

Examples 2 in Circles

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Examples 2 in Circles

Doing algebra, we often get to solve equations that are in connection with circles.

Many are in fact, equations for circles whereas others are not, that is, they do not indicate any circle. And many look quite simple, but they only look simple.

Solve for x and y in each equation below.

0. $x^2 + y^2 = 0$.

1. $ax^2 + ay^2 = 0$ where a is a constant.

2. $ax^2 + ay^2 = a$ where a is a constant.

3. $ax^2 + ay^2 = a^2$ where a is a constant.

4. $ax^2 + ay^2 = a^3$ where a is constant.

5. $x^2 + (y - c)^2 = 0$ where c is constant.

6. $\frac{x^2}{a} + \frac{(y-c)^2}{d} = 0$ where c and d are constants.

7. $(x - a)^2 + (y - b)^2 = 0$ where a and b are constant.

8. $w(x - u)^2 + w(y - v)^2 = 0$ where u , v , and w are constant.

**Suggestions or Solutions
To the Problem in the Example 0**

Solve for x and y the equation as follows: $x^2 + y^2 = 0$.

To begin with, neither of x^2 and y^2 can be negative.

Next, if $x \neq 0$, we get $x^2 > 0$, and if $y \neq 0$, we get $y^2 > 0$.

So anyway, if $x \neq 0$, we get $x^2 \neq 0$, and also, if $y \neq 0$, we get $y^2 \neq 0$.

So if either of x and y is not 0, we get $x^2 + y^2 \neq 0$.

Next, if $x = 0$, we get $x^2 = 0$, and if $y = 0$, we get $y^2 = 0$.

That is, if $x = 0$, and $y = 0$, we get $x^2 = 0$, and $y^2 = 0$.

So if $x = 0$, and $y = 0$, we get $x^2 + y^2 = 0$.

And we know if either of x and y is not 0, we get $x^2 + y^2 \neq 0$.

So only if $x = 0$, and $y = 0$, we get $x^2 + y^2 = 0$.

In other words, if $x^2 + y^2 = 0$, we get $x = 0$, and $y = 0$.

Thus, the solution is $x = 0$, and $y = 0$.

And in this case, we say that if and only if $x = 0$, and $y = 0$, we get $x^2 + y^2 = 0$, and also, we say that if and only if $x^2 + y^2 = 0$, we get $x = 0$, and $y = 0$.

And in math, symbolically, we put it this way: $(x = 0, \text{ and } y = 0) \Leftrightarrow x^2 + y^2 = 0$.

Also, we can put it this way, too: $x^2 + y^2 = 0 \Leftrightarrow (x = 0, \text{ and } y = 0)$.

And it is not rare to see similar cases when we do algebra.

For instance, we can have a situation where $(x - 2)^2 + (y^2 + 3y - 4)^2 = 0$.

Then, we need to have $x - 2 = 0$, and $y^2 + 3y - 4 = 0$.

So we get $x = 2$, and also, get $y + 3y - 4 = (y - 1)(y + 4) = 0 \Rightarrow y = 1$ or -4 .

Thus, we get $x = 2$, and $y = 1$ or -4 , which means $x = 2$ and $y = 1$, or $x = 2$ and $y = -4$.

In sum, we have $A^2 + B^2 = 0 \Leftrightarrow A = 0$ and $B = 0$. In other words, we have

$$A^2 + B^2 = 0 \Rightarrow A = 0 \text{ and } B = 0, \text{ and also, } (A = 0 \text{ and } B = 0) \Rightarrow A^2 + B^2 = 0.$$

And more generally, $A^2 + B^2 + C^2 + \dots = 0 \Leftrightarrow A = 0, B = 0, C = 0, \dots$

If m and n are positive even numbers, we get $A^m + B^n = 0 \Leftrightarrow A = 0$ and $B = 0$.

And if p , q , and r are positive and even, we get

$$A^p + B^q + C^r + \dots = 0 \Leftrightarrow A = 0, B = 0, C = 0, \dots$$

Suggestions or Solutions To the Problem in the Example 1

Solve for x and y the equation as follows: $ax^2 + ay^2 = 0$ where a is a constant.

