Examples 6 in Parabolas

Click this to start.

Examples 6 in Circles

Find the circle in each case below.

0. The circle has **(3, 1)**, and is tangent to the y-axis, and the center is in a line $y = 1 - x$.

1. Two points **(1, 0)** and **(3, 0)** are in the circle tangent to the y-axis.

2. Two points **(0, 2)** and **(0, 4)** are in the circle tangent to the x-axis.

3. Two points **(1, 2)** and **(3, 5)** are in the circle tangent to the x-axis.

4. Two points **(1, 2)** and **(3, 5)** are in the circle tangent to the y-axis.

Suggestions or Solutions
To the Problem in the Example 0

The circle has (3, 1), and is tangent to the *y*-axis, and the center is in a line $y = 1 - x$.

Suppose the radius is R, and the center is (a, b), where a and b are constant.

Then, first, since the circle is tangent to the y-axis, the tangent point is $(0, b)$.

Next, we get

First, the distance from the center to $(3, 1) \Rightarrow R^2 = (a - 3)^2 + (b - 1)^2$.

Next, the distance from the center to $(0, b) \Rightarrow R^2 = (a - 0)^2 + (b - b)^2 = a^2 \Rightarrow R^2 = a^2$.

So we get $(a - 3)^2 + (b - 1)^2 = a^2$.

Besides, since (a, b) is in the given line $y = 1 - x$, we get $b = 1 - a$.

So we get $(a - 3)^2 + (b - 1)^2 = a^2 \Rightarrow (a - 3)^2 + (1 - a - 1)^2 = a^2 \Rightarrow (a - 3)^2 + a^2 = a^2$
$\Rightarrow (a - 3)^2 = 0 \Rightarrow a = 3$.

Thus, we get $b = 1 - a = 1 - 3 = -2$, and $R^2 = a^2 = 3^2$.

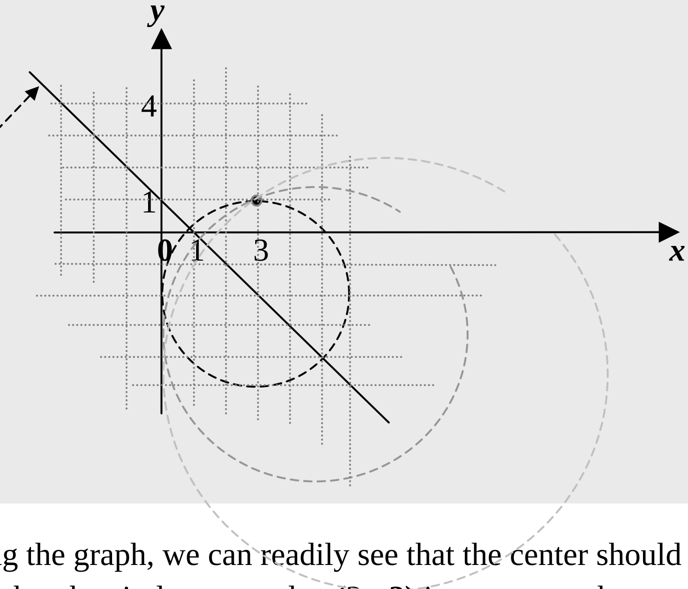
Therefore, the center is $(3, -2)$, and the circle is $(x - 3)^2 + (y + 2)^2 = 9$.

If not quite sure of the idea behind the processes above, follow the steps below.

Putting the problem in a graph, we can get the solution safely as well as quickly.

So to begin with, let's put in a graph the line, the point given, and some probable circles, tangent to the y-axis.

Fig. 0



Now, examining the graph, we can readily see that the center should be at **(3, -2)**.
It is quite clear that the circle centered at **(3, -2)** is tangent to the ***y***-axis, and is passing through the point **(3, 1)**.

So the radius should be 3, and thus, the circle seems to be $(x - 3)^2 + (y + 2)^2 = 9$.

We want to show however, that it is the case, since it is not completely clear, because math is exact science.

It looks like in fact, there can be some other circle that can be the solution, too.

Examining the graph more closely, we can see that if a circle tangent to the ***y***-axis passes through the point **(3, 1)**, and the radius is bigger than 3, then the bigger the radius gets, the more the center gets away from the line $y = 1 - x$.
We want to make sure though, that only the circle above can be the solution.

Suppose now, ***C*** is the circle we are after, the radius is ***R***, and the center is **(*a*, *b*)**, where ***a*** and ***b*** are constant.

Then, since the circle ***C*** is tangent to the ***y***-axis, the tangent point is **(0, *b*)**.   Why?

If a circle is tangent to the ***y***-axis, the ***y***-coordinate at the center has to be the same as the ***y***-coordinate at the tangent point.

And if a circle is tangent to the ***x***-axis, the ***x***-coordinate at the center has to be the same as the ***x***-coordinate at the tangent point.

Now, by the definition for circles, the tangent point $\mathbf{(0,\ \mathit{b})}$ and the given point $\mathbf{(3,\ 1)}$ are respectively $\boldsymbol{R}$ away from the center. So putting the ideas above in equations, we get

First, the distance from the center to $\mathbf{(3,\ 1)} \Rightarrow \boldsymbol{R^2 = (a - 3)^2 + (b - 1)^2}$.

Next, the distance from the center to $\mathbf{(0,\ \mathit{b})} \Rightarrow \boldsymbol{R^2 = (a - 0)^2 + (b - b)^2 = a^2 \Rightarrow R^2 = a^2}$.

So we get $\boldsymbol{(a - 3)^2 + (b - 1)^2 = a^2}$.

Besides, since the center $\boldsymbol{(a,\ b)}$ is in the given line $\boldsymbol{y = 1 - x}$, we get $\boldsymbol{b = 1 - a}$.

So we get $\boldsymbol{(a - 3)^2 + (b - 1)^2 = a^2 \Rightarrow (a - 3)^2 + (1 - a - 1)^2 = a^2 \Rightarrow (a - 3)^2 + a^2 = a^2}$

$\boldsymbol{\Rightarrow (a - 3)^2 = 0 \Rightarrow a = 3 \Rightarrow b = 1 - a = 1 - 3 = -2}$.

Therefore, the center of the circle $\boldsymbol{C}$ is $\mathbf{(3,\ -2)}$.

Meanwhile, we get $\boldsymbol{R^2 = a^2 = 3^2}$. Therefore, the circle $\boldsymbol{C}$ is $\boldsymbol{(x - 3)^2 + (y + 2)^2 = 9}$.

In short:

Suppose the radius is $\boldsymbol{R}$, and the center is $\boldsymbol{(a,\ b)}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant.

Then, first, since the circle is tangent to the ***y***-axis, the tangent point is $\mathbf{(0,\ \mathit{b})}$.
Next, we get

First, the distance from the center to $\mathbf{(3,\ 1)} \Rightarrow \boldsymbol{R^2 = (a - 3)^2 + (b - 1)^2}$.
Next, the distance from the center to $\mathbf{(0,\ \mathit{b})} \Rightarrow \boldsymbol{R^2 = (a - 0)^2 + (b - b)^2 = a^2 \Rightarrow R^2 = a^2}$.

So we get $\boldsymbol{(a - 3)^2 + (b - 1)^2 = a^2}$.
Besides, since $\boldsymbol{(a,\ b)}$ is in the given line $\boldsymbol{y = 1 - x}$, we get $\boldsymbol{b = 1 - a}$.
So we get $\boldsymbol{(a - 3)^2 + (b - 1)^2 = a^2 \Rightarrow (a - 3)^2 + (1 - a - 1)^2 = a^2 \Rightarrow (a - 3)^2 + a^2 = a^2}$
$\boldsymbol{\Rightarrow (a - 3)^2 = 0 \Rightarrow a = 3}$.
Thus, we get $\boldsymbol{b = 1 - a = 1 - 3 = -2}$, and $\boldsymbol{R^2 = a^2 = 3^2}$.
Therefore, the center is $\mathbf{(3,\ -2)}$, and the circle is $\boldsymbol{(x - 3)^2 + (y + 2)^2 = 9}$.

