $\boldsymbol{y = n - p = \tfrac{17}{16} - (-\tfrac{1}{4}) = \tfrac{17+4}{16} = \tfrac{21}{16}}$.

If not quite sure of the idea behind the processes above, follow the steps below.

If the axis of symmetry is perpendicular to the $\boldsymbol{x}$-axis, the vertex form often used is as follows. $\boldsymbol{x^2 = 4py}$, or $\boldsymbol{(x - m)^2 = 4p(y - n)}$.

In a parabola $\boldsymbol{x^2 = 4py}$, the vertex = $\mathbf{(0, 0)}$, the focus = $\boldsymbol{(0, p)}$, and the directrix is $\boldsymbol{y = -p}$.

Moving the parabola above by $\boldsymbol{m}$ in the direction of the $\boldsymbol{x}$-axis and by $\boldsymbol{n}$ in the direction of the $\boldsymbol{y}$-axis, we get a new parabola $\boldsymbol{(x - m)^2 = 4p(y - n)}$.

Then, the vertex, focus, and directrix, etc. get moved in the same manner, too.
So we get

The vertex = $\boldsymbol{(0 + m, 0 + n) = (m, n)}$.
The focus = $\boldsymbol{(0 + m, p + n) = (m, p + n)}$, and the directrix is $\boldsymbol{y = -p + n = n - p}$.

Putting the first equation in the vertex form, we get
$\boldsymbol{y = 2x^2 - 4x + 2 = 2(x^2 - 2x + 1) = 2(x-1)^2 \Rightarrow y = 2(x-1)^2 \Rightarrow (x-1)^2 = 4 \cdot \frac{1}{8} y}$.

Then, comparing the equation above with $\boldsymbol{(x - m)^2 = 4p(y - n)}$, we can see that
$\boldsymbol{p = \frac{1}{8}, m = 1}$, and $\boldsymbol{n = 0}$.

So we get

The vertex = $\boldsymbol{(m, n) = (1, 0)}$.

The focus = $\boldsymbol{(m, p + n) = (1, \frac{1}{8} + 0) = (1, \frac{1}{8})}$.

The directrix is $\boldsymbol{y = n - p = 0 - \frac{1}{8} = -\frac{1}{8}}$.

Next, putting the second equation in the vertex form, we get

$$\boldsymbol{2x^2 - 3x + 2y - 1 = 0 \Rightarrow 2y = -2x^2 + 3x + 1 \Rightarrow y = -x^2 + \frac{3}{2}x + \frac{1}{2} = -(x^2 - \frac{3}{2}x) + \frac{1}{2}}$$
$$\boldsymbol{= -\{x^2 - \frac{3}{2}x + (\frac{3}{4})^2 - (\frac{3}{4})^2\} + \frac{1}{2} = -(x - \frac{3}{4})^2 + (\frac{3}{4})^2 + \frac{1}{2} = -(x - \frac{3}{4})^2 + \frac{9+8}{16}}$$
$$\boldsymbol{\Rightarrow y = -(x - \frac{3}{4})^2 + \frac{17}{16} \Rightarrow (x - \frac{3}{4})^2 = -y + \frac{17}{16} = -4 \cdot \frac{1}{4}(y - \frac{17}{16}) \Rightarrow (x - \frac{3}{4})^2 = 4 \cdot (-\frac{1}{4})(y - \frac{17}{16})}.$$

Then, comparing the equation above with $\boldsymbol{(x - m)^2 = 4p(y - n)}$, we can see that
$\boldsymbol{p = -\frac{1}{4}}$, $\boldsymbol{m = \frac{3}{4}}$, and $\boldsymbol{n = \frac{17}{16}}$.

