Examples 4 in the Vertex Form

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Examples 4 in the Vertex Form

In this set of examples, you are going to do practices on the vertex form just about the same way as you did in **Examples 3 in the Vertex Form**.

Assuming ***V*** is the vertex of a parabola, ***F*** is the focus, and ***D*** is the directrix, find the equation of each of the parabolas as follows.

0. ***V*****(0, 0)** and ***F*****(0, 3)**

1. ***V*****(-1, 0)** and ***F*****(-1, 3)**

2. ***V*****(0, 1)** and ***F*****(0, 3)**

3. ***V*****(-1, 1)** and ***F*****(-1, 3)**

4. ***V*****(3, 1)** and ***D*** is ***x* = -2**.

5. ***F*****(*m*, *n*)** and ***D*** is ***y* = *d***.

Suggestions or Solutions
To the Problem in the Example 0

The vertex is the origin, and the focus is (0, 3).

Assuming first, the directrix is a line $\boldsymbol{y = d}$, we get $\mathbf{0} = \frac{d+3}{2} \Rightarrow \boldsymbol{d} = \mathbf{-3}$.

So assuming next, $\boldsymbol{P(x, y)}$ an arbitrary point in the parabola, since $\boldsymbol{P}$ is the same distance away from the focus and directrix, we get

$$(x - x)^2 + \{y - (-3)\}^2 = (x - 0)^2 + (y - 3)^2 \Rightarrow (y + 3)^2 = s^2 + (y - 3)^2$$
$$\Rightarrow s^2 = (y + 3)^2 - (y - 3)^2 = (y + 3 + y - 3)(y + 3 - y + 3) = 2y \cdot 6 = 12y \Rightarrow x^2 = 12y.$$

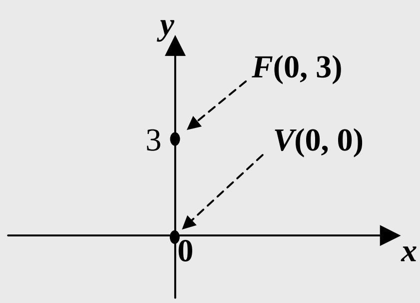
So the equation of the parabola is $\boldsymbol{x^2 = 12y}$.

If not quite sure of the idea behind the processes above, follow the steps below.

As mentioned in the previous examples, putting ideas in a graph, we can see better the solution's whereabouts. So it's a good idea to begin with a graph.

We can start with information as points or values given, or lines or curves, etc.
So first, we can begin a graph the way as follows.

Fig. 0



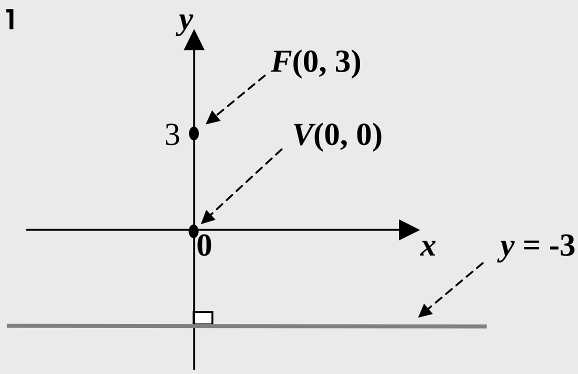
Then, we can now see that the axis of symmetry is the $\boldsymbol{y}$-axis, since the vertex and the focus are in the axis of symmetry.

We can get the parabola using the definition, which says each point in a parabola is the same distance away from the focus and the directrix. So we want to get the directrix.

We know the directrix meets the axis of symmetry at 90^o, and the vertex is the midpoint between the focus and the directrix.

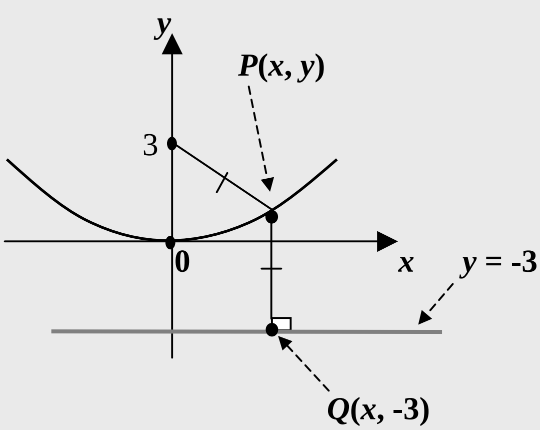
So we can put the directrix in the graph, the way below.