Suggestions or Solutions
To the Problem in the Example 1

Two points (1, 0) and (3, 0) are in the circle tangent to the *y*-axis.

Suppose $\boldsymbol{C}$ is the circle we want, and is tangent to the $\boldsymbol{y}$-axis at $\mathbf{(0,}\ \boldsymbol{b}\mathbf{)}$, where $\boldsymbol{b}$ is constant. Then, the center of $\boldsymbol{C}$ is in the line $\boldsymbol{y} = \boldsymbol{b}$.

The center is also, in the line passing through the midpoint between the two points given since the two points are in the circle $\boldsymbol{C}$.

The midpoint is $\left(\frac{3+1}{2}, \frac{0+0}{2}\right) = (2, 0)$, so the center is in a line $x = 2$, also.
Therefore, the center is at $(2, b)$, where the two lines $x = 2$ and $y = b$ meet each other.

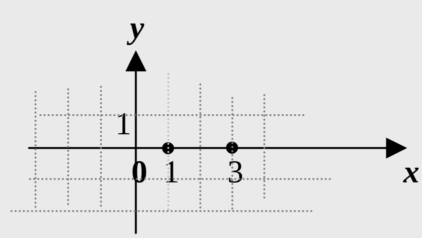
Thus, the tangent point is 2 away from the center, so the radius of the circle C is 2.
The two points given are in C, so the two are respectively 2 away from the center $(2, b)$.
Thus, 2 is the distance from the point $(1, 0)$ to the center of C.

So we get $2^2 = (2-1)^2 + (b-0)^2 \Rightarrow 4 = 1 + b^2 \Rightarrow b = \pm\sqrt{3}$.
Thus, the center of C is $(2, \sqrt{3})$ or $(2, -\sqrt{3})$.
Of the two centers above, $(2, \sqrt{3})$ is the center of a circle above the x-axis, and the other, $(2, -\sqrt{3})$ is the center of a circle below the x-axis.
Therefore, the circle C is $(x-2)^2 + (y \pm \sqrt{3})^2 = 4$.

If not quite sure of the idea behind the processes above, follow the steps below.

Let's first, put in a graph the two points given. Then, we can see better how a circle has to be placed so that it can be tangent to the *y*-axis passing through the two points.

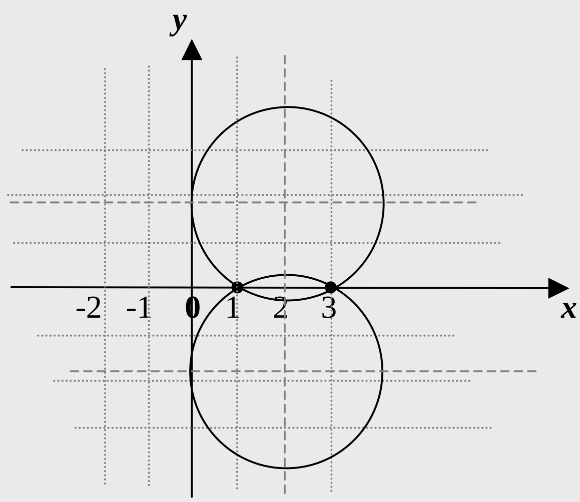
Fig. 0



Then, adding some circles to the graph, we will see a circle likely to satisfy the problem.

And we will quickly see that not one but two circles can satisfy the problem. Putting a circle in a graph, it's always a good idea to indicate (or show, mark, etc.) the point where the center is. For instance, we can put two dashed lines perpendicular to each other and meeting at the center.

Fig. 1



Now, we have two circles tangent to the ***y***-axis, so we may want to find the points where the circles are tangent to the ***y***-axis.

Finding each tangent point, we will have three points in total to work with since we are already given two points **(1, 0)** and **(3, 0)**. So we can find each of the two circles at a time, for three points not in a line can determine a circle. Of course, finding the center and radius, we can get the same, too, by the standard form.

Let's now, begin with the upper circle in the graph above.
So assuming ***C*** is the circle, and examining the graph, we can clearly see that the center is in the line $\boldsymbol{x = 2}$.   Why is the center in the line?

• Suppose ***L*** is a line that passes through the midpoint between two points in a circle, and is perpendicular to the line segment connecting the two points. Then, the line ***L*** passes through the center of the circle. Put some circles in a graph, and try it.

Now, the midpoint between **(1, 0)** and **(3, 0)** is **(2, 0)**, so the center of the circle ***C*** has to be in the line $x = 2$, which is parallel to the ***y***-axis.

Therefore, the tangent point is 2 away from the center, so the radius of the circle ***C*** is 2.

Suppose now, that the circle ***C*** is tangent to the ***y***-axis at $(0, b)$, where ***b*** is constant.

Then, the center of ***C*** is in the line $y = b$, too. Why?

• A line tangent to a circle is perpendicular to a line passing through the tangent point and the center of the circle. Put some circles in a graph again, and try it.

Now, the ***y***-axis is a line tangent to the circle ***C*** at the point $(0, b)$.

The line $y = b$ is perpendicular to the ***y***-axis and passes through the tangent point $(0, b)$.

So the line $y = b$ passes through the center, which therefore, is in the line $y = b$.

Thus, the center of ***C*** is not only in the line $x = 2$ but in the line $y = b$, too.
Where is the center, then?

The center is at a point $(2, b)$, where the two lines $x = 2$ and $y = b$ meet each other.

Now, finding the center (that is, finding ***b***), we can find the circle ***C*** by means of the standard form since we know the radius, which is 2. How do we find the center, then?

We can find it by means of the definition for circles.

• All the points in a circle are respectively the radius away from the center.

So the two points given are respectively the radius away from the center $(2, b)$.

The radius is 2, and is the distance from the point **(1, 0)** to the center $(2, b)$, so we get
$2^2 = (2 - 1)^2 + (b - 0)^2 \Rightarrow 4 = 1 + b^2 \Rightarrow b = \pm\sqrt{3}$.

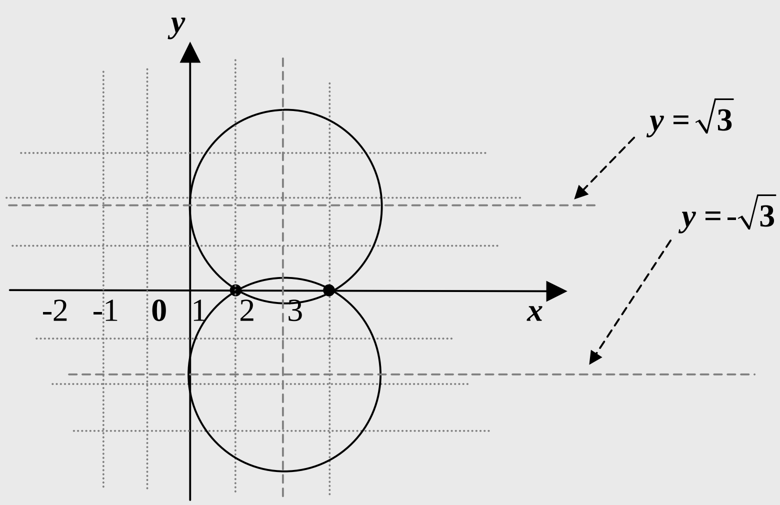
Thus, we can see that the center of the circle ***C*** is **$(2, \sqrt{3})$** or **$(2, -\sqrt{3})$**.

