

Examples 4 in Circles

Copyright © 2018 by Seong Ryeol Kim. All rights reserved

Click this to start.

Examples 4 in Circles

Find the circle in each case below assume the circle is in the x - y plane.

0. The circle has the center in a line $y = \frac{x}{2} + 1$, and is tangent to both coordinate axes.
1. The circle includes two points $(1, 2)$ and $(2, -1)$, and the center is in a line $y = \frac{x}{2} + 1$.
2. Two points $(0, 3)$ and $(2, -1)$ are in the circle where the center is in a line $y = x$.
3. The circle has two points $(1, 4)$ and $(3, 2)$, and the center is in a line $y = x + 1$.

Suggestions or Solutions To the Problem in the Example 0

The circle has the center in a line $y = \frac{x}{2} + 1$, and is tangent to both coordinate axes.

If a circle is tangent to both of the axes, the center is the same distance away from the two axes. So the center is in either of the two lines $y = x$ and $y = -x$.

Suppose L is the line $y = \frac{x}{2} + 1$, and C is the circle to be found.

Then, since the center of C is in L , the center is either of two points where L meets either of the two lines above.

To begin with, finding the point where the line L , $y = \frac{x}{2} + 1$ meets the line $y = -x$, we get $\frac{x}{2} + 1 = -x \Rightarrow \frac{3x}{2} + 1 = 0 \Rightarrow x = -\frac{2}{3} \Rightarrow y = -x = -(-\frac{2}{3}) = \frac{2}{3}$.

So the center of C can be $(-\frac{2}{3}, \frac{2}{3})$, and thus, the radius can be $\frac{2}{3}$.

Therefore, the equation of C can be $(x + \frac{2}{3})^2 + (y - \frac{2}{3})^2 = \frac{4}{9}$.

Next, finding the point where the line L meets the other line $y = x$, we get $\frac{x}{2} + 1 = x \Rightarrow \frac{x}{2} = 1 \Rightarrow x = 2 \Rightarrow y = x = 2$.

So the center of C can be $(2, 2)$, and thus, the radius can be 2.

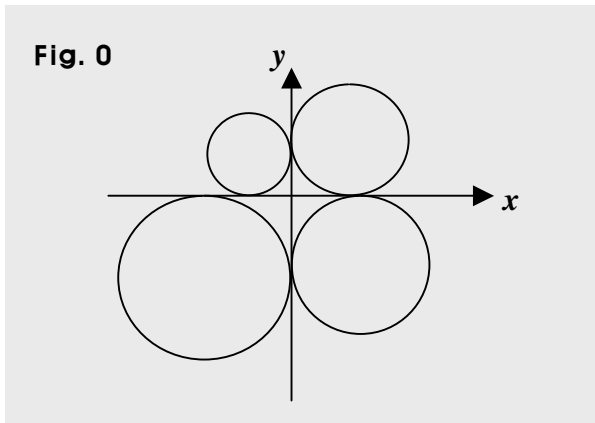
Thus, the equation of C can be $(x - 2)^2 + (y - 2)^2 = 2^2$.

Therefore, the circle C is $(x + \frac{2}{3})^2 + (y - \frac{2}{3})^2 = \frac{4}{9}$ or $(x - 2)^2 + (y - 2)^2 = 2^2$.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's begin with putting in a graph some circles tangent to both coordinate axes.

Then, we can see better where the solution is around.

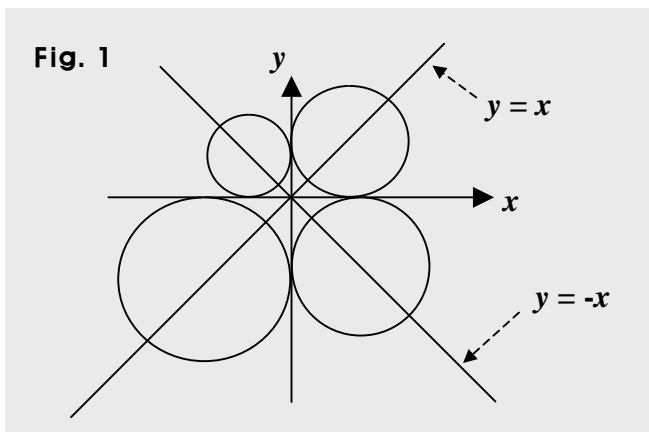


If a circle is tangent to both of the axes, the center of the circle is the same distance away from the two axes respectively. Then, where does the center have to be?

