Examples 5 in Circles

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Find the circle in each case below assume the circle is in the ***x-y*** plane.

0. The circle passes through **(1, 4)** and **(3, 2)**, and the center is in a line $x = 0$.

1. Two points **(1, 3)** and **(2, -1)** are in the circle where the center is in a line $y = 0$.

2. The center of the circle is in a line $y = 1 - x$, and a point **(1, 3)** is in the circle.

3. The circle has **(1, 3)**, and is tangent to the ***x***-axis, and the center is in a line $y = 1 - x$.

4. The circle has **(1, 3)**, and is tangent to the ***y***-axis, and the center is in a line $y = 1 - x$.

Suggestions or Solutions
To the Problem in the Example 0

The circle passes through (1, 4) and (3, 2), and the center is in a line *x* = 0.

Suppose that $\boldsymbol{R}$ is the radius, and that the center is $(0, b)$, where b is constant since the center is in the y-axis.

Then, $R^2 = (0 - 1)^2 + (b - 3)^2$, and also, $R^2 = (0 - 2)^2 + (b + 1)^2$. So we get

$$(0 - 1)^2 + (b - 3)^2 = (0 - 2)^2 + (b + 1)^2 \Rightarrow 1 + (b - 3)^2 = 4 + (b + 1)^2$$
$$\Rightarrow (b - 3)^2 - (b + 1)^2 - 3 = 0 \Rightarrow b^2 - 6b + 9 - b^2 - 2b - 1 - 3 = -8b + 5 = 0 \Rightarrow b = \tfrac{5}{8}.$$

Thus, the center is $(0, \frac{5}{8})$, and we get

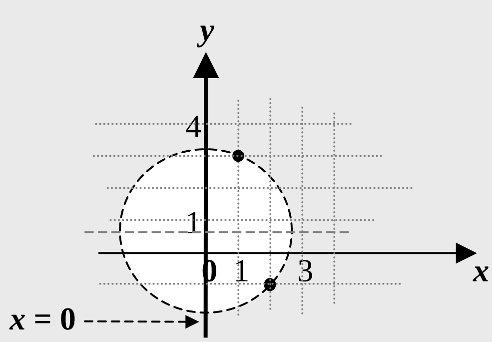
$$R^2 = 4 + (b+1)^2 = 4 + (\tfrac{5}{8} + 1)^2 = 4 + (\tfrac{13}{8})^2 = 4 + \tfrac{169}{64} = \tfrac{256+169}{64} = \tfrac{425}{64}.$$

Therefore, the circle is $x^2 + (y - \frac{5}{8})^2 = (\frac{5\sqrt{17}}{8})^2$.

If not quite sure of the idea behind the processes above, follow the steps below.

Putting in a graph the line and two points given, together with a probable circle, we can see better where the solution can be around. The line $x = 0$ is the y-axis. It's quite easy to put a probable circle in the graph since the center of the circle to be found is in the y-axis.

Fig. 0



Even looking at the graph only, we can see that the center should be somewhere between 0 and 1 in the ***y***-axis. So the ***x***-coordinate at the center is **0**, and **0** < the ***y***-coordinate < **1**.

Let's now, begin with finding precisely where the center is.

The definition for circles says the two points **(1, 3)** and **(2, -1)** are respectively the same distance away from the center, and the same distance is the radius.

Suppose now, that ***C*** is the circle we are after, that ***b*** is constant, that the center of the circle ***C*** is **(0, *b*)** since it is in the ***y***-axis, and that ***R*** is the radius.

Then, beginning with the distance from **(0, *b*)** to **(1, 3)**, we get $R^2 = (0 - 1)^2 + (b - 3)^2$.

Next, we have another point **(2, -1)** in the circle ***C***, so we get $R^2 = (0 - 2)^2 + (b + 1)^2$.

Thus, we get $(0 - 1)^2 + (b - 3)^2 = (0 - 2)^2 + (b + 1)^2 \Rightarrow 1 + (b - 3)^2 = 4 + (b + 1)^2$

$\Rightarrow (b - 3)^2 - (b + 1)^2 - 3 = 0 \Rightarrow b^2 - 6b + 9 - b^2 - 2b - 1 - 3 = -8b + 5 = 0 \Rightarrow b = \frac{5}{8}$.

Therefore, the center of the circle ***C*** is $(0, \frac{5}{8})$.