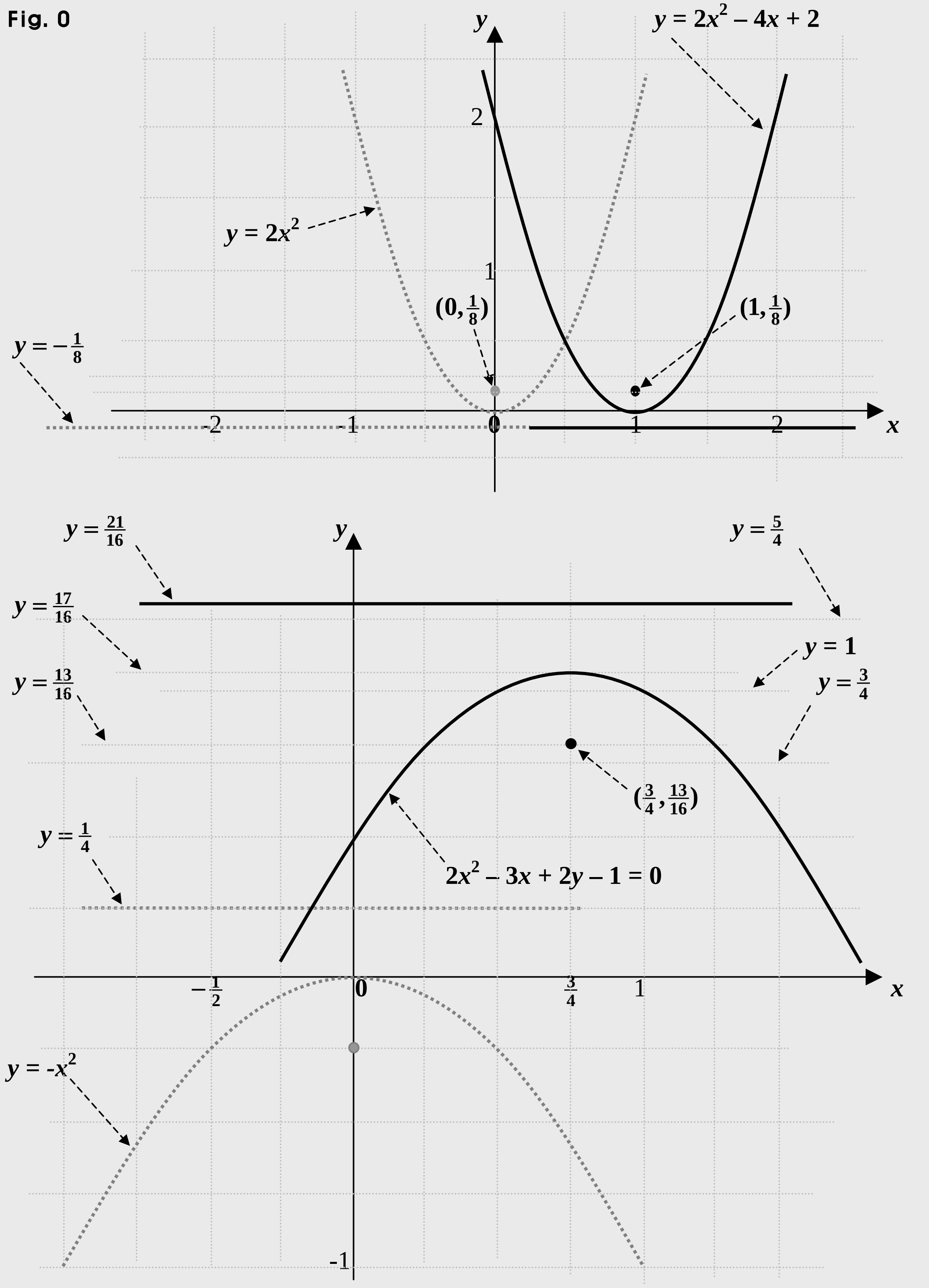
So we get

The vertex = $\boldsymbol{(m, n) = (\frac{3}{4}, \frac{17}{16})}$.

The focus = $\boldsymbol{(m, p + n) = (\frac{3}{4}, -\frac{1}{4} + \frac{17}{16}) = (\frac{3}{4}, \frac{13}{16})}$.

The directrix is $\boldsymbol{y = n - p = \frac{17}{16} - (-\frac{1}{4}) = \frac{17+4}{16} = \frac{21}{16}}$.