Since both of the axes are the same distance away from the center, the magnitudes of the coordinates of the center are the same as each other. So at the center, the magnitude of the x -coordinate equals that of the y -coordinate.

In short, $|y| = |x|$. In other words, we get $y = x$ or $y = -x$.

Therefore, the center of the circle can be in either of the two lines $y = x$ and $y = -x$. So let's put the two lines in the graph, too.



Suppose now, C is the circle we are after in this problem, and L is the line $y = \frac{x}{2} + 1$.

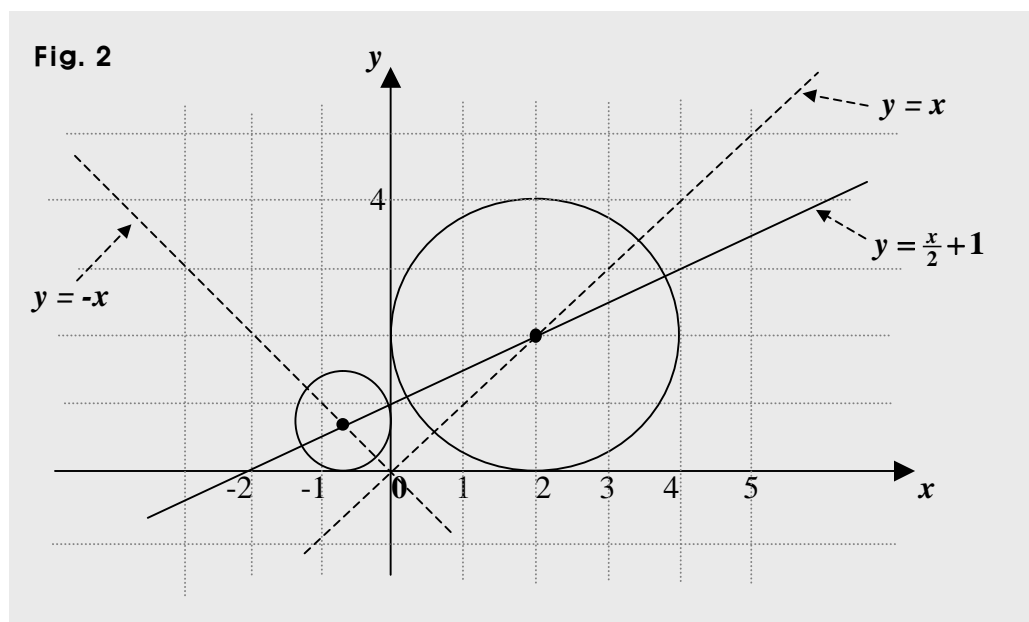
Then, the circle C is either of the four circles in the graph above, and has the center in either of the two lines $y = x$ and $y = -x$.

And also, the problem says that the line L passes through the center of C .

Thus, we can see that the line L meets either of the two lines at the center of C .

So let's add the line L to the graph above, and see how L can meet either of the two lines.

Since the y -intercept of the line L is positive, L cannot pass through the centers of the two circles below the x -axis. So we want to consider the two circles above the x -axis.



Then, we can see that the line L meets the two lines $y = x$ and $y = -x$ at two points, and that the two points can be the center of the circle C .

That is to say that the two circles above can be the circle C .

In sum, L passes through two centers of two circles tangent to both coordinate axes.

Specifically,

In one circle, the center is at the point where the line L meets the line $y = -x$.

In the other, the center is at the point where the line L meets the other line $y = x$.

So let's now find the two points where the line L meets the two lines.

To begin with, let's find the point where the line L , $y = \frac{x}{2} + 1$ meets the line $y = -x$.

Then, the circle C will be the smaller of the two in the graph above, and we get

$$\frac{x}{2} + 1 = -x \Rightarrow \frac{3x}{2} + 1 = 0 \Rightarrow x = -\frac{2}{3} \Rightarrow y = -x = -(-\frac{2}{3}) = \frac{2}{3}.$$

So the center of the circle C can be at $(-\frac{2}{3}, \frac{2}{3})$, and thus, its radius is $\frac{2}{3}$.

How is the radius $\frac{2}{3}$, though?

Since the circle C is tangent to both axes, the radius of C is the same as the magnitude of either of the two coordinates at the center, which is the point where the line L meets the line $y = -x$. Why either of the two?

The center is a point in the line $y = -x$, where every point has a pair of coordinates of the same magnitude. For instance, $(1, -1)$, $(-1, 1)$, $(3, -3)$, $(-5, 5)$, etc. are in the line $y = -x$.