We can put it this way: $ax^2 + ay^2 = 0 \Rightarrow a(x^2 + y^2) = 0$.

So to begin with, if $a = 0$, then x and y both can take any real number.

Next, if $a \neq 0$, we get $a(x^2 + y^2) = 0 \Rightarrow x^2 + y^2 = 0$, which means x and y both are 0.

Therefore, the solution is as follows.

If $a = 0$, x and y are any real numbers.

If $a \neq 0$, we get $x = y = 0$.

(Note that the material below is for advanced students.)

Now, what if $ax^2 + by^2 = 0$, where a and b are constant?

To begin with, if $a = 0$, and $b = 0$, then x and y both can take any real number.

Next, if $a = 0$, and $b \neq 0$, then x can take any real number, but $y = 0$.

Next, if $a \neq 0$, and $b = 0$, then $x = 0$, but y can take any real number.

Next, if $a \neq 0$, and $b \neq 0$, then x and y both are 0. Are they 0 only? Are they?

It's not quite the case, because we can have a case where $\frac{a}{b} = -1$.

In other words, we can have a situation where $a + b = 0$, and both a and b are not 0.

Then, we can get $a = -b$. So what?

So we get $ax^2 + by^2 = ax^2 - ay^2 = a(x^2 - y^2) = 0 \Rightarrow x^2 - y^2 = 0$ since $a \neq 0$.

Thus, we get $x^2 - y^2 = 0 \Rightarrow (x + y)(x - y) = 0 \Rightarrow y = x$ or $y = -x$.

What then, do we mean by this: $y = x$ or $y = -x$?

Both are lines, that is, each of the two equations indicates a line in the x - y plane.
So what?

The two lines $y = x$ and $y = -x$ both pass through the origin.

And we have a fact that each point in a line has a pair of coordinates.
One is the x -coordinate, and the other is the y -coordinate.

So we can say this:

- At each point in the line, $y = x$, the x -coordinate is the *value* of x , and the y -coordinate is the *value* of y , and they satisfy the equation, $y = x$, and thus, satisfy $ax^2 + by^2 = 0$.

And the same is true for the line $y = -x$, too. So we can say this:

- At each point in the line, $y = -x$, the x -coordinate is the value of x , and the y -coordinate is the value of y , and they satisfy the equation, $y = -x$, and thus, satisfy $ax^2 + by^2 = 0$.

So the coordinates of each of all the points in the two lines $y = x$ and $y = -x$ are respectively the values of x and y in $ax^2 + by^2 = 0$. Therefore, the solution is as follows.

- If $a = 0$, and $b = 0$, then x and y both can take any real number.
- If $a = 0$, and $b \neq 0$, then x can take any real number, but $y = 0$.
- If $a \neq 0$, and $b = 0$, then $x = 0$, but y can take any real number.
- And if $a \neq 0$, and $b \neq 0$, then

If $a + b \neq 0$, then x and y both are 0.

If $a + b = 0$, then the coordinates of each of all the points in two lines $y = x$ and $y = -x$ are respectively the values of x and y in $ax^2 + by^2 = 0$.

So more specifically, if $a \neq 0$, $b \neq 0$, and $a + b = 0$, the values of x and y are as follows.

...(-2, -2), ... (-0.3, -0.3), ... (0, 0), (0.1, 0.1), ... ($\frac{1}{3}$, $\frac{1}{3}$), ... (1, 1), ... (3, 3), ...

...(-2, 2), ... (-0.3, 0.3), ... (-0.1, 0.1), ... ($-\frac{1}{3}$, $\frac{1}{3}$), ... (1, -1), ... (3, -3), ...