Looking at the ***y***-coordinates at the two centers above, we can readily see that **$(2, \sqrt{3})$** is the center of the upper circle, and **$(2, -\sqrt{3})$** is the center of the lower circle.

Therefore, the circle ***C*** is as follows. **$(x - 2)^2 + (y - \sqrt{3})^2 = 4$** or **$(x - 2)^2 + (y + \sqrt{3})^2 = 4$**.

The circle ***C*** can be put this way, too: **$(x - 2)^2 + (y \pm \sqrt{3})^2 = 4$**.

Fig. 2



Suggestions or Solutions
To the Problem in the Example 2

Two points (0, 2) and (0, 4) are in the circle tangent to the *x*-axis.

Suppose $\boldsymbol{C}$ is the circle we want, and is tangent to the $\boldsymbol{x}$-axis at $\mathbf{(}\boldsymbol{a}\mathbf{, 0)}$, where $\boldsymbol{a}$ is constant.

Then, the center of $\boldsymbol{C}$ is in a line $\boldsymbol{x = a}$, and also, is in a line passing through the midpoint between the two points given, and perpendicular to the line segment connecting the two.

The midpoint is $\mathbf{(\frac{0+0}{2}, \frac{4+2}{2}) = (0, 3)}$, and the line perpendicular to the line segment is parallel to the $\boldsymbol{x}$-axis, so the center is in a line $\boldsymbol{y} \mathbf{= 3}$, too. Thus, the center is $\mathbf{(}\boldsymbol{a}\mathbf{, 3)}$.

Thus, the tangent point is 3 away from the center, so the radius is 3.

The point $\mathbf{(0, 2)}$ is in the circle $\boldsymbol{C}$, so the point is 3 away from the center $\mathbf{(}\boldsymbol{a}\mathbf{, 3)}$.

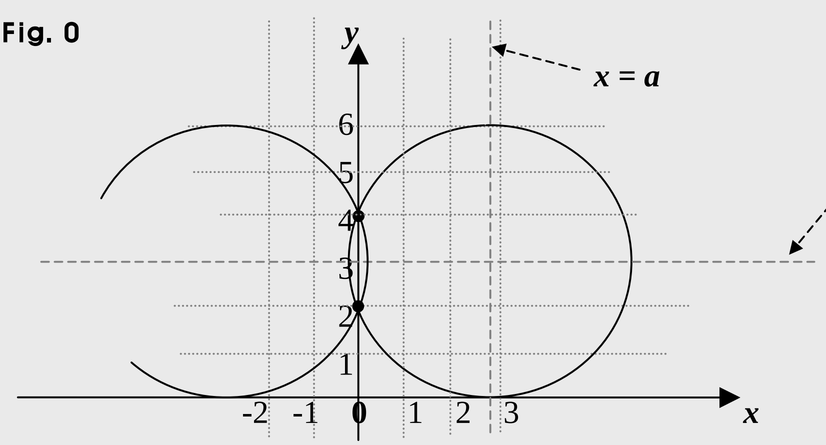
Thus, we get $\mathbf{3^2 = (}\boldsymbol{a}\mathbf{ - 0)^2 + (3 - 2)^2 \Rightarrow 9 = }\boldsymbol{a}^2\mathbf{ + 1 \Rightarrow }\boldsymbol{a}\mathbf{ = \pm\sqrt{8} = \pm 2\sqrt{2}}$. So the center is $\mathbf{(2\sqrt{2}, 3)}$, which is for the circle on the right of the $\boldsymbol{y}$-axis, or is $\mathbf{(-2\sqrt{2}, 3)}$, which is for the one on the left. Therefore, the circle $\boldsymbol{C}$ is $\mathbf{(}\boldsymbol{x}\mathbf{ \pm 2\sqrt{2})^2 + (}\boldsymbol{y}\mathbf{ - 3)^2 = 3^2}$.

If not quite sure of the idea behind the processes above, follow the steps below.

The solution to this problem is just about the same as the one to the previous problem. Comparing the two solutions, we will see another way of handling problems with circles.

Now, let's put in a graph the two points given, together with probable circles, of course.

Fig. 0



Then, we can see two circles can be the solution. Each of the two is tangent to the x-axis.

Considering the tangent point, along with the two points given, we can work with three points in total which are not in a line, so we can find the circle. And of course, finding the center and radius, we can get the circle, too, by the template standard.

Let's now, begin with the circle on the right in the graph above.

Assuming C is the circle, and looking at the graph, we can clearly see that the center is in a line $y = 3$. Why is the center in the line, though?

• A line passing through the midpoint between two points in a circle and perpendicular to the line segment connecting the two points passes through the center of the circle.

Now, the midpoint between **(0, 2)** and **(0, 4)** is **(0, 3)**, so the center of the circle C has to be in the line $y = 3$, which is parallel to the x-axis. Thus, the tangent point is 3 away from the center, so the radius is 3. Suppose now, C is tangent to the x-axis at $(a, 0)$, where a is constant. Then, the center is in a line $x = a$, too. Why?

• A line tangent to a circle is perpendicular to a line passing through the tangent point and the center of the circle.

The line perpendicular to the x-axis and passing through the tangent point $(a, 0)$ is the line $x = a$.
So the center of C is in the line $x = a$, and is a point $(a, 3)$ where the two lines $x = a$ and $y = 3$ meet each other.
Thus, finding the center (that is, finding a), we can find the circle using the standard form since we know the radius, which is 3.
And we can get the center by means of the definition, every point is the radius away from the center.

So the two given points are respectively the radius away from the center $(a, 3)$.

The radius is 3, and is the distance from the point **(0, 2)** to the center $(a, 3)$, so we get

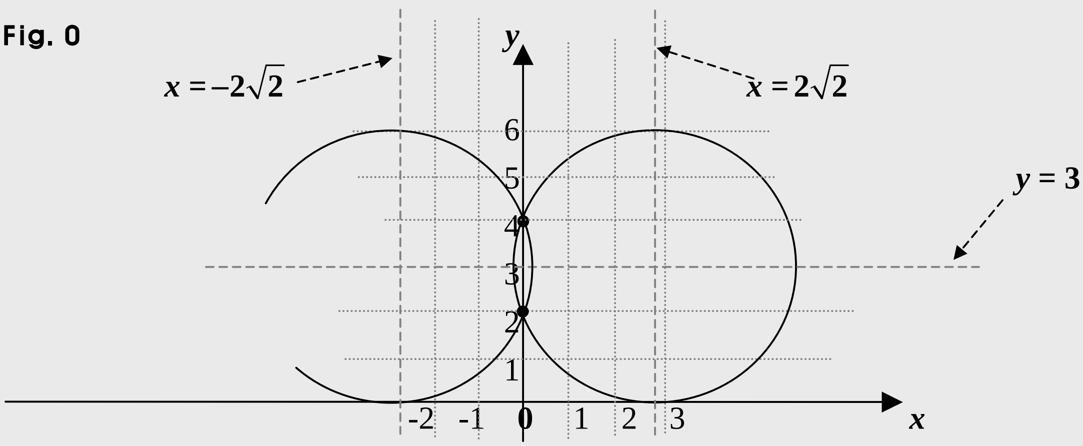
$$3^2 = (a - 0)^2 + (3 - 2)^2 \Rightarrow 9 = a^2 + 1 \Rightarrow a = \pm\sqrt{8} = \pm 2\sqrt{2}\ .$$

Therefore, we can see that the center is $(2\sqrt{2},\ 3)$, which is for the circle on the right, or $(-2\sqrt{2},\ 3)$, which is for the one on the left.