Therefore, the equation of the circle C can be $\{x - (-\frac{2}{3})\}^2 + (y - \frac{2}{3})^2 = (\frac{2}{3})^2$.

Next, let's find the point where the line L , $y = \frac{x}{2} + 1$ meets the other line $y = x$.

Then, the circle C will be the larger of the two circles in the graph above, and we get

$$\frac{x}{2} + 1 = x \Rightarrow \frac{x}{2} = 1 \Rightarrow x = 2 \Rightarrow y = x = 2.$$

Thus, the center of C can be $(2, 2)$, and thus, its radius can be 2.

So the equation of the circle C can be $(x - 2)^2 + (y - 2)^2 = 2^2$.

Therefore, the solution is $(x + \frac{2}{3})^2 + (y - \frac{2}{3})^2 = \frac{4}{9}$ or $(x - 2)^2 + (y - 2)^2 = 2^2$.

In short:

If a circle is tangent to both of the axes, the center is the same distance away from the two axes. So the center is in either of the two lines $y = x$ and $y = -x$.

Suppose L is the line $y = \frac{x}{2} + 1$, and C is the circle to be found.

Then, since the center of C is in L , the center is either of two points where L meets either of the two lines above.

To begin with, finding the point where the line L , $y = \frac{x}{2} + 1$ meets the line $y = -x$, we get

$$\frac{x}{2} + 1 = -x \Rightarrow \frac{3x}{2} + 1 = 0 \Rightarrow x = -\frac{2}{3} \Rightarrow y = -x = -(-\frac{2}{3}) = \frac{2}{3}.$$

So the center of C can be $(-\frac{2}{3}, \frac{2}{3})$, and thus, the radius can be $\frac{2}{3}$.

Therefore, the equation of C can be $(x + \frac{2}{3})^2 + (y - \frac{2}{3})^2 = \frac{4}{9}$.

Next, finding the point where the line L meets the other line $y = x$, we get

$$\frac{x}{2} + 1 = x \Rightarrow \frac{x}{2} = 1 \Rightarrow x = 2 \Rightarrow y = x = 2.$$

So the center of C can be $(2, 2)$, and thus, the radius can be 2.

Thus, the equation of C can be $(x - 2)^2 + (y - 2)^2 = 2^2$.

Therefore, the circle C is $(x + \frac{2}{3})^2 + (y - \frac{2}{3})^2 = \frac{4}{9}$ or $(x - 2)^2 + (y - 2)^2 = 2^2$.

Suggestions or Solutions
To the Problem in the Example 1

The circle includes two points (1, 2) and (2, -1), and the center is in a line $y = \frac{x}{2} + 1$.

Suppose C is the circle to be found, (s, t) is the center, and D is the distance from the center of C to each of the two points given.

Then, we get $D^2 = (s - 1)^2 + (t - 2)^2$, and $D^2 = (s - 2)^2 + \{t - (-1)\}^2 = (s - 2)^2 + (t + 1)^2$.

So we get $(s - 1)^2 + (t - 2)^2 = (s - 2)^2 + (t + 1)^2$.

Thus, we get $(s - 1)^2 - (s - 2)^2 = (t + 1)^2 - (t - 2)^2$

$\Rightarrow \{(s - 1) + (s - 2)\}\{(s - 1) - (s - 2)\} = \{(t + 1) + (t - 2)\}\{(t + 1) - (t - 2)\}$

$\Rightarrow (2s - 3)1 = 2s - 3 = (2t - 1)3 = 6t - 3 \Rightarrow 2s = 6t \Rightarrow s = 3t$.

We know that the center (s, t) is in the given line $y = \frac{x}{2} + 1$. So we get $t = \frac{s}{2} + 1$.

Thus, we get $s = 3t \Rightarrow t = \frac{s}{2} + 1 = \frac{3t}{2} + 1 \Rightarrow t = \frac{3t}{2} + 1 \Rightarrow \frac{t}{2} = -1 \Rightarrow t = -2 \Rightarrow s = -6$.

Therefore, the center of C is $(-6, -2)$.

Since a radius is the distance from the center to a point in a circle, the radius is D .

$D^2 = (s - 1)^2 + (t - 2)^2 = (-6 - 1)^2 + (-2 - 2)^2 = 49 + 16 = 65$.

Therefore, the circle C is $(x + 6)^2 + (y + 2)^2 = 65$.

If not quite sure of the idea behind the processes above, follow the steps below.

Basically, finding a circle, we need the center and radius. Then, using the standard form, we can get the circle. What about the general form, then?