Next, since we now have the value of ***b***, the radius ***R*** is as follows.

$$R^2 = 4 + (b+1)^2 = 4 + (\tfrac{5}{8} + 1)^2 = 4 + (\tfrac{13}{8})^2 = 4 + \tfrac{169}{64} = \tfrac{256+169}{64} = \tfrac{425}{64}.$$

Therefore, using the standard form, we can see that the circle ***C*** is as follows.

$$x^2 + (y - \tfrac{5}{8})^2 = \tfrac{425}{64} = (\tfrac{5\sqrt{17}}{8})^2.$$

Suggestions or Solutions
To the Problem in the Example 1

Two points (1, 3) and (2, -1) are in the circle where the center is in a line $y = 0$.

Suppose R is the radius, and the center is $(a, 0)$ where a is constant.

Then, $R^2 = (a - 1)^2 + (0 - 3)^2$, and also, $R^2 = (a - 2)^2 + (0 + 1)^2$.

So we get $(a - 1)^2 + (0 - 3)^2 = (a - 2)^2 + (0 + 1)^2 \Rightarrow 9 + (a - 1)^2 = 1 + (a - 2)^2$

$\Rightarrow (a - 1)^2 - (a - 2)^2 + 8 = 0 \Rightarrow a^2 - 2a + 1 - a^2 + 4a - 4 + 8 = 2a + 5 = 0 \Rightarrow a = -\frac{5}{2}$.

Thus, the center is $(-\frac{5}{2}, 0)$, so we get

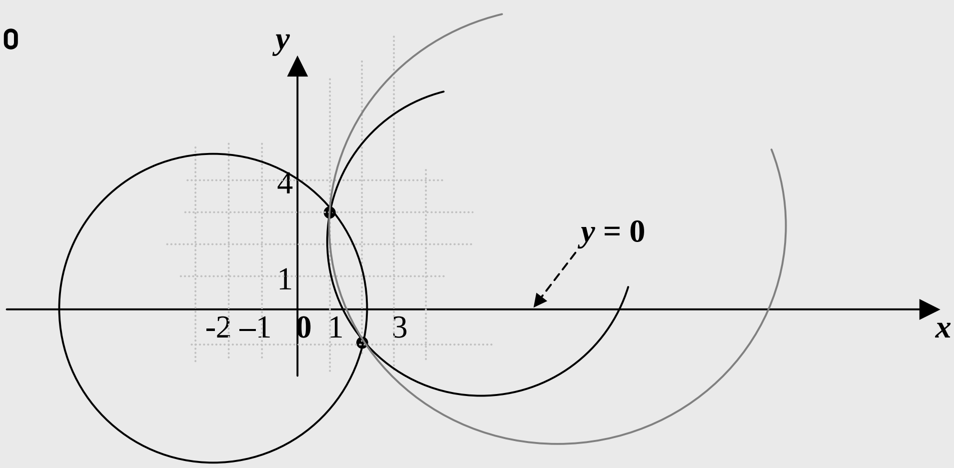
$R^2 = (a - 2)^2 + (0 + 1)^2 = (-\frac{5}{2} - 2)^2 + 1 = (-\frac{9}{2})^2 + 1 = \frac{81}{4} + 1 = \frac{85}{4}$.

Therefore, the circle is $(x + \frac{5}{2})^2 + y^2 = \frac{85}{4}$.

If not quite sure of the idea behind the processes above, follow the steps below.

Making the problem visible, we can see better the solution's whereabouts. So we may want to begin with putting in a graph the line and two points given, together with some probable circles. The line is $y = 0$, which is the x-axis itself, so the center is in the x-axis.

Fig. 0



Looking at the graph, we can see that the center should be on the left of the origin in the ***x***-axis, and quite clearly see that the center cannot be on the right of the origin.

It looks like the center should be somewhere between -3 and -2 in the ***x***-axis.

So let's now find exactly where the center is positioned.

Suppose that ***C*** is the circle we want, that ***R*** is the radius of the circle ***C***, and that since the center is in the ***x***-axis, the center is $(a, 0)$ where ***a*** is constant.

By the definition for circles, the two points **(1, 3)** and **(2, -1)** are respectively the radius away from the center.

So beginning with the distance from the center to **(1, 3)**, we get $R^2 = (a - 1)^2 + (0 - 3)^2$, and thus, finding ***a***, we can get the radius ***R***, too.