So the circle C is $(x-2\sqrt{2})^2 + (y-3)^2 = 9$ or $(x+2\sqrt{2})^2 + (y-3)^2 = 9$.

We can put it this way, too: $(x \pm 2\sqrt{2})^2 + (y-3)^2 = 3^2$.

Fig. 0



In short:

Suppose C is the circle we want, and is tangent to the x-axis at $(a, 0)$, where a is constant. Then, the center of C is in a line $x = a$, and also, is in a line passing through the midpoint between the two points given, and perpendicular to the line segment connecting the two.

The midpoint is $(\frac{0+0}{2}, \frac{4+2}{2}) = (0, 3)$, and the line perpendicular to the line segment is parallel to the x-axis, so the center is in a line $y = 3$, too. Thus, the center is $(a, 3)$.

Thus, the tangent point is 3 away from the center, so the radius is 3.

The point $(0, 2)$ is in the circle C, so the point is 3 away from the center $(a, 3)$.

Thus, we get $3^2 = (a-0)^2 + (3-2)^2 \Rightarrow 9 = a^2 + 1 \Rightarrow a = \pm\sqrt{8} = \pm 2\sqrt{2}$. So the center is $(2\sqrt{2},\ 3)$, which is for the circle on the right of the y-axis, or is $(-2\sqrt{2},\ 3)$, which is for the one on the left. Therefore, the circle C is $(x \pm 2\sqrt{2})^2 + (y-3)^2 = 3^2$.

Suggestions or Solutions
To the Problem in the Example 3

Two points (1, 2) and (3, 5) are in the circle tangent to the *x*-axis.

Suppose the center is in a line $y = b$, where b is constant, and the point where the circle is tangent to the x-axis is $(a, 0)$, where a is constant.

Then, the center is at (a, b), where a line $x = a$ meets a line $y = b$, and the radius is $|b|$.

The two points given are respectively the radius $|b|$ away from the center (a, b).

So taking the distances from (a, b) to $(1, 2)$ and $(3, 5)$, we get

$b^2 = (a-1)^2 + (b-2)^2$, and $b^2 = (a-3)^2 + (b-5)^2$. Then first, we get

$(a-1)^2 + (b-2)^2 = (a-3)^2 + (b-5)^2 \Rightarrow (a-1)^2 - (a-3)^2 + (b-2)^2 - (b-5)^2 = 0$

$\Rightarrow \{(a-1)-(a-3)\}\{(a-1)+(a-3)\} + \{(b-2)-(b-5)\}\{(b-2)+(b-5)\} = 0$

$\Rightarrow 2(2a-4) + 3(2b-7) = 4a + 6b - 29 = 0$. Next, we get

$(a-1)^2 + (b-2)^2 = b^2 \Rightarrow (a-1)^2 + b^2 - 4b + 4 = b^2 \Rightarrow (a-1)^2 + 4 - 4b = 0$.

So we need to solve a system where $4a + 6b - 29 = 0$ and $(a-1)^2 + 4 - 4b = 0$.

Then, we get first, $4a + 6b - 29 = 0 \Rightarrow b = \frac{29-4a}{6}$. Next, we get

$$(a-1)^2 + 4 - 4b = a^2 - 2a + 1 + 4 - 4\frac{(29-4a)}{6} = a^2 - 2a + \frac{8a}{3} + 5 - \frac{58}{3} = 0 \Rightarrow a^2 + \frac{2a}{3} - \frac{43}{3} = 0.$$

Then, $a^2 + \frac{2a}{3} - \frac{43}{3} = a^2 + \frac{2a}{3} + \frac{1}{9} - \frac{1}{9} - \frac{43}{3} = (a + \frac{1}{3})^2 - \frac{1}{9} - \frac{43}{3} = (a + \frac{1}{3})^2 - \frac{130}{9} = 0$

$$\Rightarrow (a + \tfrac{1}{3})^2 = \frac{130}{9} \Rightarrow a + \frac{1}{3} = \pm\sqrt{\frac{130}{9}} \Rightarrow a = -\frac{1}{3} \pm \frac{\sqrt{130}}{3}.$$

So we get $a = \frac{-1+\sqrt{130}}{3}$ or $a = \frac{-1-\sqrt{130}}{3}$. Also, we have $b = \frac{29-4a}{6}$. So we get

$$a = \frac{-1+\sqrt{130}}{3} \Rightarrow b = \frac{1}{6}(29 - 4 \cdot \frac{-1+\sqrt{130}}{3}) = \frac{1}{6} \cdot \frac{87+4-4\sqrt{130}}{3} = \frac{91-4\sqrt{130}}{18}.$$

$$a = \frac{-1-\sqrt{130}}{3} \Rightarrow b = \frac{1}{6}(29 - 4 \cdot \frac{-1-\sqrt{130}}{3}) = \frac{1}{6} \cdot \frac{87+4+4\sqrt{130}}{3} = \frac{91+4\sqrt{130}}{18}.$$

And we know that the radius is $|b|$, and that the center is (a, b).

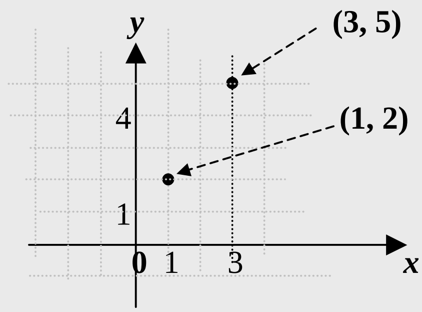
Therefore, the circle is $(x - a)^2 + (y - b)^2 = b^2$ where

$a = \frac{-1+\sqrt{130}}{3}$ and $b = \frac{91-4\sqrt{130}}{18}$ or $a = \frac{-1-\sqrt{130}}{3}$ and $b = \frac{91+4\sqrt{130}}{18}$.

If not quite sure of the idea behind the processes above, follow the steps below.

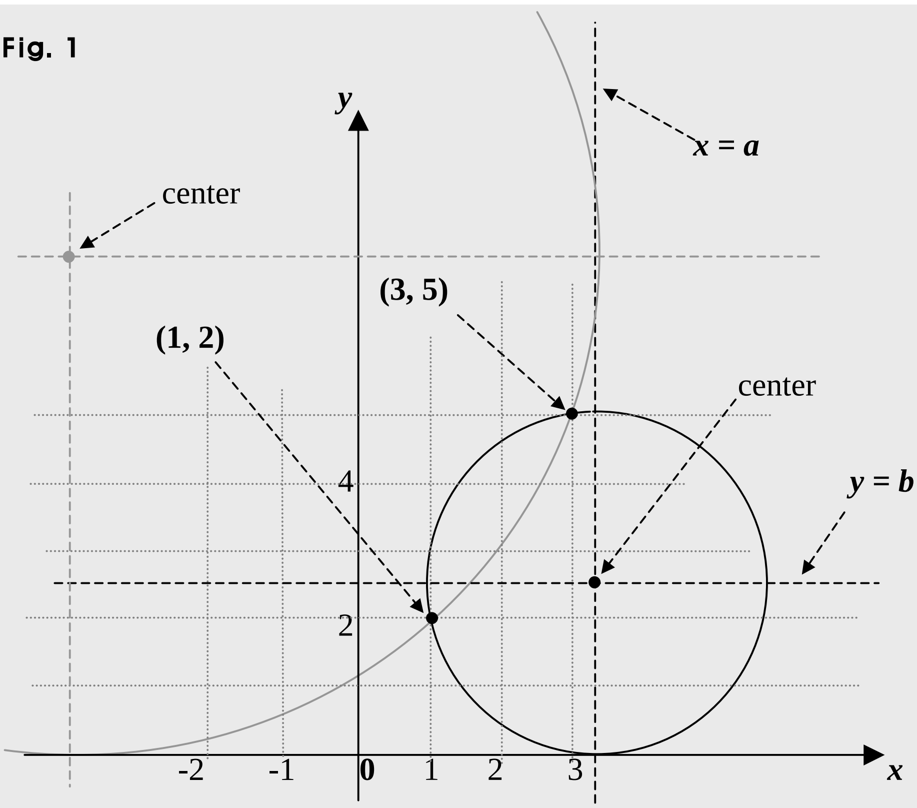
This example is quite similar to the Example E. So putting in a graph the two points first, we can see better what circle can be tangent to the ***x***-axis passing through the two points.