That is to say that

... $x = -2$ and $y = -2$ $x = -0.3$ and $y = -0.3$ $x = 0$ and $y = 0$

... $x = 0.1$ and $y = 0.1$ $x = \frac{1}{3}$ and $y = \frac{1}{3}$ $x = 1$ and $y = 1$

... $x = -2$ and $y = 2$ $x = -0.3$ and $y = 0.3$ $x = -0.1$ and $y = 0.1$

... $x = -\frac{1}{3}$ and $y = \frac{1}{3}$ $x = 1$ and $y = -1$ $x = 3$ and $y = -3$

**Suggestions or Solutions
To the Problem in the Example 2**

Solve for x and y the equation as follows: $ax^2 + ay^2 = a$ where a is a constant.

We can put it this way: $ax^2 + ay^2 = a \Rightarrow a(x^2 + y^2) = a$.

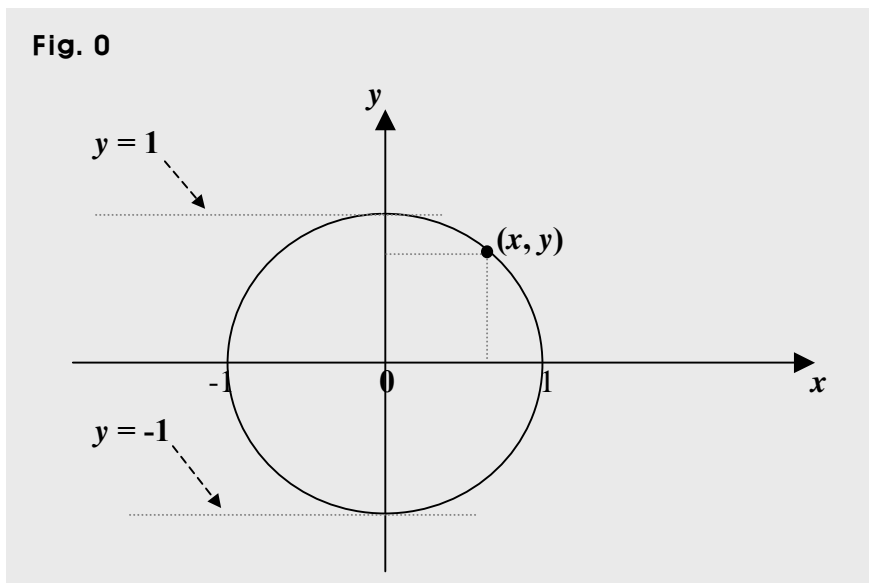
So to begin with, if $a = 0$, then x and y both can take any real numbers.

Next, if $a \neq 0$, we get $a(x^2 + y^2) = a \Rightarrow x^2 + y^2 = 1$.

Then, what do we mean by this: $x^2 + y^2 = 1$?

It is a circle, and is the circle of radius 1 centered at the origin.

That is, the equation indicates a unit circle centered at the origin in the x - y plane.



So what?

Just about the same story as in the example 1 above

Each point in a circle has a pair of coordinates.

One is the x -coordinate, and the other is the y -coordinate.

So at each and every point in the circle, $x^2 + y^2 = 1$, the x -coordinate is the value of x in $ax^2 + ay^2 = a$, and the y -coordinate is the value of y in $ax^2 + ay^2 = a$.

Thus, the coordinates of each of all the points in the circle, $x^2 + y^2 = 1$, are respectively the values of x and y in the equation $ax^2 + ay^2 = a$. For instance,

If we choose 1 for x , then we get $x^2 + y^2 = 1 \Rightarrow 1 + y^2 = 1$.

So we have to choose 0 for y .

Next, if $\frac{1}{2}$ or $-\frac{1}{2}$ is chosen for x , we get $(\frac{1}{2})^2 + y^2 = 1$.

So next, we want to get the solution to the equation, $(\frac{1}{2})^2 + y^2 = 1$, that is, the value of y .

Then, we get $(\frac{1}{2})^2 + y^2 = 1 \Rightarrow y^2 = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow y = \pm \frac{\sqrt{3}}{2}$.

So we get $x = \pm \frac{1}{2} \Rightarrow y = \pm \frac{\sqrt{3}}{2}$.

Thus, one solution is $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$. And another is $x = -\frac{1}{2}$ and $y = -\frac{\sqrt{3}}{2}$.