Next, we have another point **(2, -1)** in the circle ***C***, so we get $R^2 = (a - 2)^2 + (0 + 1)^2$.

Thus, we get $(a - 1)^2 + (0 - 3)^2 = (a - 2)^2 + (0 + 1)^2 \Rightarrow 9 + (a - 1)^2 = 1 + (a - 2)^2$

$\Rightarrow (a - 1)^2 - (a - 2)^2 + 8 = 0 \Rightarrow a^2 - 2a + 1 - a^2 + 4a - 4 + 8 = 2a + 5 = 0 \Rightarrow a = -\frac{5}{2}$.

Therefore, the center of the circle ***C*** is $(-\frac{5}{2}, 0)$.

Next, now that we have found ***a***, the radius ***R*** is as follows.

$R^2 = (a - 2)^2 + (0 + 1)^2 = (-\frac{5}{2} - 2)^2 + 1 = (-\frac{9}{2})^2 + 1 = \frac{81}{4} + 1 = \frac{85}{4} = (\frac{\sqrt{85}}{2})^2 \approx 4.61^2$.

Therefore, using the standard form, we can see that the circle ***C*** is $(x + \frac{5}{2})^2 + y^2 = \frac{85}{4}$.

Suggestions or Solutions
To the Problem in the Example 2

The center of the circle is in a line $y = 1 - x$, and a point (1, 3) is in the circle.

Suppose now, C is the circle to be found, the center of the circle C is (a, b), where a and b are constant, and the radius is R.

Then, first, (a, b) is in the line $y = 1 - x$, so we get $b = 1 - a$, and $(a, b) = (a, 1 - a)$.

Next, the point **(1, 3)** is the circle C, so it is R away from the center.

So we get

$$R^2 = (a - 1)^2 + (b - 3)^2 = (a - 1)^2 + (1 - a - 3)^2 = (a - 1)^2 + (-a - 2)^2$$

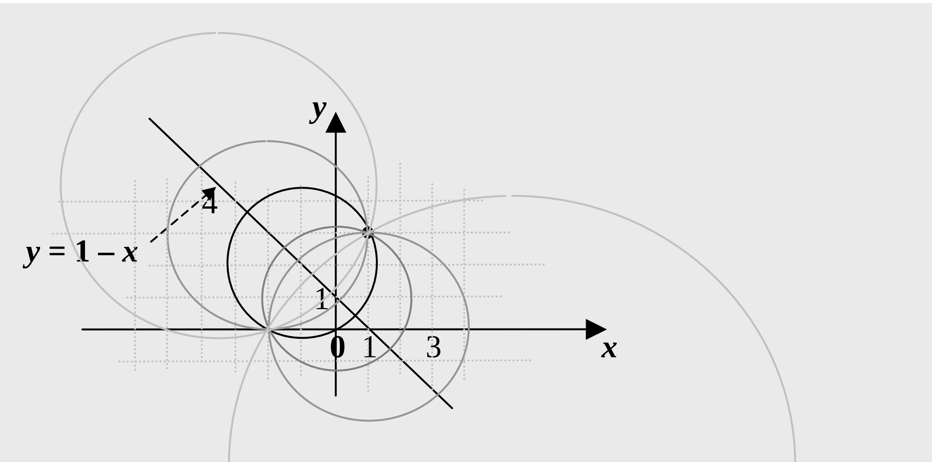
$$= (a - 1)^2 + (a + 2)^2 = 2a^2 + 2a + 5.$$

Therefore, the circle C is $(x - a)^2 + (y - 1 + a)^2 = 2a^2 + 2a + 5$, where a is constant.

If not quite sure of the idea behind the processes above, follow the steps below.

To begin with, let's put in a graph the line and point given, along with some probable circles. Then, we can see better where the solution can be around, and quite frequently, we can even see the solution right away, or in just a few steps

Fig. 0



Now, we can see not just one but many circles, too, can be the solution. In fact, it can be infinitely many. So we are looking for not one particular circle but a group of circles.

Every circle in the group passes through the point **(1, 3)**, but is centered at a different point, which however, is always in the line $\boldsymbol{y = 1 - x}$. So we can put it the way below.

• As a circle moves along the line $\boldsymbol{y = 1 - x}$ keeping its center in the line, the circle shrinks or expands, but always passes through the point **(1, 3)**.