Using the general form, we basically need three points in the circle. We are given only two points only though. Thus, we may want to try the standard form.

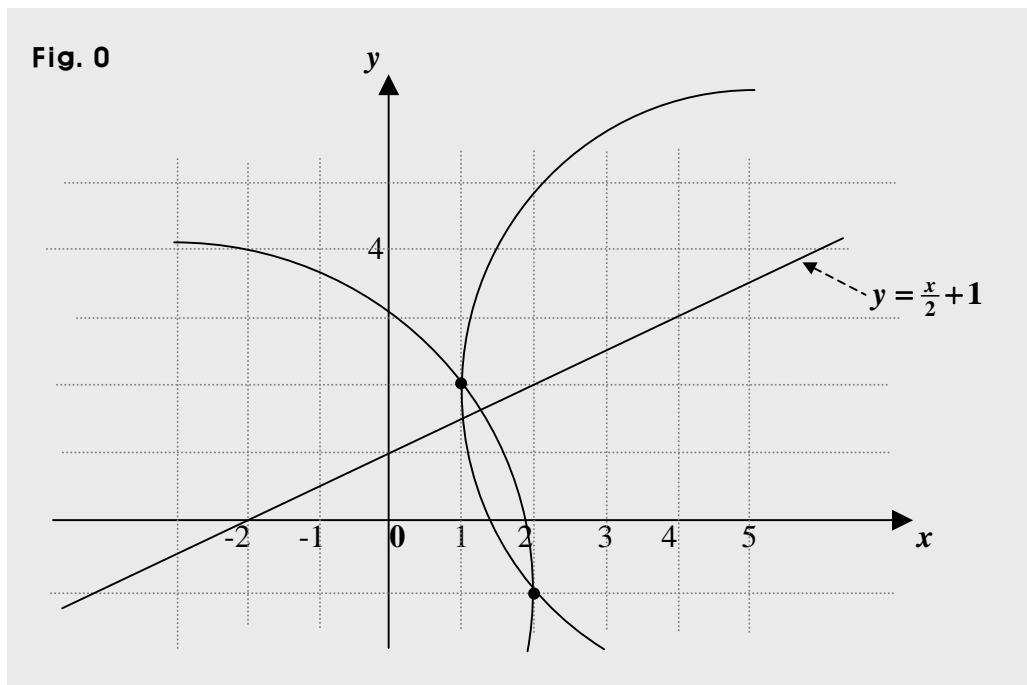
So let's get the center and radius. Then to begin with, where is the radius?

Every point in a circle is the radius away from the center.

So the radius is between the center and either of the two points $(1, 2)$ and $(2, -1)$.

We have only the whereabouts of the center, though, which is somewhere in the line $y = \frac{x}{2} + 1$. Where then, exactly is it?

Let's put in a graph the line and the two points, together with some candidates for the circle we are after. Such a candidate doesn't have to be a complete circle. Just some part of a circle can help, too. Then, we can see better where the solution can be around.



Now, whereabouts does the center of the circle have to be?

It seems to be somewhere in the third quadrant, and doesn't seem to be in the first.

Anyway, it's not quite easy to see it. Then, what should we do?

Getting stuck, we may want to get back to the basics, which is the definition for circles.

The definition says these:

- Each of all the points in a circle is the same distance away from the center.
- The same distance is the radius, of course.

So each of the two points given in the problem needs to be the same distance away from the center, too. What then do we mean by the same distance?

It means the radius, of course. What else can we think of, though?

The distance from the center to one of the two points is the same as that from the center to the other point. So what?

We can find the center by means of the famous distance formula.

Suppose now, C is the circle we are after, (s, t) is the center of C , and D is the distance from the center to either of the two points given.

Then, beginning with the distance between the center and the point $(1, 2)$, we get

$$D^2 = (s - 1)^2 + (t - 2)^2.$$

Next is the distance between the center and the point $(2, -1)$.

$$D^2 = (s - 2)^2 + \{t - (-1)\}^2 = (s - 2)^2 + (t + 1)^2.$$

Now, we know the two distances above have to be the same.

$$\text{So we get } (s - 1)^2 + (t - 2)^2 = (s - 2)^2 + (t + 1)^2.$$

Meanwhile, we have a factorization identity, $A^2 - B^2 = (A + B)(A - B)$.