Suppose this time, that we choose $\frac{\pm\sqrt{2}}{2}$ for x .

Then, we want to get the solution to the equation, $(\frac{\pm\sqrt{2}}{2})^2 + y^2 = 1$, that is, the value of y .

Then, we get $(\frac{\pm\sqrt{2}}{2})^2 + y^2 = 1 \Rightarrow (\frac{\sqrt{2}}{2})^2 + y^2 = 1 \Rightarrow \frac{2}{4} + y^2 = 1 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm\sqrt{\frac{1}{2}}$.

Meanwhile, we have $\frac{1}{2} = \frac{2}{4}$. So we get $y = \pm\sqrt{\frac{1}{2}} = \pm\sqrt{\frac{2}{4}} = \pm\frac{\sqrt{2}}{2} = \frac{\pm\sqrt{2}}{2}$.

Thus, y has to be $\frac{\pm\sqrt{2}}{2}$, too.

So another solution is $x = \frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$. And another is $x = -\frac{\sqrt{2}}{2}$ and $y = -\frac{\sqrt{2}}{2}$.

Therefore, the entire solution is as follows.

If $a = 0$, then x and y both can take any real numbers.

If $a \neq 0$, then the coordinates of each of all the points in the circle, $x^2 + y^2 = 1$, are respectively the values of x and y in the equation $ax^2 + ay^2 = a$. And note that in this case, we have $|x| \leq 1$, and $|y| \leq 1$, that is, $-1 \leq x \leq 1$, and $-1 \leq y \leq 1$.

**Suggestions or Solutions
To the Problem in the Example 3**

Solve for x and y the equation as follows: $ax^2 + ay^2 = a^2$ where a is a constant.

We can put it this way, too: $ax^2 + ay^2 = a^2 \Rightarrow a(x^2 + y^2) = a^2$.

So to begin with, if $a = 0$, then x and y can be any real numbers.

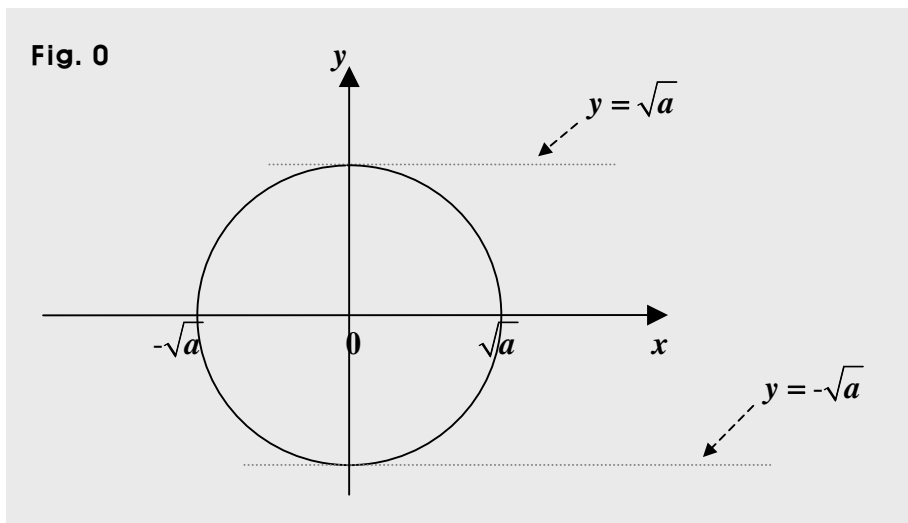
And next, if $a > 0$, we get $a(x^2 + y^2) = a^2 \Rightarrow x^2 + y^2 = a$. So what?

The equation represents a circle centered at the origin with the radius of \sqrt{a} .

Therefore, in $ax^2 + ay^2 = a^2$, the values of x and y are respectively the coordinate values at each of all the points in the circle. That is, the x -coordinate at every point in the circle is the value of x in $ax^2 + ay^2 = a^2$, and the y -coordinate is the value of y .

And note that in this case, we have

$|x| \leq \sqrt{a}$, and $|y| \leq \sqrt{a}$, that is, $-\sqrt{a} \leq x \leq \sqrt{a}$, and $-\sqrt{a} \leq y \leq \sqrt{a}$.