Besides, every circle in the group seems to pass through another particular point, which is on the other sides of the line $\boldsymbol{y = 1 - x}$, and the particular point looks like **(-2, 0)**.

Suppose now, ***C*** is the circle to be found, the center of the circle ***C*** is $\boldsymbol{(a, b)}$, where ***a*** and ***b*** are constant, and the radius is ***R***. Then, we want to find two things. What are the two?

One is the radius, and the other is the center, of course. However, we don't really want to find the values of ***a*** and ***b***, do we? That's because, there can be infinitely many circles that can be the solution. What then, do we need to do with ***a*** and ***b***?

In fact, we should indicate the center $\boldsymbol{(a, b)}$ more specifically rather than find it.

We know $\boldsymbol{(a, b)}$ is a point in a plane, of course, but it is not just any point in the plane.

We know $\boldsymbol{(a, b)}$ is the center of the circle ***C***, and the center is in the line $\boldsymbol{y = 1 - x}$.

However, (a, b) alone doesn't show the fact that the center is in the line $y = 1 - x$.

So we want to put the center (a, b) in a different manner. How?

We can do so explaining the relationship between the coordinates of the center (a, b).

How do we explain it?

Coming up with an expression in terms of a and b, we explain the relationship.

Due to the relationship, (a, b) is in the line $y = 1 - x$.

A connective expression can explain such a relationship, and the connective expression connects a and b, of course. So we need the connective expression between a and b.

We can readily find such an expression by means of the equation of the line given.

That's because (a, b) is the center of the circle C, and is in the line given.

The line is $y = 1 - x$, and (a, b) is in that line, so we get $b = 1 - a$, which is the very connective expression between a and b.

So we can put the center (a, b) in this way: $(a, 1 - a)$.

Now, $(a, 1 - a)$ is the center, and $(a, 1 - a)$ alone can show the fact that the center is in the line $y = 1 - x$.

Next, let's move on to the radius R.

All the points in a circle are respectively the radius away from the center.

So the point $(1, 3)$ in the circle C is R away from the center (a, b), which is $(a, 1 - a)$.

We can find the radius R by means of the distance formula.

Taking the distance from the point **(1, 3)** to the center $(a, b) = (a, 1 - a)$, we get

$R^2 = (a - 1)^2 + (b - 3)^2 = (a - 1)^2 + (1 - a - 3)^2 = (a - 1)^2 + (-a - 2)^2$

$= (a - 1)^2 + (a + 2)^2 = 2a^2 + 2a + 5$.

Now, the center is $(a, 1 - a)$, and the radius is $\sqrt{2a^2 + 2a + 5}$, where a is constant.

Thus, using the standard form, we can get the circle C, which is as follows.

$(x - a)^2 + \{y - (1 - a)\}^2 = (x - a)^2 + (y - 1 + a)^2 = 2a^2 + 2a + 5$.

Therefore, the circle C can be any of all the circles in the group represented by such an equation as follows. $(x - a)^2 + (y - 1 + a)^2 = 2a^2 + 2a + 5$, where a is constant.

By the way, we have

$2a^2 + 2a + 5 = 2(a^2 + a) + 5 = 2(a^2 + a + \frac{1}{4} - \frac{1}{4}) + 5 = 2(a + \frac{1}{2})^2 - \frac{1}{2} + 5 = 2(a + \frac{1}{2})^2 + \frac{9}{2}$

$= 2(a + \frac{1}{2})^2 + (\frac{3}{\sqrt{2}})^2 = 2(a + \frac{1}{2})^2 + (\frac{3\sqrt{2}}{2})^2$.

So when $a = -\frac{1}{2}$, such a circle has the minimum radius, which is $\frac{3\sqrt{2}}{2} \approx 2.12$.

In fact, all the circles in the group share the two points **(1, 3)** and **(-2, 0)**, and the smallest of all the circles has a diameter where the two points above are the two endpoints. That is, the distance from **(1, 3)** to **(-2, 0)** is the diameter of the smallest circle in the group.

Suggestions or Solutions
To the Problem in the Example 3

The circle has (1, 3), and is tangent to the x-axis, and the center is in a line $y = 1 - x$.

Suppose that the center is (a, b), where a and b are constant, that $(a, 0)$ is the point where the circle is tangent to the x-axis, and that the radius is R.

Then, we get $R^2 = (a - 1)^2 + (b - 3)^2$, and also, $R^2 = (a - a)^2 + (b - 0)^2 = b^2 \Rightarrow R^2 = b^2$.