So taking advantage of it, we get

$$\begin{aligned}
(s-1)^2 - (s-2)^2 &= (t+1)^2 - (t-2)^2 \\
\Rightarrow \{(s-1) + (s-2)\}\{(s-1) - (s-2)\} &= \{(t+1) + (t-2)\}\{(t+1) - (t-2)\} \\
\Rightarrow (2s-3)1 = 2s-3 = (2t-1)3 = 6t-3 &\Rightarrow 2s = 6t \Rightarrow s = 3t.
\end{aligned}$$

Now, we have one equation, but we have two unknowns. What then, about the other equation?

The problem says that the center is in the line $y = \frac{x}{2} + 1$.

That is, the center (s, t) is in the line $L, y = \frac{x}{2} + 1$. So we get $t = \frac{s}{2} + 1$.

$$\text{So we get } s = 3t \Rightarrow t = \frac{s}{2} + 1 = \frac{3t}{2} + 1 \Rightarrow t = \frac{3t}{2} + 1 \Rightarrow \frac{t}{2} = -1 \Rightarrow t = -2 \Rightarrow s = -6.$$

Therefore, the center is $(-6, -2)$, which is in the third quadrant.

Now, what's left is the radius.

Since a radius is the distance from the center to a point in a circle, the radius is D . So we can make use of D^2 , which has been set above, and is as follows.

$$D^2 = (s-1)^2 + (t-2)^2 = (-6-1)^2 + (-2-2)^2 = 49 + 16 = 65.$$

Now, we have the radius squared, and the center.

So we can use the standard form, and put the circle C in such a way as follows.

$$(x+6)^2 + (y+2)^2 = 65.$$

By the way, we can find the center a bit differently, too.

To begin with, the center (s, t) is in the line $y = \frac{x}{2} + 1$, so we get $t = \frac{s}{2} + 1$.

Therefore, the center can be put this way: $(s, \frac{s}{2} + 1)$.

Next, the distance D from the center to $(1, 2)$ is the same as that from the center to $(2, -1)$.

So we get $D^2 = (s-1)^2 + (\frac{s}{2} + 1 - 2)^2 = (s-1)^2 + (\frac{s}{2} - 1)^2$, and also,

$$D^2 = (s-2)^2 + \{\frac{s}{2} + 1 - (-1)\}^2 = (s-2)^2 + (\frac{s}{2} + 2)^2.$$

Therefore, we get $(s-1)^2 + (\frac{s}{2}-1)^2 = (s-2)^2 + (\frac{s}{2}+2)^2$.

Then again, we get to use the identity, $A^2 - B^2 = (A+B)(A-B)$.

(The identity above is used quite frequently, and often convenient, so we may want to put it in our memory. We should be able to know how to get it, too, though.)

So we get $(s-1)^2 - (s-2)^2 + (\frac{s}{2}-1)^2 - (\frac{s}{2}+2)^2 = 0$

$\Rightarrow \{(s-1) + (s-2)\}\{(s-1) - (s-2)\} + \{(\frac{s}{2}-1) + (\frac{s}{2}+2)\}\{(\frac{s}{2}-1) - (\frac{s}{2}+2)\} = 0$

$\Rightarrow (2s-3)1 + (s+1)(-3) = 2s-3-3s-3 = -s-6 = 0 \Rightarrow s = -6$.

Thus, we get $s = -6 \Rightarrow t = \frac{s}{2} + 1 = -\frac{6}{2} + 1 = -2$.

Therefore, the center is **(-6, -2)**.

In short:

Suppose C is the circle to be found, (s, t) is the center, and D is the distance from the center of C to each of the two points given.

Then, we get $D^2 = (s-1)^2 + (t-2)^2$, and $D^2 = (s-2)^2 + \{t-(-1)\}^2 = (s-2)^2 + (t+1)^2$.

So we get $(s-1)^2 + (t-2)^2 = (s-2)^2 + (t+1)^2$.

Thus, we get $(s-1)^2 - (s-2)^2 = (t+1)^2 - (t-2)^2$

$\Rightarrow \{(s-1) + (s-2)\}\{(s-1) - (s-2)\} = \{(t+1) + (t-2)\}\{(t+1) - (t-2)\}$

$\Rightarrow (2s-3)1 = 2s-3 = (2t-1)3 = 6t-3 \Rightarrow 2s = 6t \Rightarrow s = 3t$.

We know that the center (s, t) is in the given line $y = \frac{x}{2} + 1$. So we get $t = \frac{s}{2} + 1$.

Thus, we get $s = 3t \Rightarrow t = \frac{s}{2} + 1 = \frac{3t}{2} + 1 \Rightarrow t = \frac{3t}{2} + 1 \Rightarrow \frac{t}{2} = -1 \Rightarrow t = -2 \Rightarrow s = -6$.