Fig. 1



And assuming $\boldsymbol{P(x, y)}$ is an arbitrary point in the parabola, and $\boldsymbol{Q(x, -3)}$ is a point in the line $\boldsymbol{y = -3}$, we can add to the graph, $\boldsymbol{P}$, $\boldsymbol{Q}$, and the parabola the way below.

Fig. 2



Note that the line segment $\boldsymbol{PQ}$ is perpendicular to the line $\boldsymbol{y = -3}$.

So both points share the same $\boldsymbol{x}$-coordinate.

We now have the directrix, so we can now get the parabola using the fact that each point in a parabola is the same distance away from the focus and the directrix.

So assuming now, $\boldsymbol{U}$ is the distance from $\boldsymbol{P(x, y)}$ to the directrix, $\boldsymbol{V}$ is the distance from $\boldsymbol{P(x, y)}$ to the focus, and using the distance formula, we get

$\boldsymbol{U^2 = (x - x)^2 + \{y - (-3)\}^2 = (y + 3)^2}$, and $\boldsymbol{V^2 = (x - 0)^2 + (y - 3)^2 = x^2 + (y - 3)^2}$.

And next, since the two distances are the same, we get $\boldsymbol{(y + 3)^2 = x^2 + (y - 3)^2}$

$\boldsymbol{\Rightarrow x^2 = (y + 3)^2 - (y - 3)^2 = (y + 3 + y - 3)(y + 3 - y + 3) = 2y{\cdot}6 = 12y \Rightarrow x^2 = 12y}$.

And the equation above is the connective equation between the coordinates of $\boldsymbol{P(x, y)}$.

So $\boldsymbol{x^2 = 12y}$ is the equation of the parabola in the figure above.

In short:

Assuming first, the directrix is a line $\boldsymbol{y = d}$, we get $\boldsymbol{0 = \frac{d+3}{2} \Rightarrow d = -3}$.

So assuming next, $\boldsymbol{P(x, y)}$ an arbitrary point in the parabola, since $\boldsymbol{P}$ is the same distance away from the focus and directrix, we get

$\boldsymbol{(x - x)^2 + \{y - (-3)\}^2 = (x - 0)^2 + (y - 3)^2 \Rightarrow (y + 3)^2 = s^2 + (y - 3)^2}$

$\boldsymbol{\Rightarrow s^2 = (y + 3)^2 - (y - 3)^2 = (y + 3 + y - 3)(y + 3 - y + 3) = 2y{\cdot}6 = 12y \Rightarrow x^2 = 12y}$.

So the equation of the parabola is $\boldsymbol{x^2 = 12y}$.

Suggestions or Solutions
To the Problem in the Example 1

The vertex is (-1, 0), and the focus is (-1, 3).

Assuming first, the directrix is a line $\boldsymbol{y = d}$, we get $\mathbf{0} = \frac{d+3}{2} \Rightarrow \boldsymbol{d} = \mathbf{-3}$.

So assuming next, $\boldsymbol{P(x, y)}$ is an arbitrary point in the parabola, since $\boldsymbol{P}$ is the same distance away from the focus and directrix, we get

$$(x - x)^2 + \{y - (-3)\}^2 = \{x - (-1)\}^2 + (y - 3)^2 \Rightarrow (y + 3)^2 = (x + 1)^2 + (y - 3)^2$$

$$\Rightarrow (x + 1)^2 = (y + 3)^2 - (y - 3)^2 = (y + 3 + y - 3)(y + 3 - y + 3) = 2y \cdot 6 = 12y$$

$$\Rightarrow (x + 1)^2 = 12y.$$

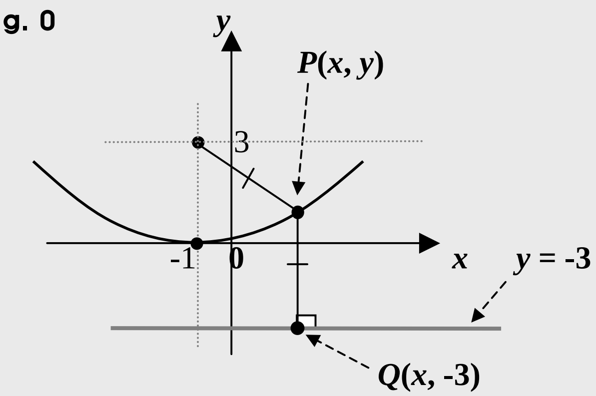
So the equation of the parabola is $\boldsymbol{(x + 1)^2 = 12y}$.