So we get $(a - 1)^2 + (b - 3)^2 = b^2$.

Since (a, b) is in the given line $y = 1 - x$, we get $b = 1 - a$.

So $(a - 1)^2 + (b - 3)^2 = b^2 \Rightarrow b^2 + (b - 3)^2 = b^2 \Rightarrow b = 3 \Rightarrow R^2 = 9$ since $R^2 = b^2$.

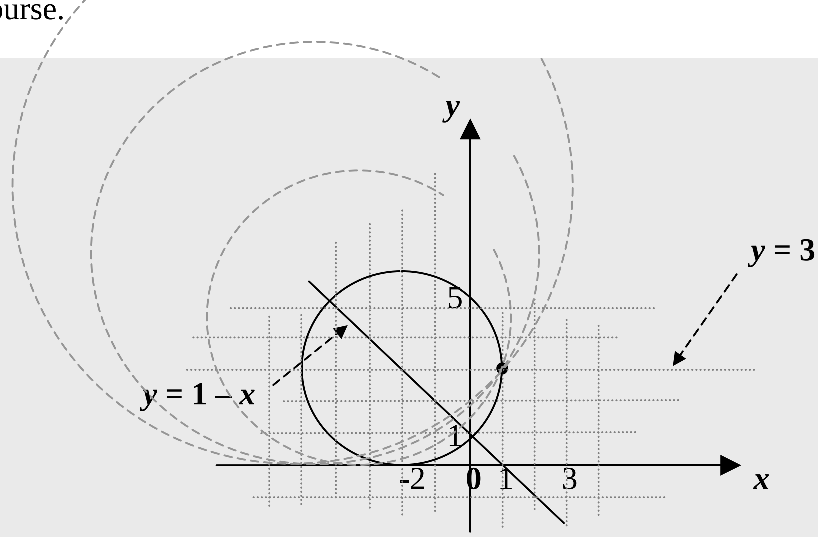
Thus, $b = 1 - a \Rightarrow a = 1 - b = 1 - 3 = -2$. So the center is $(-2, 3)$.

Therefore, the circle is $(x + 2)^2 + (y - 3)^2 = 3^2$.

If not quite sure of the idea behind the processes above, follow the steps below.

Putting the problem in a graph, we can see better where the solution can be around. Not only that, but we can reduce mistakes approaching the solution, too. Besides, it is often the case where we can even see the solution right away. So let's first, put in a graph the line, the point given, and some probable circles, together with a circle tangent to the x-axis, of course.

Fig. 0



Even without examining the graph closely, it's not hard to see that the center is **(-2, 3)**.

So a circle passing through the point **(1, 3)** and tangent to the ***x***-axis is centered at **(-2, 3)**.

What about the radius, then?

The ***y***-coordinate of the center is 3. So is the radius. That's because the circle is tangent to the ***x***-axis.

That is, if a circle is tangent to the ***x***-axis, the radius is the magnitude of the ***y***-coordinate of the center.

And if a circle is tangent to the ***y***-axis, the radius is the magnitude of the ***x***-coordinate of the center. So for instance, if a circle is centered at **(-1, 2)**, and is tangent to the ***y***-axis, the radius is **|-1| = 1**. What if a circle is tangent to both axes, then?

Both coordinates at the center are the same in magnitude since the center is in either of two lines $y = x$, and $y = -x$. So the radius is the magnitude of either coordinate.

Why is the circle centered at **(-2, 3)**, though? In other words, how do we know if it is the only circle centered at a point in the line given, passing through **(1, 3)**, and tangent to the ***x***-axis? It's not quite clear that some other circle cannot be the solution, is it?

Suppose ***A*** is a circle tangent to the ***x***-axis and centered at a point in the line $y = 1 – x$, and the radius is bigger than 3.

Then, it seems that the bigger the radius gets, the more the center gets away from the line $y = 1 – x$. So it seems that only the circle centered at **(-2, 3)** is the solution.

However, we still need to show the fact analytically.

Suppose now, ***C*** is the circle we are after, the center is (a, b), where ***a*** and ***b*** are constant, $(a, 0)$ is the point where the circle is tangent to the ***x***-axis, and the radius is ***R***.

Then, **(1, 3)** and **(a, 0)** are in the circle $\boldsymbol{C}$, so each of the two points is $\boldsymbol{R}$ away from **(a, b)**. Why?