That's not it though, because we have another case where $a < 0$.

If $a < 0$, we get $x^2 + y^2 = a < 0$, which however, is not possible, and thus, we have no numbers for x and y . Why not?

That's because $x^2 + y^2$ cannot be < 0 , since any real number squared is ≥ 0 .

Therefore, the solution to $ax^2 + ay^2 = a^2$ is as follows.

First, if $a = 0$, then x and y can be any real numbers.

Next, if $a > 0$, then x is the x -coordinate, and y is the y -coordinate of each of all the points in the circle $x^2 + y^2 = a$.

And next, if $a < 0$, there are no values for x and y .

**Suggestions or Solutions
To the Problem in the Example 4**

Solve for x and y the equation as follows: $ax^2 + ay^2 = a^3$ where a is a constant.

We can have $ax^2 + ay^2 = a^3 \Rightarrow a(x^2 + y^2) = a^3$.

So first, if $a = 0$, then x and y can take any real numbers.

Next, if $a \neq 0$, we get $a(x^2 + y^2) = a^3 \Rightarrow x^2 + y^2 = a^2$.

We know $x^2 + y^2 = a^2$ is an equation of a circle of radius a centered at the origin.

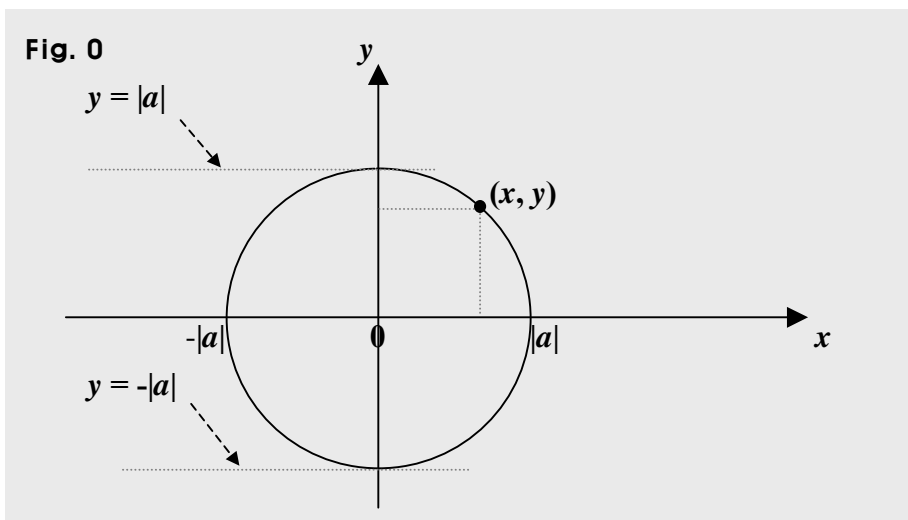
So the coordinate values of each of all the points in the circle above can be respectively the values of x and y . And in this case, note that we have

$|x| \leq |a|$, and $|y| \leq |a|$, that is, $-|a| \leq x \leq |a|$, and $-|a| \leq y \leq |a|$.

Why not just a but $|a|$, though?

In the problem definition, a is said to be just a constant.

That is, it is not the case where a has to be positive or 0 only.



So what if a is negative?

Even if a is negative, the equation of the circle above still holds since $a^2 > 0$, and thus, there is no problem with it.

Therefore, the solution to $ax^2 + ay^2 = a^3$ is as follows.

First, if $a = 0$, then x and y can be any real numbers.

Next, if $a \neq 0$, the value of x is the x -coordinate, and the value of y is the y -coordinate of each of all the points in the circle $x^2 + y^2 = a^2$.

**Suggestions or Solutions
To the Problem in the Example 5**

Solve for x and y the equation as follows: $x^2 + (y - c)^2 = 0$ where c is constant.

To begin with, if $x^2 + (y - c)^2 = 0$, both x^2 and $(y - c)^2$ have to be 0 at the same time.