If not quite sure of the idea behind the processes above, follow the steps below.

As mentioned in the previous examples, putting ideas in a graph, we can see better the solution's whereabouts. So it's a good idea to begin with a graph.

If getting used to this type of problems, and doing practices enough, you can quickly get to the graph as shown in the figure below, beginning with the vertex and the focus, of course.

Fig. 0



Note that the line segment $\boldsymbol{PQ}$ is perpendicular to the line $\boldsymbol{y = -3}$.

And the line $\boldsymbol{y = -3}$ is the directrix.

Not quite sure how to get it?

You can get the graph above using the facts below.

Each point in a parabola is the same distance away from the focus and the directrix.

Between the focus and the directrix, the midpoint is the vertex.

And the vertex and the focus are in the axis of symmetry perpendicular to the directrix.

And in fact, once you've got the directrix, using the definition for parabolas, we can get the parabola.

So assuming now, $\boldsymbol{P(x, y)}$ is an arbitrary point in the parabola, $\boldsymbol{U}$ is the distance from $\boldsymbol{P}$ to the directrix, and $\boldsymbol{V}$ is the distance from $\boldsymbol{P}$ to the focus, and using the distance formula, we get

$\boldsymbol{U^2 = (x - x)^2 + \{y - (-3)\}^2 = (y + 3)^2}$, and $\boldsymbol{V^2 = \{x - (-1)\}^2 + (y - 3)^2 = (x + 1)^2 + (y - 3)^2}$.

And next, since the two distances are the same, we get $\boldsymbol{(y + 3)^2 = (x + 1)^2 + (y - 3)^2}$

$\Rightarrow \boldsymbol{(x + 1)^2 = (y + 3)^2 - (y - 3)^2 = (y + 3 + y - 3)(y + 3 - y + 3) = 2y \cdot 6 = 12y}$

$\Rightarrow \boldsymbol{(x + 1)^2 = 12y}$, which is the equation of the parabola in the figure above.

• And of course, we can get the solution with no use of a graph, too.

So for practice purposes, let's get this time, the solution with reasoning processes only with no graph.

A parabola is symmetric about a line, so the line is called the axis of symmetry.
And the axis of symmetry passes through the vertex and the focus.
What then, is the axis of symmetry, in this example, of course?

The vertex is **(-1, 0)**, and the focus is **(-1, 3)**.
So both points share the same $\boldsymbol{x}$-coordinate, which is -1.

Thus, both are in a line $\boldsymbol{x = -1}$, since the $\boldsymbol{x}$-coordinate of every point in the line is -1.
And the two points are located in the axis of symmetry.
So the axis of symmetry is a line $\boldsymbol{x = -1}$, which is parallel to the $\boldsymbol{y}$-axis.

Next, the directrix is perpendicular to the axis of symmetry, which is parallel to the $\boldsymbol{y}$-axis in this case.

So the directrix is perpendicular to the $\boldsymbol{y}$-axis, and thus, is passing through a point in the $\boldsymbol{y}$-axis at 90^o. So we can assume that the directrix is a line $\boldsymbol{y = d}$.

Since the vertex is the midpoint between the directrix and focus, we get

$\boldsymbol{0 = \frac{d+3}{2} \Rightarrow d = -3}$. Thus, the directrix is a line $\boldsymbol{y = -3}$.

So next, we want to use the fact that each point in a parabola is the same distance away from a particular point called a focus and a particular line called a directrix.

So assuming now, $\boldsymbol{P(x, y)}$ is an arbitrary point in the parabola, $\boldsymbol{U}$ is the distance from $\boldsymbol{P}$ to the directrix, and $\boldsymbol{V}$ is the distance from $\boldsymbol{P}$ to the focus, and using the distance formula, we get

$\boldsymbol{U^2 = (x - x)^2 + \{y - (-3)\}^2 = (y + 3)^2}$, and $\boldsymbol{V^2 = \{x - (-1)\}^2 + (y - 3)^2 = (x + 1)^2 + (y - 3)^2}$.