Fig. 0



We don't really have to use compasses, and a part of a circle can serve the purpose, too. Sketching some probable circles, we can see quickly that two circles can be the solution.

Fig. 1



Let's for now, begin with the circle on the right, which is the smaller of the two.

And since the circle is tangent to the ***x***-axis, we may want to begin with the tangent point.

Suppose ***C*** is the circle we want, and is tangent to the ***x***-axis at a point **(*a*, 0)**, where ***a*** is constant. Then, the center of ***C*** is in a line ***x* = *a***. Why?

• A line tangent to a circle is perpendicular to the line passing through the tangent point and the center of the circle.

The ***x***-axis is the line tangent to the circle ***C*** at the point **(*a*, 0)**, and the line perpendicular to the ***x***-axis and passing through the tangent point **(*a*, 0)** is the line ***x* = *a***.

So the center of ***C*** is in the line ***x* = *a***. That is, the ***x***-coordinate at the center is ***a***.
What then, about the ***x***-coordinate at the center?

We want to find it, too, of course, so suppose now, the center is in the line ***y* = *b***, too.
That is, we assume that the ***y***-coordinate at the center is ***b***.

Then, the center is at a point **(*a*, *b*)**, where the two lines ***x* = *a*** and ***y* = *b*** meet each other, and the tangent point **(*a*, 0)** is |***b***| away from the center, so the radius of the circle ***C*** is |***b***|.

Thus, finding the values of ***a*** and ***b*** (that is, finding the center), we can get the radius, too.

Then, by means of the standard form, we can get the circle ***C***.

We can find the center of ***C*** by means of the definition for circles, which keeps repeating the story about the same distance, which is the radius. So according to the definition, the two points given are respectively the radius, |***b***| away from the center **(*a*, *b*)**.

So beginning with the distance from **(*a*, *b*)** to **(1, 2)**, we get $b^2 = (a - 1)^2 + (b - 2)^2$.

Next, we have another point **(3, 5)** in ***C***, so we get $b^2 = (a - 3)^2 + (b - 5)^2$, too.

Therefore, we get $(a - 1)^2 + (b - 2)^2 = (a - 3)^2 + (b - 5)^2$.

Then, do we get a system of three equations for ***a*** an ***b***?

No, we don't. The three systems shown below are actually equivalent to each other.

- $b^2 = (a-1)^2 + (b-2)^2$, and $b^2 = (a-3)^2 + (b-5)^2$.
- $b^2 = (a-1)^2 + (b-2)^2$, and $(a-1)^2 + (b-2)^2 = (a-3)^2 + (b-5)^2$.
- $b^2 = (a-3)^2 + (b-5)^2$, and $(a-1)^2 + (b-2)^2 = (a-3)^2 + (b-5)^2$.

So solving either of the three above, we get the same solution. Let's solve the second one of the three. Then, beginning with the first of the two equations, we get

$$(a-1)^2 + (b-2)^2 = b^2 \Rightarrow (a-1)^2 + b^2 - 4b + 4 = b^2 \Rightarrow (a-1)^2 + 4 - 4b = 0.$$

Next, moving on to the other equation, we get

$$(a-1)^2 + (b-2)^2 = (a-3)^2 + (b-5)^2 \Rightarrow (a-1)^2 - (a-3)^2 + (b-2)^2 - (b-5)^2 = 0$$

$$\Rightarrow \{(a-1)-(a-3)\}\{(a-1)+(a-3)\} + \{(b-2)-(b-5)\}\{(b-2)+(b-5)\} = 0$$

$$\Rightarrow 2(2a-4) + 3(2b-7) = 4a + 6b - 29 = 0.$$

Now, the system has been reduced to another system for a and b as follows.
$4a + 6b - 29 = 0$ and $(a-1)^2 + 4 - 4b = 0$.

The system above is equivalent to any of the three systems stated above, of course. The system above is now smaller, but is still quadratic. So we get to do some algebra. To begin with, $4a + 6b - 29 = 0 \Rightarrow b = \frac{29-4a}{6}$. Then, we get

$$(a-1)^2 + 4 - 4b = a^2 - 2a + 1 + 4 - 4\frac{(29-4a)}{6} = a^2 - 2a + \frac{8a}{3} + 5 - \frac{58}{3} = 0 \Rightarrow a^2 + \frac{2a}{3} - \frac{43}{3} = 0.$$

Then, we get $a^2 + \frac{2a}{3} - \frac{43}{3} = a^2 + \frac{2a}{3} + \frac{1}{9} - \frac{1}{9} - \frac{43}{3} = (a+\frac{1}{3})^2 - \frac{1}{9} - \frac{43}{3} = (a+\frac{1}{3})^2 - \frac{130}{9} = 0$

$$\Rightarrow (a+\tfrac{1}{3})^2 = \tfrac{130}{9} \Rightarrow a + \tfrac{1}{3} = \pm\sqrt{\tfrac{130}{9}} \Rightarrow a = -\tfrac{1}{3} \pm \tfrac{\sqrt{130}}{3}.$$

So we get $a = \frac{-1+\sqrt{130}}{3}$ or $a = \frac{-1-\sqrt{130}}{3}$. In short, we have $a = \frac{-1\pm\sqrt{130}}{3}$.

And looking at the beginning above, we can see we have $b = \frac{29-4a}{6}$, too. So we get

$$a = \tfrac{-1+\sqrt{130}}{3} \Rightarrow b = \tfrac{1}{6}(29 - 4\cdot\tfrac{-1+\sqrt{130}}{3}) = \tfrac{1}{6}\cdot\tfrac{87+4-4\sqrt{130}}{3} = \tfrac{91-4\sqrt{130}}{18}.$$

$$a = \tfrac{-1-\sqrt{130}}{3} \Rightarrow b = \tfrac{1}{6}(29 - 4\cdot\tfrac{-1-\sqrt{130}}{3}) = \tfrac{1}{6}\cdot\tfrac{87+4+4\sqrt{130}}{3} = \tfrac{91+4\sqrt{130}}{18}.$$

Now, we know that the radius is $|\boldsymbol{b}|$, and that the center is $(\boldsymbol{a}, \boldsymbol{b})$.