Therefore, we get $x = 0$, and $y - c = 0 \Rightarrow y = c$.

Since there is no circle of radius 0, the equation does not indicate any circle at all.

What if $x^2 + a(y - c)^2 = 0$ where a and c are constant?

Then, x^2 and $a(y - c)^2$ both have to be 0 at the same time.

Thus, we need to take care of a case where $a = 0$, and another case where $a \neq 0$.

Then, we get

First, if $a \neq 0$, we get $x^2 = 0 \Rightarrow x = 0$, and $(y - c)^2 = 0 \Rightarrow y - c = 0 \Rightarrow y = c$.

So we get $x = 0$ and $y = c$.

And next, if $a = 0$, we get $x^2 = 0 \Rightarrow x = 0$, and $(y - c)^2$ can be any real number.

So x can be 0 only, but y can be any real number. Why can y be any real number?

If $(y - c)^2$ can be any real number, then $y - c$ can be any real number, too.

By the same token, if $y - c$ can be any real number, then y can be any real number, also.

What about c , though?

It is just a constant, so no matter what value c may get, it is fixed, and thus, y can get any value since $y - c$ can be any real number. Therefore, the solution to $x^2 + a(y - c)^2 = 0$ is

If $a \neq 0$, we get $x = 0$ and $y = c$.

If $a = 0$, then x can be 0 only, but y can be any real number.

Suggestions or Solutions To the Problem in the Example 6

Solve for x and y the equation $\frac{x^2}{d} + \frac{(y-c)^2}{d} = 0$ where c and d are constants.

First of all, there is no division by 0.

So we need to have $d \neq 0$ since d is the denominator in the equation given.

Next, multiplying by d both sides of the equation, we get $x^2 + (y - c)^2 = 0$.

So we get $x^2 + (y - c)^2 = 0 \Rightarrow x = 0$, and $y - c = 0$, that is, $y = c$.

Therefore, the solution is $x = 0$, and $y = c$.

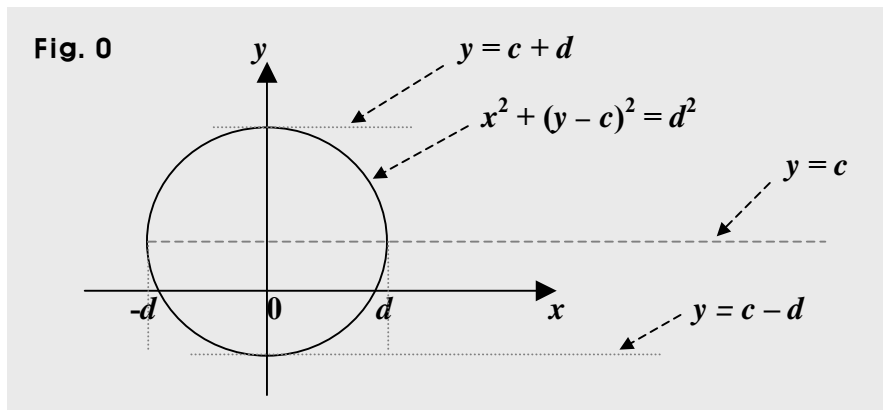
What if we have $\frac{x^2}{d} + \frac{(y-c)^2}{d} = d$?

Then, multiplying by d both sides of the equation, we get $x^2 + (y - c)^2 = d^2$, which is an equation of a circle, and the circle has a radius of d , and is centered at $(0, c)$.

So if $d > 0$, the value of x is the x -coordinate, and that of y is the y -coordinate of each of all the points in the circle above. In this case, too, however, we may want to note that

$|x| \leq |d|$, and $|y - c| \leq |d|$, that is, $-d \leq x \leq d$, and $-d \leq y - c \leq d$.

In other words, $-d \leq x \leq d$, and $c - d \leq y \leq c + d$.



What if $d < 0$?

Even if d is negative, we still get a circle, which is the same as the one above.

That's because the negative sign gets canceled out.

Suppose for instance, $d = -2$.