And next, since the two distances are the same, we get $\boldsymbol{(y + 3)^2 = (x + 1)^2 + (y - 3)^2}$

$\boldsymbol{\Rightarrow (x + 1)^2 = (y + 3)^2 - (y - 3)^2 = (y + 3 + y - 3)(y + 3 - y + 3) = 2y \cdot 6 = 12y}$

$\boldsymbol{\Rightarrow (x + 1)^2 = 12y}$, which is the equation of the parabola in the figure above.

Suggestions or Solutions
To the Problem in the Example 2

The vertex is (0, 1), and the focus is (0, 3).

Assuming first, the directrix is a line $\boldsymbol{y = d}$, we get $\mathbf{1} = \frac{d+3}{2} \Rightarrow \boldsymbol{d} = \mathbf{-1}$.

So next, assuming $\boldsymbol{P(x, y)}$ is an arbitrary point in the parabola, since $\boldsymbol{P}$ is the same distance away from the focus and directrix, we get

$$(x - x)^2 + \{y - (-1)\}^2 = (x - 0)^2 + (y - 3)^2 \Rightarrow (y + 1)^2 = s^2 + (y - 3)^2$$

$$\Rightarrow s^2 = (y + 1)^2 - (y - 3)^2 = (y + 1 + y - 3)(y + 1 - y + 3) = 2(y - 1)4 = 8(y - 1)$$

$$\Rightarrow x^2 = 8(y - 1).$$

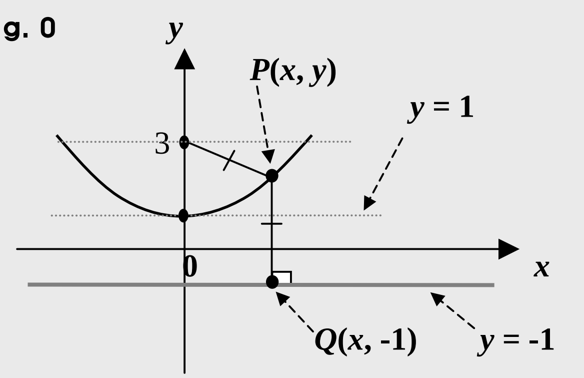
So the equation of the parabola is $\boldsymbol{x^2 = 8(y - 1)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

As mentioned in the previous examples, putting ideas in a graph, we can see better the solution's whereabouts. So it's a good idea to begin with a graph.

If getting used to this type of problems, and doing practices enough, you can quickly get to the graph as shown in the figure below, beginning with the vertex and the focus, of course.

Fig. 0



Note that the line segment $\boldsymbol{PQ}$ is perpendicular to the line $\boldsymbol{y = -1}$.

And the line $\boldsymbol{y = -1}$ is the directrix.

Not quite sure how to get it?

You can get the graph above using the facts below.

Each point in a parabola is the same distance away from the focus and the directrix.

Between the focus and the directrix, the midpoint is the vertex.

And the vertex and the focus are in the axis of symmetry perpendicular to the directrix.

And in fact, once you've got the directrix, using the definition for parabolas, we can get the parabola.

So assuming now, $\boldsymbol{P(x, y)}$ is an arbitrary point in the parabola, $\boldsymbol{U}$ is the distance from $\boldsymbol{P}$ to the directrix, and $\boldsymbol{V}$ is the distance from $\boldsymbol{P}$ to the focus, and using the distance formula, we get

$\boldsymbol{U^2 = (x - x)^2 + \{y - (-1)\}^2 = (y + 1)^2}$, and $\boldsymbol{V^2 = (x - 0)^2 + (y - 3)^2 = s^2 + (y - 3)^2}$.

And next, since the two distances are the same, we get

$\boldsymbol{(y + 1)^2 = s^2 + (y - 3)^2 \Rightarrow x^2 = (y + 1)^2 - (y - 3)^2 = (y + 1 + y - 3)(y + 1 - y + 3)}$

$\boldsymbol{= 2(y - 1)4 = 8(y - 1) \Rightarrow x^2 = 8(y - 1)}$.

So the equation of the parabola is $\boldsymbol{x^2 = 8(y - 1)}$.

• And of course, we can get the solution with no use of a graph, too.

So for practice purposes, let's get this time, the solution with reasoning processes only with no graph.

A parabola is symmetric about a line, so the line is called the axis of symmetry.
And the axis of symmetry passes through the vertex and the focus.

What then, is the axis of symmetry, in this example, of course?