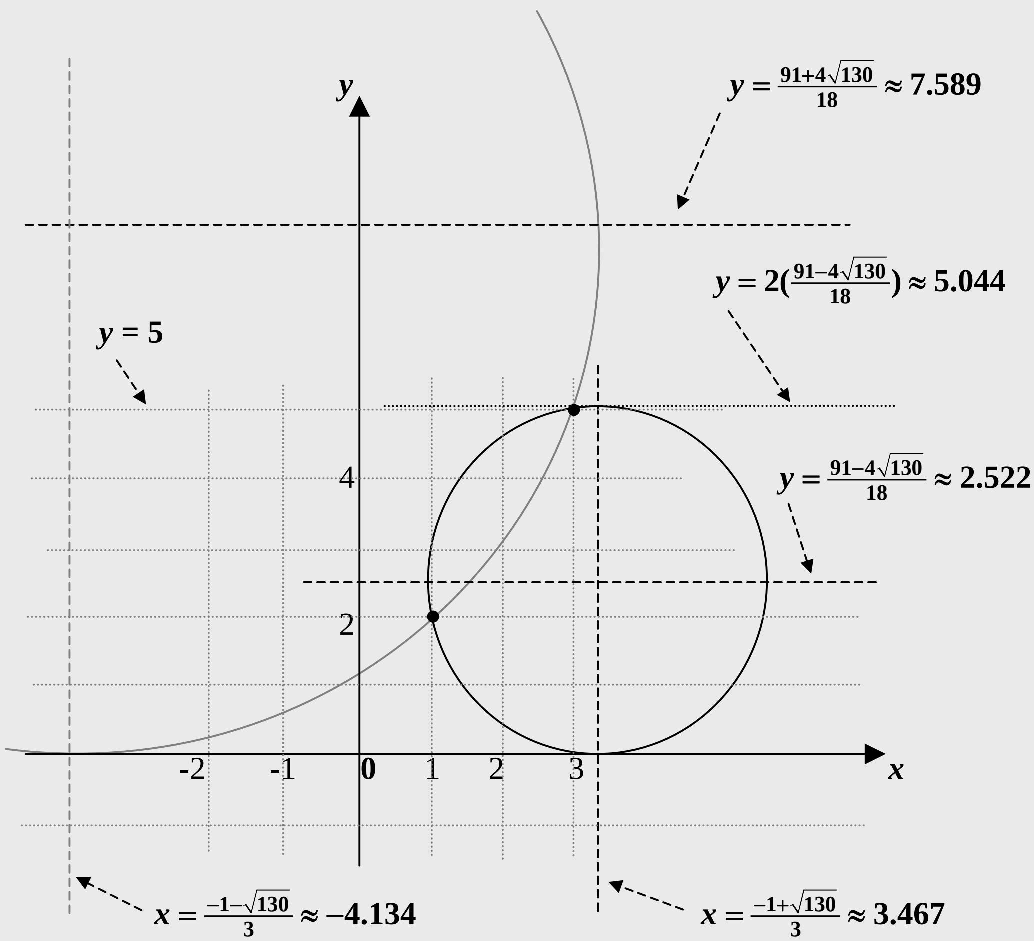
Therefore, the circle $\boldsymbol{C}$ is $(\boldsymbol{x}-\boldsymbol{a})^2 + (\boldsymbol{y}-\boldsymbol{b})^2 = \boldsymbol{b}^2$ where

$$\boldsymbol{a} = \frac{-1+\sqrt{130}}{3} \approx \mathbf{3.467} \text{ and } \boldsymbol{b} = \frac{91-4\sqrt{130}}{18} \approx \mathbf{2.522}$$

or $\boldsymbol{a} = \frac{-1-\sqrt{130}}{3} \approx \mathbf{-4.134}$ and $\boldsymbol{b} = \frac{91+4\sqrt{130}}{18} \approx \mathbf{7.589}$.

And putting the circles in a graph, we can put them the way below.

Fig. 2



Suggestions or Solutions
To the Problem in the Example 4

Two points (1, 2) and (3, 5) are in the circle tangent to the *y*-axis

Suppose that the center is in a line $x = a$, and that the point where the circle is tangent to the y-axis is $(0, b)$. Then, the center is at a point (a, b), where the line $x = a$ meets the line $y = b$, and the radius is $|a|$.

Each of the two points given is the radius $|a|$ away from (a, b).
So taking the distances from (a, b) to $(1, 2)$ and $(3, 5)$, we get
$b^2 = (a - 1)^2 + (b - 2)^2$, and $b^2 = (a - 3)^2 + (b - 5)^2$.

Then first, we get
$(a - 1)^2 + (b - 2)^2 = a^2 \Rightarrow a^2 - 2a + 1 + b^2 - 4b + 4 = a^2 \Rightarrow b^2 - 4b - 2a + 5 = 0$.

Next, we get
$(a - 1)^2 + (b - 2)^2 = (a - 3)^2 + (b - 5)^2 \Rightarrow (a - 1)^2 - (a - 3)^2 + (b - 2)^2 - (b - 5)^2 = 0$
$\Rightarrow \{(a - 1) - (a - 3)\}\{(a - 1) + (a - 3)\} + \{(b - 2) - (b - 5)\}\{(b - 2) + (b - 5)\} = 0$
$\Rightarrow 2(2a - 4) + 3(2b - 7) = 4a + 6b - 29 = 0$.

Thus, we need to solve a system where $4a + 6b - 29 = 0$ and $b^2 - 4b - 2a + 5 = 0$.

Then first, we get $4a + 6b - 29 = 0 \Rightarrow a = \frac{29-6b}{4}$. Next, we get

$$b^2 - 4b - 2a + 5 = b^2 - 4b - 2\cdot\frac{(29-6b)}{4} + 5 = b^2 - 4b + 3b + 5 - \frac{29}{2} = 0 \Rightarrow b^2 - b - \frac{19}{2} = 0.$$

Then, $b^2 - b - \frac{19}{2} = b^2 - b + \frac{1}{4} - \frac{1}{4} - \frac{19}{2} = (b - \frac{1}{2})^2 - \frac{1}{4} - \frac{19}{2} = (b - \frac{1}{2})^2 - \frac{39}{4} = 0$

$$\Rightarrow (b - \tfrac{1}{2})^2 = \tfrac{39}{4} \Rightarrow b - \tfrac{1}{2} = \pm\sqrt{\tfrac{39}{4}} \Rightarrow b = \tfrac{1}{2} \pm \tfrac{\sqrt{39}}{2}.$$

So we get $b = \frac{1+\sqrt{39}}{2}$ or $b = \frac{1-\sqrt{39}}{2}$. And we have $a = \frac{29-6b}{4}$, too. Thus, we get

$$b = \frac{1+\sqrt{39}}{2} \Rightarrow a = \frac{1}{4}(29 - 6\cdot\frac{1+\sqrt{39}}{2}) = \frac{1}{4}\cdot\frac{58-6-6\sqrt{39}}{2} = \frac{52-6\sqrt{39}}{8} = \frac{26-3\sqrt{39}}{4}.$$

$$b = \frac{1-\sqrt{39}}{2} \Rightarrow a = \frac{1}{4}(29 - 6\cdot\frac{1-\sqrt{39}}{2}) = \frac{1}{4}\cdot\frac{58-6+6\sqrt{39}}{2} = \frac{52+6\sqrt{39}}{8} = \frac{26+3\sqrt{39}}{4}.$$

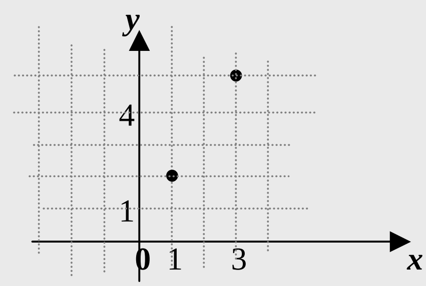
We know that the radius is $|a|$, and that the center is (a, b).

Therefore, the circle C is $(x - a)^2 + (y - b)^2 = a^2$, where $a = \frac{26\mp 3\sqrt{39}}{4}$ and $b = \frac{1\pm\sqrt{39}}{2}$.

If not quite sure of the idea behind the processes above, follow the steps below.