Then, we get $\frac{x^2}{d} + \frac{(y-c)^2}{d} = d \Rightarrow \frac{x^2}{-2} + \frac{(y-c)^2}{-2} = -2 \Rightarrow x^2 + (y-c)^2 = (-2)^2 = 4 = 2^2$

$\Rightarrow x^2 + (y-c)^2 = 2^2$, which is the equation of a circle of radius 2 centered at $(0, c)$.

**Suggestions or Solutions
To the Problem in the Example 7**

Solve for x and y the equation $(x - a)^2 + (y - b)^2 = 0$ where a and b are constant.

To begin with, if $(x - a)^2 + (y - b)^2 = 0$, we get $x - a = 0$, and $y - b = 0$ at the same time.

Therefore, the solution is that $x = a$ and $y = b$.

What if we want to solve for x , y , and z the equation as follows?

$(x - a)^2 + (y - b)^2 + (z - c)^2 = 0$ where a , b , and c are constant.

First of all, in the equation given, x , y , z , a , b , and c can have real numbers only.

So $x - a$, $y - b$, and $z - c$ are real, too, and thus, neither of $(x - a)^2$, $(y - b)^2$, and $(z - c)^2$ can be negative.

So first, if $x - a = 0$, we get $(x - a)^2 = 0$, if $y - b = 0$, we get $(y - b)^2 = 0$, and if $z - c = 0$, we get $(z - c)^2 = 0$.

Next, if $x - a \neq 0$, we get $(x - a)^2 > 0$, if $y - b \neq 0$, we get $(y - b)^2 > 0$, and if $z - c \neq 0$, we get $(z - c)^2 > 0$.

Thus, we can get three cases as follows.

- If either of $(x - a)$, $(y - b)$, and $(z - c)$ is 0, and the others are not, then either of $(x - a)^2$, $(y - b)^2$, and $(z - c)^2$ is 0, and the others are positive, so we get

$$(x - a)^2 + (y - b)^2 + (z - c)^2 > 0.$$

- If $(x - a)$, $(y - b)$, and $(z - c)$ are all non-zero, then all of $(x - a)^2$, $(y - b)^2$, and $(z - c)^2$ are positive, so we get $(x - a)^2 + (y - b)^2 + (z - c)^2 > 0$.
- If $(x - a)$, $(y - b)$, and $(z - c)$ all are 0, then all of $(x - a)^2$, $(y - b)^2$, and $(z - c)^2$ are 0, too, so we get $(x - a)^2 + (y - b)^2 + (z - c)^2 = 0$.

Thus, $(x - a)$, $(y - b)$, and $(z - c)$ all have to be 0 at the same time.

In other words, unless $(x - a)$, $(y - b)$, and $(z - c)$ all are 0 at the same time, we get $(x - a)^2 + (y - b)^2 + (z - c)^2 > 0$.

Thus, we need to have $x - a = 0$, $y - b = 0$, and $z - c = 0$.

Therefore, the solution is $x = a$, $y = b$, and $z = c$.

**Suggestions or Solutions
To the Problem in the Example 8**

Solve for x and y the equation as follows.

$w(x - u)^2 + w(y - v)^2 = 0$ where u , v , and w are constant.

We can have $w(x - u)^2 + w(y - v)^2 = 0 \Rightarrow w\{(x - u)^2 + (y - v)^2\} = 0$.

So first, if $w = 0$, then $(x - u)$ and $(y - v)$ can be any real numbers

Thus, if $(x - u)$ is any real number, we can say that $(x = u + \text{any real number})$ is any real number.

So x can be any real number.

And the same is true for y , too.

Thus, x and y can be any real number.

Next, if $w \neq 0$, then we get

$$w\{(x - u)^2 + (y - v)^2\} = 0 \Rightarrow (x - u)^2 + (y - v)^2 = 0$$

So we get $x - u = 0$ and $y - v = 0$, and thus, we get $x = u$ and $y = v$.

Therefore, the solution to $w(x - u)^2 + w(y - v)^2 = 0$ is

First, if $w = 0$, then x and y are any real numbers.

Next, if $w \neq 0$, we get $x = u$, and $y = v$.