Fig. 0



$y = 2x^2 - 4x + 2$
$y = 2x^2$
$(0, \frac{1}{8})$
$(1, \frac{1}{8})$
$y = -\frac{1}{8}$
y
x
2
1
-2
-1
0
1
2
$y = \frac{21}{16}$
$y = \frac{5}{4}$
$y = \frac{17}{16}$
$y = 1$
$y = \frac{13}{16}$
$y = \frac{3}{4}$
$(\frac{3}{4}, \frac{13}{16})$
$y = \frac{1}{4}$
$2x^2 - 3x + 2y - 1 = 0$
$-\frac{1}{2}$
0
$\frac{3}{4}$
1
$y = -x^2$
-1

Suggestions or Solutions
To the Problem in the Example 2

Find the vertices, the focuses, and the directrices of parabolas as follows.
$x = 3y^2 - 6y + 3$ and $2y^2 + x - y - 1 = 0$.

$(y - n)^2 = 4p(x - m)$
$\Rightarrow$ the vertex = (m, n), the focus = $(p + m, n)$, and the directrix is $x = m - p$.

So first, putting the first equation into the vertex form, we get

$$x = 3y^2 - 6y + 3 = 3(y^2 - 2y + 1) = 3(y-1)^2 \Rightarrow x = 3(y-1)^2 \Rightarrow (y-1)^2 = 4\cdot\tfrac{1}{12}x.$$

Then, comparing the equation above with $(y - n)^2 = 4p(x - m)$, we can see that
$p = \frac{1}{12}$, $m = 0$, and $n = 1$.

So we get
The vertex = $(m, n) = (0, 1)$.
The focus = $(p + m, n) = (\frac{1}{12} + 0, 1) = (\frac{1}{12}, 1)$.
The directrix is $x = m - p = 0 - \frac{1}{12} = -\frac{1}{12}$.

Next, putting the second equation into the vertex form, we get

$$2y^2 + x - y - 1 = 0 \Rightarrow x = -2y^2 + y + 1 = -2(y^2 - \tfrac{1}{2}y) + 1 = -2\{y^2 - \tfrac{1}{2}y + (\tfrac{1}{4})^2 - (\tfrac{1}{4})^2\} + 1$$
$$= -2(y - \tfrac{1}{4})^2 + 2(\tfrac{1}{4})^2 + 1 = -2(y - \tfrac{1}{4})^2 + \tfrac{2+16}{16} \Rightarrow x = -2(y - \tfrac{1}{4})^2 + \tfrac{9}{8}$$
$$\Rightarrow -2(y - \tfrac{1}{4})^2 = (x - \tfrac{9}{8}) \Rightarrow (y - \tfrac{1}{4})^2 = -\tfrac{1}{2}(x - \tfrac{9}{8}) = 4\cdot(-\tfrac{1}{8})(x - \tfrac{9}{8}).$$

Then, comparing the equation above with $(y - n)^2 = 4p(x - m)$, we can see that
$p = -\frac{1}{8}$, $m = \frac{9}{8}$, and $n = \frac{1}{4}$.

So we get
The vertex = $(m, n) = (\frac{9}{8}, \frac{1}{4})$.
The focus = $(p + m, n) = (-\frac{1}{8} + \frac{9}{8}, \frac{1}{4}) = (1, \frac{1}{4})$.
The directrix is $x = m - p = \frac{9}{8} + \frac{1}{8} = \frac{10}{8} = \frac{5}{4}$.

If not quite sure of the idea behind the processes above, follow the steps below.

If the axis of symmetry is perpendicular to the y-axis, the vertex form often used is as follows. $y^2 = 4px$, or $(y - n)^2 = 4p(x - m)$.

In a parabola $y^2 = 4px$, the vertex = $(0, 0)$, the focus = $(p, 0)$, and the directrix is $x = -p$.

Moving the parabola above by m in the direction of the x-axis, and by n in the direction of the y-axis, we get a new parabola $(y - n)^2 = 4p(x - m)$. Then, the vertex, focus, and directrix, etc. get moved in the same manner, too. So we get

The vertex = $(0 + m, 0 + n) = (m, n)$.
The focus = $(p + m, 0 + n) = (p + m, n)$, and the directrix is $x = -p + m = m - p$.