Every point in a circle is the radius away from the center.

So beginning with the distance from the center to **(1, 3)**, we get $R^2 = (a - 1)^2 + (b - 3)^2$.

And next, we have another point **(a, 0)** in the circle $\boldsymbol{C}$, too, so we get

$R^2 = (a - a)^2 + (b - 0)^2 = b^2 \Rightarrow R^2 = b^2$.

So we can see that $(a - 1)^2 + (b - 3)^2 = b^2$.

Next, since **(a, b)** is in the given line $y = 1 - x$, we get $b = 1 - a$.

Therefore, we get $(a - 1)^2 + (b - 3)^2 = b^2 \Rightarrow b^2 + (b - 3)^2 = b^2 \Rightarrow b = 3$.

Thus, we get $b = 1 - a \Rightarrow a = 1 - b = 1 - 3 = -2$. So the center is **(-2, 3)**.

Besides, we have $R^2 = b^2$, too, so we get $R^2 = 9$.

Therefore, the circle $\boldsymbol{C}$ is $(x + 2)^2 + (y - 3)^2 = 3^2$.

In short:

Suppose that the center is **(a, b)**, where a and b are constant, that **(a, 0)** is the point where the circle is tangent to the x-axis, and that the radius is R.

Then, we get $R^2 = (a - 1)^2 + (b - 3)^2$, and also, $R^2 = (a - a)^2 + (b - 0)^2 = b^2 \Rightarrow R^2 = b^2$.

So we get $(a - 1)^2 + (b - 3)^2 = b^2$.

Since **(a, b)** is in the given line $y = 1 - x$, we get $b = 1 - a$.

So $(a - 1)^2 + (b - 3)^2 = b^2 \Rightarrow b^2 + (b - 3)^2 = b^2 \Rightarrow b = 3 \Rightarrow R^2 = 9$ since $R^2 = b^2$.

Thus, $b = 1 - a \Rightarrow a = 1 - b = 1 - 3 = -2$. So the center is **(-2, 3)**.

Therefore, the circle is $(x + 2)^2 + (y - 3)^2 = 3^2$.

Suggestions or Solutions
To the Problem in the Example 4

The circle has (1, 3), and is tangent to the *y*-axis, and the center is in a line $y = 1 - x$.

Suppose the radius is R, and the center is (a, b), where a and b are constant. Then, first, since the circle is tangent to the y-axis, the tangent point is $(0, b)$. Next, we get

The distance from the center to $(1, 3) \Rightarrow R^2 = (a - 1)^2 + (b - 3)^2$
The distance from the center to $(0, b) \Rightarrow R^2 = (a - 0)^2 + (b - b)^2 = a^2$.
So we get $(a - 1)^2 + (b - 3)^2 = a^2$.

Also, since (a, b) is in the given line $y = 1 - x$, we get $b = 1 - a$.

So $(a - 1)^2 + (b - 3)^2 = a^2 \Rightarrow (a - 1)^2 + (1 - a - 3)^2 = a^2 \Rightarrow (a - 1)^2 + (a + 2)^2 = a^2$.

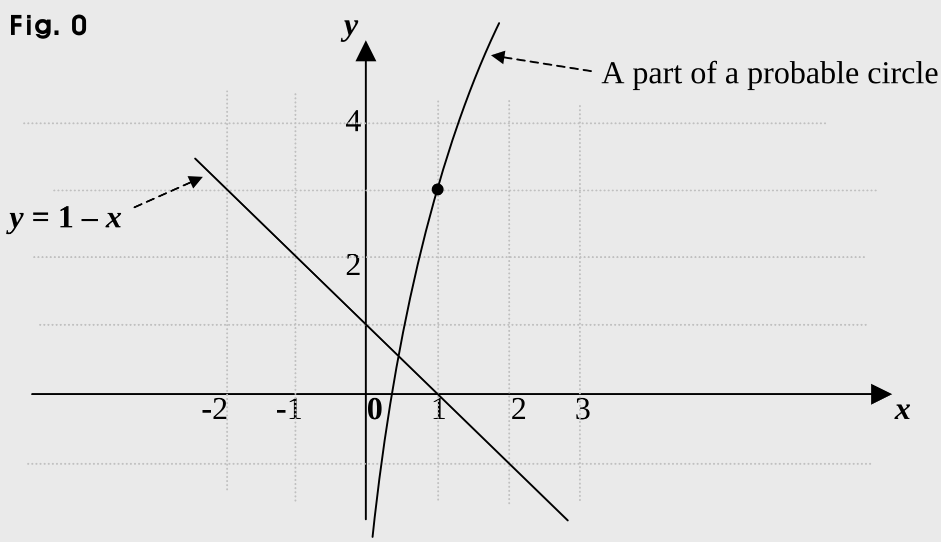
However, $a^2 - 2a + 1 + a^2 + 4a + 4 = a^2 \Rightarrow a^2 + 2a + 5 = (a + 1)^2 + 5$, which cannot be 0.