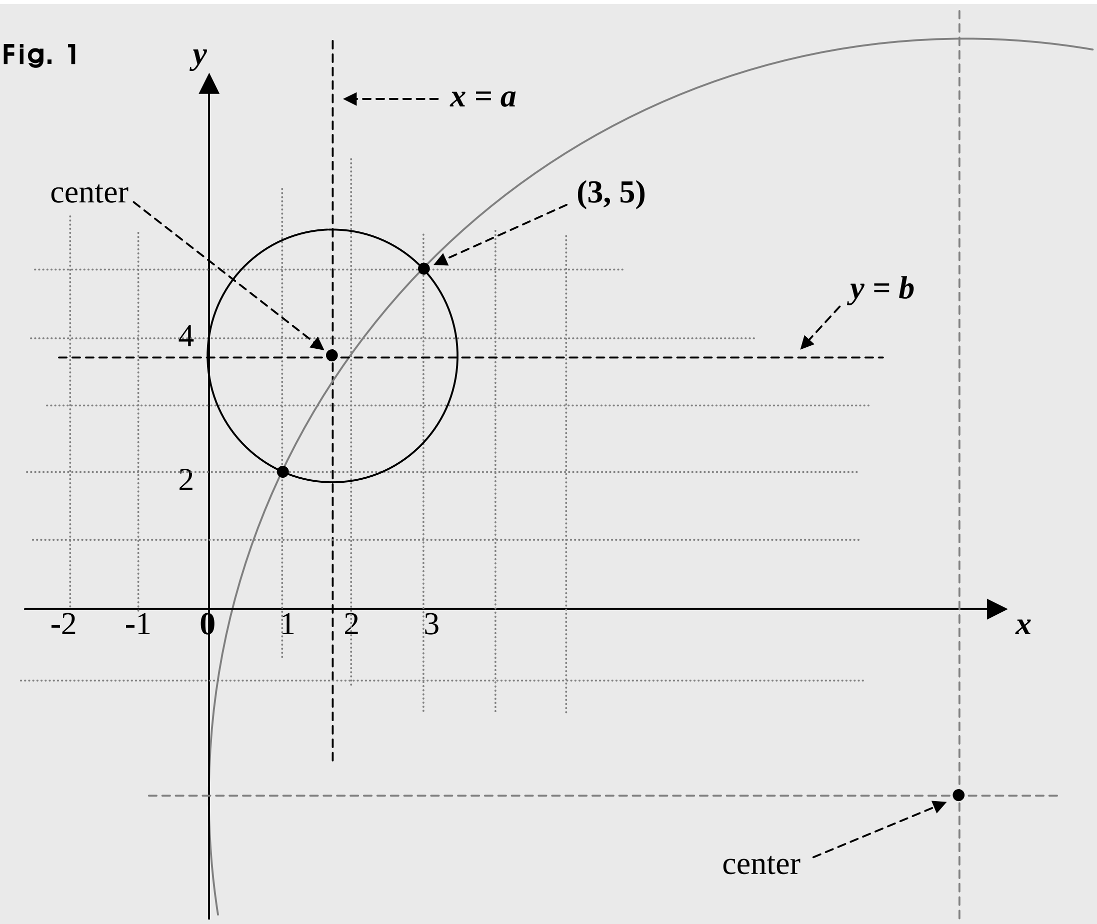
First, putting in a graph the two points given, we can see better what circle can be tangent to the ***x***-axis passing through the two points.

Fig. 0



This example is in fact, almost the same as the previous one. Sketching some probable circles in the graph, we can see that two circles can be the solution.

Fig. 1



Let's begin with the smaller of the two circles above.

Suppose $\boldsymbol{C}$ is the circle, and is tangent to the $\boldsymbol{y}$-axis at $\mathbf{(0,}\ \boldsymbol{b}\mathbf{)}$, where $\boldsymbol{b}$ is constant.

Then, the center of the circle $\boldsymbol{C}$ is in a line $\boldsymbol{y = b}$. Why?

We have a fact that a line tangent to a circle is perpendicular to a line passing through the tangent point and the center.

Now, the line $\boldsymbol{y = b}$ is perpendicular to the $\boldsymbol{y}$-axis, which is tangent to $\boldsymbol{C}$, and passes through the tangent point $\mathbf{(0,}\ \boldsymbol{b}\mathbf{)}$. So the center of $\boldsymbol{C}$ is in the line $\boldsymbol{y = b}$.

Suppose now, that the center of $\boldsymbol{C}$ is in a line $\boldsymbol{x = a}$, too.

Then, the center is a point $\boldsymbol{(a, b)}$, where the line $\boldsymbol{x = a}$ meets the line $\boldsymbol{y = b}$.

So the tangent point is $|\boldsymbol{a}|$ away from the center, and therefore, the radius of $\boldsymbol{C}$ is $|\boldsymbol{a}|$.

Thus, finding the values of $\boldsymbol{a}$ and $\boldsymbol{b}$ (that is, finding the center), we can get the radius, too.

Then, by means of the standard form, we can get the circle $\boldsymbol{C}$.

We can find the center of $\boldsymbol{C}$ using the definition for circles. By the definition, each of the two points given is the radius $|\boldsymbol{a}|$ away from the center $\boldsymbol{(a, b)}$.

So beginning with the distance from $\boldsymbol{(a, b)}$ to $\mathbf{(1, 2)}$, we get $\boldsymbol{a^2 = (a - 1)^2 + (b - 2)^2}$.

Next, we have another point $\mathbf{(3, 5)}$ in $\boldsymbol{C}$, so we get $\boldsymbol{a^2 = (a - 3)^2 + (b - 5)^2}$, too.

Therefore, we get $\boldsymbol{(a - 1)^2 + (b - 2)^2 = (a - 3)^2 + (b - 5)^2}$.

Then first, we get

$$\boldsymbol{(a - 1)^2 + (b - 2)^2 = a^2 \Rightarrow a^2 - 2a + 1 + b^2 - 4b + 4 = a^2 \Rightarrow b^2 - 4b - 2a + 5 = 0}.$$

Next, we get

$(a-1)^2 + (b-2)^2 = (a-3)^2 + (b-5)^2 \Rightarrow (a-1)^2 - (a-3)^2 + (b-2)^2 - (b-5)^2 = 0$

$\Rightarrow \{(a-1)-(a-3)\}\{(a-1)+(a-3)\} + \{(b-2)-(b-5)\}\{(b-2)+(b-5)\} = 0$

$\Rightarrow 2(2a-4) + 3(2b-7) = 4a + 6b - 29 = 0$.

Thus, we get to solve a system of two equations for a and b as follows.

$4a + 6b - 29 = 0$ and $b^2 - 4b - 2a + 5 = 0$.

To begin with, $4a+6b-29=0 \Rightarrow a=\frac{(29-6b)}{4}$. Then, we get

$b^2-4b-2a+5 = b^2-4b-2\cdot\frac{(29-6b)}{4}+5 = b^2-4b+3b+5-\frac{29}{2}=0 \Rightarrow b^2-b-\frac{19}{2}=0$.

Then, we get $b^2-b-\frac{19}{2} = b^2-b+\frac{1}{4}-\frac{1}{4}-\frac{19}{2} = (b-\frac{1}{2})^2-\frac{1}{4}-\frac{19}{2} = (b-\frac{1}{2})^2-\frac{39}{4}=0$

$\Rightarrow (b-\frac{1}{2})^2=\frac{39}{4} \Rightarrow b-\frac{1}{2}=\pm\sqrt{\frac{39}{4}} \Rightarrow b=\frac{1}{2}\pm\frac{\sqrt{39}}{2}$.