The vertex is **(0, 1)**, and the focus is **(0, 3)**.
So both points share the same x-coordinate, which is 0.

Thus, both are in a line $x = 0$, since the x-coordinate of every point in the line is 0.
And the two points are located in the axis of symmetry.

So the axis of symmetry is a line $x = 0$, which is the y-axis.

Next, the directrix is perpendicular to the axis of symmetry, which is the y-axis in this case.

So the directrix is perpendicular to the y-axis, and thus, is passing through a point in the y-axis at 90^o. So we can assume that the directrix is a line $y = d$.

Since the vertex is the midpoint between the directrix and the focus, we get

$\mathbf{1} = \frac{d+3}{2} \Rightarrow d = -1$. Thus, the directrix is a line $y = -1$.

So next, we want to use the fact that each point in a parabola is the same distance away from a particular point called a focus and a particular line called a directrix.

So assuming now, $P(x, y)$ is an arbitrary point in the parabola, U is the distance from P to the directrix, and V is the distance from P to the focus, and using the distance formula, we get

$U^2 = (x - x)^2 + \{y - (-1)\}^2 = (y + 1)^2$, and $V^2 = (x - 0)^2 + (y - 3)^2 = s^2 + (y - 3)^2$.

And next, since the two distances are the same, we get

$(y + 1)^2 = s^2 + (y - 3)^2 \Rightarrow x^2 = (y + 1)^2 - (y - 3)^2 = (y + 1 + y - 3)(y + 1 - y + 3)$

$= 2(y - 1)4 = 8(y - 1) \Rightarrow x^2 = 8(y - 1)$, which is the parabola.

Suggestions or Solutions
To the Problem in the Example 3

The vertex is (-1, 1), and the focus is (-1, 3).

Assuming first, the directrix is a line $\boldsymbol{y = d}$, we get $\mathbf{1} = \frac{d+3}{2} \Rightarrow \boldsymbol{d} = \mathbf{-1}$.

So next, assuming $\boldsymbol{P(x, y)}$ is an arbitrary point in the parabola, since $\boldsymbol{P}$ is the same distance away from the focus and directrix, we get

$$(y + 1)^2 = (x + 1)^2 + (y - 3)^2 \Rightarrow (x + 1)^2 = (y + 1)^2 - (y - 3)^2$$

$$= (y + 1 + y - 3)(y + 1 - y + 3) = 2(y - 1)(4) = 8(y - 1) \Rightarrow (x + 1)^2 = 8(y - 1).$$

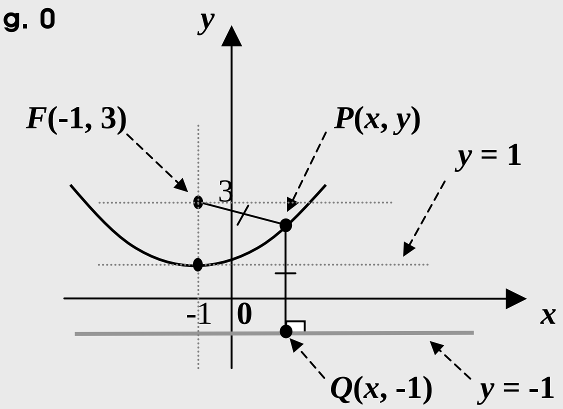
So the equation of the parabola is $\boldsymbol{(x + 1)^2 = 8(y - 1)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

As mentioned in the previous examples, putting ideas in a graph, we can see better the solution's whereabouts. So it's a good idea to begin with a graph.

If getting used to this type of problems, and doing practices enough, you can quickly get to the graph as shown in the figure below, beginning with the vertex and the focus, of course.

Fig. 0



The line segment $\boldsymbol{PQ}$ is perpendicular to the directrix $\boldsymbol{y = -1}$.

We can get the graph above using the facts below.

Each point in a parabola is the same distance away from the focus and the directrix.

Between the focus and the directrix, the midpoint is the vertex.

And the vertex and the focus are in the axis of symmetry perpendicular to the directrix.

And in fact, once you've got the directrix, using the definition for parabolas, we can get the parabola.

So assuming now, $\boldsymbol{P(x, y)}$ is an arbitrary point in the parabola, $\boldsymbol{U}$ is the distance from $\boldsymbol{P}$ to the directrix, and $\boldsymbol{V}$ is the distance from $\boldsymbol{P}$ to the focus, and using the distance formula, we get

$\boldsymbol{U^2 = (x - x)^2 + \{y - (-1)\}^2 = (y + 1)^2}$, and $\boldsymbol{V^2 = \{x - (-1)\}^2 + (y - 3)^2 = (x + 1)^2 + (y - 3)^2}$.