So to begin with, putting the first equation in the vertex form, we get

$$x = 3y^2 - 6y + 3 = 3(y^2 - 2y + 1) = 3(y - 1)^2 \Rightarrow x = 3(y - 1)^2 \Rightarrow (y - 1)^2 = 4 \cdot \tfrac{1}{12} x.$$

Then, comparing the equation above with $(y - n)^2 = 4p(x - m)$, we can see that
$p = \frac{1}{12}$, $m = 0$, and $n = 1$.

So we get

The vertex = $(m, n) = (0, 1)$.

The focus = $(p + m, n) = (\frac{1}{12} + 0, 1) = (\frac{1}{12}, 1)$.

The directrix is $x = m - p = 0 - \frac{1}{12} = -\frac{1}{12}$.

Next, putting the second equation in the vertex form, we get

$$2y^2 + x - y - 1 = 0 \Rightarrow x = -2y^2 + y + 1 = -2(y^2 - \tfrac{1}{2}y) + 1 = -2\{y^2 - \tfrac{1}{2}y + (\tfrac{1}{4})^2 - (\tfrac{1}{4})^2\} + 1$$
$$= -2(y - \tfrac{1}{4})^2 + 2(\tfrac{1}{4})^2 + 1 = -2(y - \tfrac{1}{4})^2 + \tfrac{2+16}{16} \Rightarrow x = -2(y - \tfrac{1}{4})^2 + \tfrac{9}{8}$$
$$\Rightarrow -2(y - \tfrac{1}{4})^2 = (x - \tfrac{9}{8}) \Rightarrow (y - \tfrac{1}{4})^2 = -\tfrac{1}{2}(x - \tfrac{9}{8}) = 4 \cdot (-\tfrac{1}{8})(x - \tfrac{9}{8}).$$

Then, comparing the equation above with $(y - n)^2 = 4p(x - m)$, we can see that
$p = -\frac{1}{8}$, $m = \frac{9}{8}$, and $n = \frac{1}{4}$. So we get

The vertex = $(m, n) = (\frac{9}{8}, \frac{1}{4})$.

The focus = $(p + m, n) = (-\frac{1}{8} + \frac{9}{8}, \frac{1}{4}) = (1, \frac{1}{4})$.

The directrix is $x = m - p = \frac{9}{8} + \frac{1}{8} = \frac{10}{8} = \frac{5}{4}$.

Fig. 0

The directrix is parallel to the y-axis, and the coefficient of y^2 is positive.
So the parabola is concave-right.