Therefore, there is no circle that satisfies the problem.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's first, put in a graph the line and point given, along with some probable circles. It is often the case though where we can't put in a graph a whole circle as a probable one. We may want to try however, at least a part of such a circle if a whole circle is not possible.

Fig. 0



Now, examining the graph, we can see that the center might be in the fourth quadrant if the circle exists. However, it also looks quite clear that the circle does not exist.

Running math though, we can just assume it is the case no matter how it may look clear since it only looks clear. Math is exact science.

So we want to be sure if the circle exists, and if it is the case, we want to get the precise location of the center. So let's now, pinpoint the center.

Suppose now, C is the circle we are after, the radius is R, and the center is (a, b), where a and b are constant.

Then, the circle C is tangent to the y-axis, so at the tangent point, the x-coordinate is 0, and the y-coordinate is the same as the y-coordinate at the center.

Thus, the tangent point is $(0, b)$.   Why?

The center is assumed to be (a, b).

By the definition for circles, the tangent point $(0, b)$ and the given point $(1, 3)$ are the same distance away from the center, and the same distance is R.

So putting the ideas above in equations, we get

First, the distance from the center to $(1, 3) \Rightarrow R^2 = (a - 1)^2 + (b - 3)^2$

Next, the distance from the center to $(0, b) \Rightarrow R^2 = (a - 0)^2 + (b - b)^2 = a^2$.

So we get $(a - 1)^2 + (b - 3)^2 = a^2$.

Besides, since the center (a, b) is in the given line $y = 1 - x$, we get $b = 1 - a$.

So we get $(a - 1)^2 + (b - 3)^2 = a^2 \Rightarrow (a - 1)^2 + (1 - a - 3)^2 = a^2$

$\Rightarrow (a - 1)^2 + (a + 2)^2 = a^2$.

Thus, we get $a^2 - 2a + 1 + a^2 + 4a + 4 = a^2 \Rightarrow 2a^2 + 2a + 5 = a^2$.

So we get $a^2 + 2a + 5 = 0$.

Then, the equation $a^2 + 2a + 5 = 0$ has to get at least one root if any value of a exists.

Then, the discriminant has to be ≥ 0.

And the discriminant is $2^2 - 4\cdot1\cdot5 = -16$, which is not even 0 but negative.

So a doesn't exist, and in turn, neither does b, since we have $b = 1 - a$.

Consequently, there is no circle that can satisfy this problem.

If we try putting it in a complete square, we get $(a + 1)^2 + 5$, which cannot be 0 since no real number squared can be -5.

And of course, no real number squared can be any number negative.

If a number is real, the number squared is positive or 0 only.

In short:

Suppose the radius is R, and the center is (a, b), where a and b are constant. Then, first, since the circle is tangent to the y-axis, the tangent point is $(0, b)$. Next, we get

The distance from the center to $(1, 3) \Rightarrow R^2 = (a - 1)^2 + (b - 3)^2$
The distance from the center to $(0, b) \Rightarrow R^2 = (a - 0)^2 + (b - \mathrm{b})^2 = a^2$.

So we get $(a - 1)^2 + (b - 3)^2 = a^2$.

Also, since (a, b) is in the given line $y = 1 - x$, we get $b = 1 - a$.

So $(a - 1)^2 + (b - 3)^2 = a^2 \Rightarrow (a - 1)^2 + (1 - a - 3)^2 = a^2 \Rightarrow (a - 1)^2 + (a + 2)^2 = a^2$.

However, $a^2 - 2a + 1 + a^2 + 4a + 4 = a^2 \Rightarrow a^2 + 2a + 5 = (a + 1)^2 + 5$, which cannot be 0.

Therefore, there is no circle that satisfies the problem.