Therefore, the center of C is **(-6, -2)**.

Since a radius is the distance from the center to a point in a circle, the radius is D .

$D^2 = (s-1)^2 + (t-2)^2 = (-6-1)^2 + (-2-2)^2 = 49 + 16 = 65$.

Therefore, the circle C is $(x+6)^2 + (y+2)^2 = 65$.

**Suggestions or Solutions
To the Problem in the Example 2**

Two points (0, 3) and (2, -1) are in the circle where the center is in a line $y = x$.

Suppose the radius is R , and since the center is in the line $y = x$, the center is (a, a) .

Then, $R^2 = (a - 0)^2 + (a - 3)^2$, and also, $R^2 = (a - 2)^2 + (a + 1)^2$.

So we get

$$\begin{aligned} a^2 + (a - 3)^2 &= (a - 2)^2 + (a + 1)^2 \Rightarrow a^2 - (a - 2)^2 + (a - 3)^2 - (a + 1)^2 = 0 \\ &\Rightarrow \{a + (a - 2)\}\{a - (a - 2)\} + \{(a - 3) + (a + 1)\}\{(a - 3) - (a + 1)\} \\ &= 2(2a - 2) + (-4)(2a - 2) = 4a - 4 - 8a + 8 = -4a + 4 = 0 \Rightarrow a = 1. \end{aligned}$$

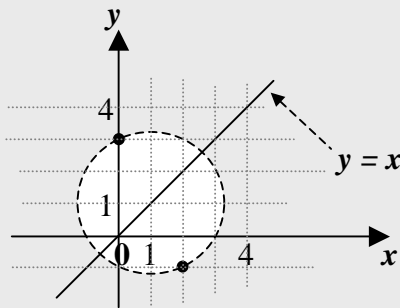
Thus, $R^2 = (a - 2)^2 + (a + 1)^2 = (1 - 2)^2 + (1 + 1)^2 = 1 + 4 = 5$.

Therefore, the circle is $(x - 1)^2 + (y - 1)^2 = 5$.

If not quite sure of the idea behind the processes above, follow the steps below.

Putting the problem in a graph, we can see better the solution's whereabouts, and reduce mistakes capturing the solution. So let's begin with putting in a graph the two points and the line given.

Fig. 0



Then, even looking at the graph only, we can readily see that the center of the circle we are after is $(1, 1)$.

Besides, by the distance formula, we can quickly see that **(the radius)² = $1^2 + 2^2 = 5$** .

Giving a name to an object in math, we can work with it conveniently.

Thus next, assuming C is the circle to be found, by means of the standard form, we can put the circle C in an equation where $(x - 1)^2 + (y - 1)^2 = 5$.

So if the problem is multiple-choice, we can get to the solution fast by simply putting the problem in a graph. What if the equation has to be in the general form, though?

Expanding the one in the standard form and simplifying the result, we can readily get it.

What if the problem is asking not only the circle but the procedure to get it, too, though?

Then, we need to show that the center is $(1, 1)$, and how to get the radius, too.

So let's see now, how to get the center and the radius analytically. What do we mean by 'analytically'?

We mean 'by calculations' if you will. That is, we get the solution by algebra.

We know all the points in a circle are respectively the same distance, that is, the radius away from the center, so the two points given have to be the radius away from the center, also. What then, do we want to take advantage of?

The distance formula, called Pythagorean theorem, can help us find the center and radius. So we can find them using the formula, together with the two points given.

Suppose that the radius of the circle C is R , and that since the center is in the line $y = x$, the center is (a, a) , where a is constant, of course.

Then, beginning with the distance from (a, a) to $(0, 3)$, we get $R^2 = (a - 0)^2 + (a - 3)^2$.

Next, we have another point $(2, -1)$ in the circle C , so we get $R^2 = (a - 2)^2 + (a + 1)^2$.

Thus, we get $a^2 + (a - 3)^2 = (a - 2)^2 + (a + 1)^2$.

Meanwhile, we have a factorization identity, where $A^2 - B^2 = (A + B)(A - B)$.

So we get

$$\begin{aligned} a^2 + (a - 3)^2 &= (a - 2)^2 + (a + 1)^2 \Rightarrow a^2 - (a - 2)^2 + (a - 3)^2 - (a + 1)^2 = 0 \\ &\Rightarrow \{a + (a - 2)\}\{a - (a - 2)\} + \{(a - 3) + (a + 1)\}\{(a - 3) - (a + 1)\} \\ &= 2(2a - 2) + (-4)(2a - 2) = 4a - 4 - 8a + 8 = -4a + 4 = 0 \Rightarrow a = 1. \end{aligned}$$

Next, using one of the two expressions for R above, we get the radius as follows.

$$R^2 = (a - 2)^2 + (a + 1)^2 = (1 - 2)^2 + (1 + 1)^2 = 1 + 4 = 5.$$

So the center is $(1, 1)$, and the radius is 5.