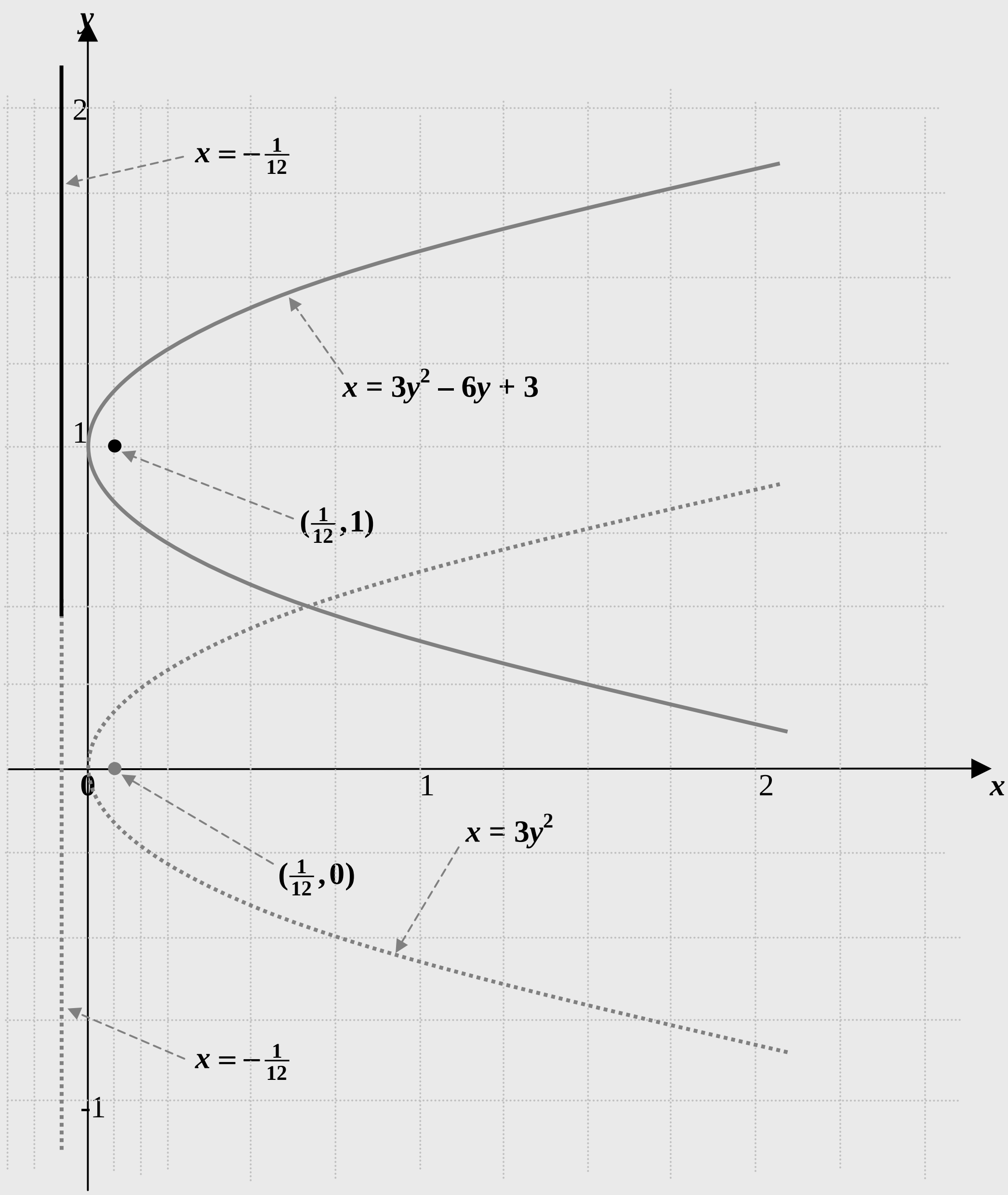
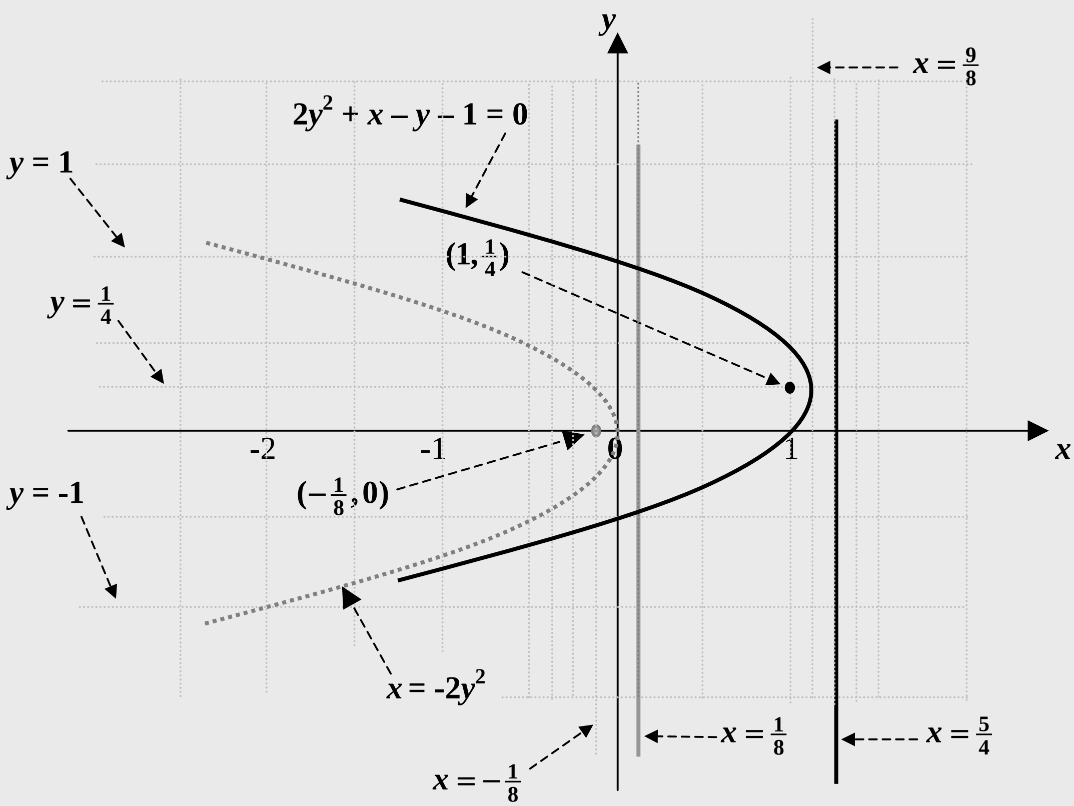