And next, since the two distances are the same, we get

$\boldsymbol{(y + 1)^2 = (x + 1)^2 + (y - 3)^2 \Rightarrow (x + 1)^2 = (y + 1)^2 - (y - 3)^2}$

$\boldsymbol{= (y + 1 + y - 3)(y + 1 - y + 3) = 2(y - 1)(4) = 8(y - 1) \Rightarrow (x + 1)^2 = 8(y - 1)}$.

So the equation of the parabola is $\boldsymbol{(x + 1)^2 = 8(y - 1)}$.

• And of course, we can get the solution with no use of a graph, too.

So for practice purposes, let's get this time, the solution with reasoning processes only with no graph.

A parabola is symmetric about a line, so the line is called the axis of symmetry.
And the axis of symmetry passes through the vertex and the focus.
What then, is the axis of symmetry, in this example, of course?

The vertex is **(-1, 1)**, and the focus is **(-1, 3)**.
So both points share the same $\boldsymbol{x}$-coordinate, which is 0.

Thus, both are in a line $\boldsymbol{x = -1}$, since the $\boldsymbol{x}$-coordinate of every point in the line is -1. And the two points are located in the axis of symmetry.

So the axis of symmetry is a line $\boldsymbol{x = -1}$, which is parallel to the $\boldsymbol{y}$-axis.

Next, the directrix is perpendicular to the axis of symmetry, which is parallel to the $\boldsymbol{y}$-axis in this case.

So the directrix is perpendicular to the $\boldsymbol{y}$-axis, and thus, is passing through a point in the $\boldsymbol{y}$-axis at 90^o. So we can assume that the directrix is a line $\boldsymbol{y = d}$.

Since the vertex is the midpoint between the directrix and the focus, we get

$\mathbf{1} = \frac{d+3}{2} \Rightarrow \boldsymbol{d} = \mathbf{-1}$. Thus, the directrix is a line $\boldsymbol{y = -1}$.

So next, we want to use the fact that each point in a parabola is the same distance away from a particular point called a focus and a particular line called a directrix.

So assuming now, $\boldsymbol{P(x, y)}$ is an arbitrary point in the parabola, $\boldsymbol{U}$ is the distance from $\boldsymbol{P}$ to the directrix, and $\boldsymbol{V}$ is the distance from $\boldsymbol{P}$ to the focus, and using the distance formula, we get

$U^2 = (x - x)^2 + \{y - (-1)\}^2 = (y + 1)^2$, and $V^2 = \{x - (-1)\}^2 + (y - 3)^2 = (x + 1)^2 + (y - 3)^2$.

And next, since the two distances are the same, we get

$(y + 1)^2 = (x + 1)^2 + (y - 3)^2 \Rightarrow (x + 1)^2 = (y + 1)^2 - (y - 3)^2$

$= (y + 1 + y - 3)(y + 1 - y + 3) = 2(y - 1)(4) = 8(y - 1) \Rightarrow (x + 1)^2 = 8(y - 1)$.

So the equation of the parabola is $\boldsymbol{(x + 1)^2 = 8(y - 1)}$.

Suggestions or Solutions
To the Problem in the Example 4

The vertex is (3, 1), and the directrix is x = -2.

Assuming first, the focus is $\boldsymbol{(m, 1)}$, we get $\boldsymbol{3 = \frac{-2+m}{2} \Rightarrow m = 8}$.

So next, assuming $\boldsymbol{P(x, y)}$ is an arbitrary point in the parabola, since $\boldsymbol{P}$ is the same distance away from the focus and directrix, we get

$$\boldsymbol{(x + 2)^2 = (x - 8)^2 + (y - 1)^2 \Rightarrow (y - 1)^2 = (x + 2)^2 - (x - 8)^2}$$
$$\boldsymbol{= (x + 2 + x - 8)(x + 2 - x + 8) = 2(x - 3)(10) = 20(x - 3) \Rightarrow (y - 1)^2 = 20(x - 3)}.$$

So the equation of the parabola is $\boldsymbol{(y - 1)^2 = 20(x - 3)}$.

If not quite sure of the idea behind the processes above, follow the steps below.