Therefore, the circle C is $(x - 1)^2 + (y - 1)^2 = 5$.

In short:

Suppose the radius is R , and since the center is in the line $y = x$, the center is (a, a) .

Then, $R^2 = (a - 0)^2 + (a - 3)^2$, and also, $R^2 = (a - 2)^2 + (a + 1)^2$.

So we get

$$\begin{aligned} a^2 + (a - 3)^2 &= (a - 2)^2 + (a + 1)^2 \Rightarrow a^2 - (a - 2)^2 + (a - 3)^2 - (a + 1)^2 = 0 \\ &\Rightarrow \{a + (a - 2)\}\{a - (a - 2)\} + \{(a - 3) + (a + 1)\}\{(a - 3) - (a + 1)\} \\ &= 2(2a - 2) + (-4)(2a - 2) = 4a - 4 - 8a + 8 = -4a + 4 = 0 \Rightarrow a = 1. \end{aligned}$$

Thus, $R^2 = (a - 2)^2 + (a + 1)^2 = (1 - 2)^2 + (1 + 1)^2 = 1 + 4 = 5$.

Therefore, the circle is $(x - 1)^2 + (y - 1)^2 = 5$.

**Suggestions or Solutions
To the Problem in the Example 3**

The center of the circle is in a line $y = x + 1$, and $(1, 4)$ and $(3, 2)$ are in the circle.

Suppose that the radius is R , and that the center is (a, b) where a and b are constant.

Then, $R^2 = (a - 1)^2 + (b - 4)^2$, and also, $R^2 = (a - 3)^2 + (b - 2)^2$. So we get

$$\begin{aligned} (a - 1)^2 + (b - 4)^2 &= (a - 3)^2 + (b - 2)^2 \Rightarrow (a - 1)^2 - (a - 3)^2 + (b - 4)^2 - (b - 2)^2 = 0 \\ \Rightarrow \{(a - 1) + (a - 3)\}\{(a - 1) - (a - 3)\} + \{(b - 4) + (b - 2)\}\{(b - 4) - (b - 2)\} \\ &= 2(2a - 4) - 2(2b - 6) = 4a - 8 - 4b + 12 = 4a - 4b + 4 = 0 \Rightarrow a - b + 1 = 0. \end{aligned}$$

Besides, the center is in the line $y = x + 1$, so we get $b = a + 1$, which however, is the same as $a - b + 1 = 0$. Therefore, each and every point in the line given can be the center, and all the circles share the two given points. The center is (a, b) , and we have $b = a + 1$.

So the center can be put in $(a, a + 1)$ where a is constant. Thus, the radius is as follows.

$$\begin{aligned} R^2 &= (a - 3)^2 + (b - 2)^2 = (a - 3)^2 + (a + 1 - 2)^2 = (a - 3)^2 + (a - 1)^2 = 2a^2 - 8a + 10 \\ &= 2(a^2 - 4a) + 10 = 2(a^2 - 4a + 4 - 4) + 10 = 2(a - 2)^2 + 2. \end{aligned}$$

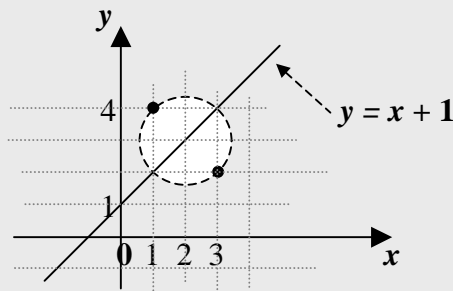
Therefore, the circle is $(x - a)^2 + (y - a - 1)^2 = 2(a - 2)^2 + 2$ where a is a constant.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's put in a graph the given line and the two points given.

Then, we can see better where the solution has to be around.

Fig. 0



Now, in this example, too, even merely seeing the graph only, we can see that the center can be $(2, 3)$, and that the radius can be $\sqrt{2}$, so the circle can be $(x - 2)^2 + (y - 3)^2 = 2$.

It can be the circle we want, of course. Is that it, though?

It is not the only circle that can be the solution. That is, another circle can be centered at a point in the line and passing through the two points. In fact, we can get infinitely many of such circles. So this time, some analytic approach to the center and radius is needed.