Fig. 1

The directrix is parallel to the ***y***-axis, and the coefficient of y^2 is negative
So the parabola is concave-left.

$$x = -2y^2 \Leftrightarrow y^2 = 4(-\tfrac{1}{8})x$$

$$2y^2 + x - y - 1 = 0 \Leftrightarrow x = -2y^2 + y + 1 \Leftrightarrow (y - \tfrac{1}{4})^2 = 4(-\tfrac{1}{8})(x - \tfrac{9}{8})$$



In short:

$(y - n)^2 = 4p(x - m)$
$\Rightarrow$ the vertex = (m, n), the focus = $(p + m, n)$, and the directrix is $x = m - p$.

So first, putting the first equation into the vertex form, we get

$$x = 3y^2 - 6y + 3 = 3(y^2 - 2y + 1) = 3(y-1)^2 \Rightarrow x = 3(y-1)^2 \Rightarrow (y-1)^2 = 4 \cdot \tfrac{1}{12}x.$$

Then, comparing the equation above with $(y - n)^2 = 4p(x - m)$, we can see that $p = \frac{1}{12}$, $m = 0$, and $n = 1$.

So we get

The vertex = $(m, n) = (0, 1)$.
The focus = $(p + m, n) = (\frac{1}{12} + 0, 1) = (\frac{1}{12}, 1)$.
The directrix is $x = m - p = 0 - \frac{1}{12} = -\frac{1}{12}$.

Next, putting the second equation into the vertex form, we get

$$2y^2 + x - y - 1 = 0 \Rightarrow x = -2y^2 + y + 1 = -2(y^2 - \tfrac{1}{2}y) + 1 = -2\{y^2 - \tfrac{1}{2}y + (\tfrac{1}{4})^2 - (\tfrac{1}{4})^2\} + 1$$
$$= -2(y - \tfrac{1}{4})^2 + 2(\tfrac{1}{4})^2 + 1 = -2(y - \tfrac{1}{4})^2 + \tfrac{2+16}{16} \Rightarrow x = -2(y - \tfrac{1}{4})^2 + \tfrac{9}{8}$$
$$\Rightarrow -2(y - \tfrac{1}{4})^2 = (x - \tfrac{9}{8}) \Rightarrow (y - \tfrac{1}{4})^2 = -\tfrac{1}{2}(x - \tfrac{9}{8}) = 4 \cdot (-\tfrac{1}{8})(x - \tfrac{9}{8}).$$

Then, comparing the equation above with $(y - n)^2 = 4p(x - m)$, we can see that $p = -\frac{1}{8}$, $m = \frac{9}{8}$, and $n = \frac{1}{4}$.

So we get
The vertex = $(m, n) = (\frac{9}{8}, \frac{1}{4})$.
The focus = $(p + m, n) = (-\frac{1}{8} + \frac{9}{8}, \frac{1}{4}) = (1, \frac{1}{4})$.
The directrix is $x = m - p = \frac{9}{8} + \frac{1}{8} = \frac{10}{8} = \frac{5}{4}$.