There can be many ways to find a parabola.
Using the set of facts below, we can get a parabola, too.

Each point in a parabola is the same distance away from the focus and the directrix.
Between the focus and the directrix, the midpoint is the vertex.
And the vertex and the focus are in a line, called the axis of symmetry.

Now in this example, we are given the vertex and the directrix, but not the focus.
Do we need the focus though?

Yes, we do, because we get to use the first fact above to get the parabola.
How then, can we get the focus?

We can get the focus using the fact that the focus and the vertex is in a line called the axis of symmetry. How then, can we get the axis of symmetry?

We can find the axis of symmetry using the fact that it is perpendicular to the directrix.

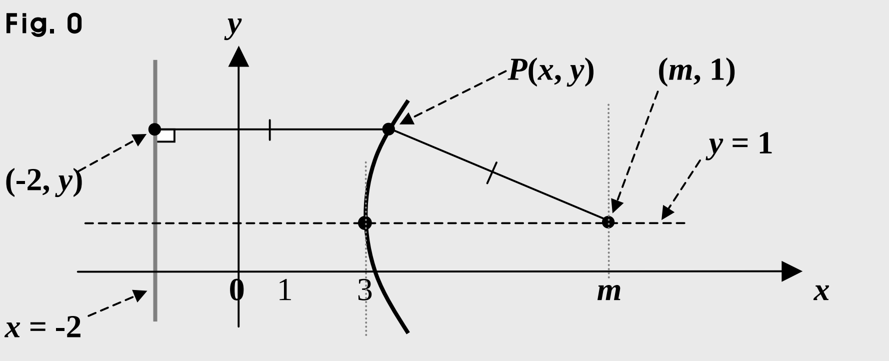
So let’s now, get the axis of symmetry.

First, the directrix is $x = -2$, which is parallel to the y-axis.
So the axis of symmetry is perpendicular to the y-axis, that is, parallel to the x-axis. So?

The y-coordinate at every point in a line parallel to the x-axis is the same.
And the vertex and the focus are two points in the line called the axis of symmetry.
So the vertex and focus share the same y-coordinate, which is 1, since the y-coordinate of the vertex is 1.

So we can assume that the focus is $(m, 1)$. And putting in a graph the directrix, along with information given, we can get

Fig. 0



Since the vertex is the midpoint between the directrix and focus, we get
$3 = \frac{-2+m}{2} \Rightarrow m = 8$. Thus, the focus is $(8, 1)$.

So assuming now, U is the distance from $P(x, y)$ to the directrix, V is the distance from $P(x, y)$ to the focus, and using the distance formula, we get

$U^2 = \{x - (-2)\}^2 + (y - y)^2 = (x + 2)^2$, and $V^2 = (x - 8)^2 + (y - 1)^2 = (x - 8)^2 + (y - 1)^2$.

And next, since the two distances are the same, we get

$(x + 2)^2 = (x - 8)^2 + (y - 1)^2 \Rightarrow (y - 1)^2 = (x + 2)^2 - (x - 8)^2$
$= (x + 2 + x - 8)(x + 2 - x + 8) = 2(x - 3)(10) = 20(x - 3)$
$\Rightarrow (y - 1)^2 = 20(x - 3)$, which is the parabola we want.

Note that when you solve problems, it's a good idea to begin with a graph and to put information in it as much as possible. Graphing matters, and it does a lot.

Suggestions or Solutions
To the Problem in the Example 5

The focus is (m, n), and the directrix is $y = d$.

Suppose first, that $P(x, y)$ is the point variable in the parabola.

Suppose next, that Q is the distance from $P(x, y)$ to the focus $F(m, n)$, and that R is the distance from $P(x, y)$ to the directrix $y = d$.

Then first, using the distance formula, we get
$Q^2 = (x - m)^2 + (y - n)^2$, and $R^2 = (x - x)^2 + (y - d)^2 = (y - d)^2$.

And next, P is the same distance away from both the focus and the directrix. So we get

$Q^2 = R^2 \Rightarrow (x - m)^2 + (y - n)^2 = (y - d)^2 \Rightarrow (x - m)^2 = (y - d)^2 - (y - n)^2$

$= \{(y - d) + (y - n)\}\{(y - d) - (y - n)\} = (2y - d - n)(n - d)$

$\Rightarrow (x - m)^2 = (2y - d - n)(n - d)$, which is the parabola.