- Finding the center and radius, we can put the circle in the standard form.
- All the points in a circle are respectively the same distance, that is, the radius away from the center. So the two points given are respectively the radius away from the center.
- We can find the distance between two points by means of the distance formula.

Suppose now, C is the circle to be found, the radius is R , and the center is (a, b) .

Then, beginning with the distance from (a, b) to $(1, 4)$, we get $R^2 = (a - 1)^2 + (b - 4)^2$.

Next, we have another point $(3, 2)$ in the circle C , so we get $R^2 = (a - 3)^2 + (b - 2)^2$.

Meanwhile, we have $A^2 - B^2 = (A + B)(A - B)$. So we get

$$\begin{aligned} (a - 1)^2 + (b - 4)^2 &= (a - 3)^2 + (b - 2)^2 \Rightarrow (a - 1)^2 - (a - 3)^2 + (b - 4)^2 - (b - 2)^2 = 0 \\ &\Rightarrow \{(a - 1) + (a - 3)\}\{(a - 1) - (a - 3)\} + \{(b - 4) + (b - 2)\}\{(b - 4) - (b - 2)\} \\ &= 2(2a - 4) - 2(2b - 6) = 4a - 8 - 4b + 12 = 4a - 4b + 4 = 0 \Rightarrow a - b + 1 = 0. \end{aligned}$$

Also, the center (a, b) is in the line $y = x + 1$. Therefore, we get $b = a + 1$.

So we now have $a - b + 1 = 0$ and $b = a + 1$, which however, are the same as each other.

Finding a and b , we need a system of two equations for a and b , and the two equations have to be different, of course. We have only one equation for a and b , though.

Where then, is the other?

It doesn't exist, which means, there can be many circles, each of which can be the circle C . There are in fact, infinitely many of such circles. Why?

We have ended up with only one equation $b = a + 1$, so the solution to that equation is any pair of values can satisfy " $b = a + 1$."

For instance, $(a, b) = (1, 2), (1.1, 2.1), (-2, -1)$, etc.

So there can be infinitely many values can be assigned to a and b . In fact, each and every point in the line $y = x + 1$ is the center of the circle C . Why?

The point (a, b) is the center of C , a is the x -coordinate, and b is the y -coordinate.

The solution to the equation $b = a + 1$ is the same as the one to the equation $y = x + 1$, which indicates a line, so the coordinates of every point in the line satisfy the equation where $y = x + 1$.

So the solution is not a particular circle but a group of infinitely many circles, and every point in the line $y = x + 1$ is the center of each circle in the group.

Of course, each of all the circles passes through the two given points, $(1, 4)$ and $(3, 2)$.

So all the circles meet each other altogether at the two given points, after all.

That is to say that all the circles share altogether the two points given.

Now, how do we put in an equation the group of such infinitely many circles?

We first, get the radius, and then, use the standard form. What then, about the center?

The center has already been chosen to be (a, b) , so we have $b = a + 1$ since the center is in the line $y = x + 1$. Thus, the center can also, be put in $(a, a + 1)$ where a is constant.

Now, since the point $(3, 2)$ is in the circle C , finding the distance from the center to the point $(3, 2)$, we get the radius.

To begin with, we get

$$R^2 = (a - 3)^2 + (b - 2)^2 = (a - 3)^2 + (a + 1 - 2)^2 \text{ since } b = a + 1.$$

So we get

$$\begin{aligned} R^2 &= (a - 3)^2 + (a - 1)^2 = 2a^2 - 8a + 10 = 2(a^2 - 4a) + 10 \\ &= 2(a^2 - 4a + 4 - 4) + 10 = 2(a - 2)^2 + 2. \end{aligned}$$

Next, using the standard form, we get $(x - a)^2 + \{y - (a + 1)\}^2 = 2(a - 2)^2 + 2$ where a is constant. So the circle C is a circle indicated by the equation below for each value of a .

$$(x - a)^2 + (y - a - 1)^2 = 2(a - 2)^2 + 2 \text{ where } a \text{ is constant.}$$

For each value of a , we get one circle by means of the equation above, and the circle passes through the two points $(1, 4)$ and $(3, 2)$, and is centered at a point in the line.

So the equation above represents a group of circles passing through the two points.

We can notice in this equation, that when $a = 2$, the radius is the smallest, so the circle is the smallest when $a = 2$. Thus, the smallest circle is centered at $(2, 3)$, and has a radius of $\sqrt{2}$. Let's now, put in a graph some of the circles in the group.

Fig. 1