So we get $b=\frac{1+\sqrt{39}}{2}$ or $b=\frac{1-\sqrt{39}}{2}$. Also, we have $a=\frac{(29-6b)}{4}$. Thus, we get

$b=\frac{1+\sqrt{39}}{2} \Rightarrow a=\frac{1}{4}(29-6\cdot\frac{1+\sqrt{39}}{2})=\frac{1}{4}\cdot\frac{58-6-6\sqrt{39}}{2}=\frac{52-6\sqrt{39}}{8}=\frac{26-3\sqrt{39}}{4}$.

$b=\frac{1-\sqrt{39}}{2} \Rightarrow a=\frac{1}{4}(29-6\cdot\frac{1-\sqrt{39}}{2})=\frac{1}{4}\cdot\frac{58-6+6\sqrt{39}}{2}=\frac{52+6\sqrt{39}}{8}=\frac{26+3\sqrt{39}}{4}$.

We know that the radius is $|a|$, and that the center is (a, b).

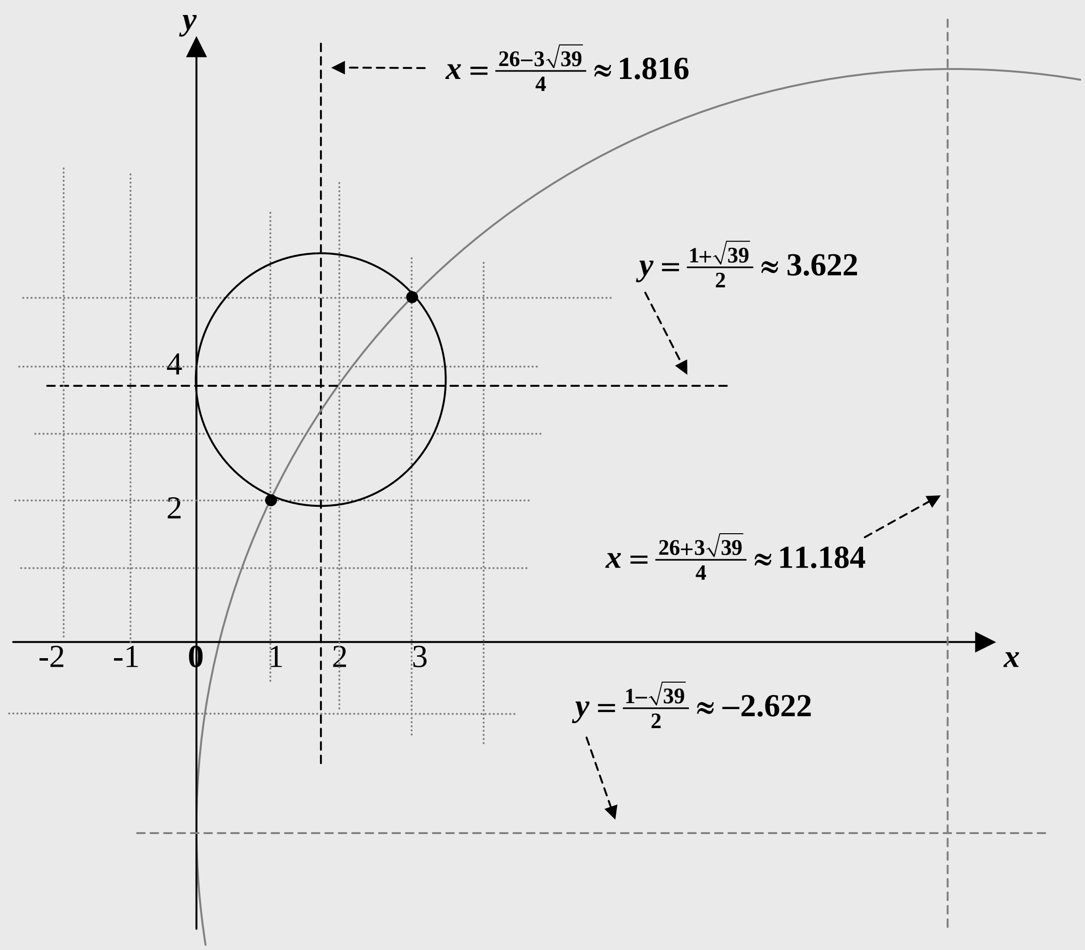
Therefore, the circle C is $(x-a)^2 + (y-b)^2 = a^2$ where

$a=\frac{26-3\sqrt{39}}{4}\approx 1.816$ and $b=\frac{1+\sqrt{39}}{2}\approx 3.622$

or $a=\frac{26+3\sqrt{39}}{4}\approx 11.184$ and $b=\frac{1-\sqrt{39}}{2}\approx -2.622$.

Let’s now, put the circles in a graph.

Fig. 2



In short:

Suppose that the center is in a line $\boldsymbol{x = a}$, and that the point where the circle is tangent to the $\boldsymbol{y}$-axis is $\boldsymbol{(0, b)}$. Then, the center is at a point $\boldsymbol{(a, b)}$, where the line $\boldsymbol{x = a}$ meets the line $\boldsymbol{y = b}$, and the radius is $|\boldsymbol{a}|$.

Each of the two points given is the radius $|\boldsymbol{a}|$ away from $\boldsymbol{(a, b)}$.

So taking the distances from (a, b) to $(1, 2)$ and $(3, 5)$, we get

$b^2 = (a - 1)^2 + (b - 2)^2$, and $b^2 = (a - 3)^2 + (b - 5)^2$.

Then first, we get

$$(a - 1)^2 + (b - 2)^2 = a^2 \Rightarrow a^2 - 2a + 1 + b^2 - 4b + 4 = a^2 \Rightarrow b^2 - 4b - 2a + 5 = 0.$$

Next, we get

$$(a - 1)^2 + (b - 2)^2 = (a - 3)^2 + (b - 5)^2 \Rightarrow (a - 1)^2 - (a - 3)^2 + (b - 2)^2 - (b - 5)^2 = 0$$

$$\Rightarrow \{(a - 1) - (a - 3)\}\{(a - 1) + (a - 3)\} + \{(b - 2) - (b - 5)\}\{(b - 2) + (b - 5)\} = 0$$

$$\Rightarrow 2(2a - 4) + 3(2b - 7) = 4a + 6b - 29 = 0.$$

Thus, we need to solve a system where $4a + 6b - 29 = 0$ and $b^2 - 4b - 2a + 5 = 0$.

Then first, we get $4a + 6b - 29 = 0 \Rightarrow a = \frac{29-6b}{4}$. Next, we get

$$b^2 - 4b - 2a + 5 = b^2 - 4b - 2\cdot\frac{(29-6b)}{4} + 5 = b^2 - 4b + 3b + 5 - \frac{29}{2} = 0 \Rightarrow b^2 - b - \frac{19}{2} = 0.$$

Then, $b^2 - b - \frac{19}{2} = b^2 - b + \frac{1}{4} - \frac{1}{4} - \frac{19}{2} = (b - \frac{1}{2})^2 - \frac{1}{4} - \frac{19}{2} = (b - \frac{1}{2})^2 - \frac{39}{4} = 0$

$$\Rightarrow (b - \tfrac{1}{2})^2 = \tfrac{39}{4} \Rightarrow b - \tfrac{1}{2} = \pm\sqrt{\tfrac{39}{4}} \Rightarrow b = \tfrac{1}{2} \pm \tfrac{\sqrt{39}}{2}.$$

So we get $b = \frac{1+\sqrt{39}}{2}$ or $b = \frac{1-\sqrt{39}}{2}$. And we have $a = \frac{29-6b}{4}$, too. Thus, we get

$$b = \frac{1+\sqrt{39}}{2} \Rightarrow a = \frac{1}{4}(29 - 6\cdot\frac{1+\sqrt{39}}{2}) = \frac{1}{4}\cdot\frac{58-6-6\sqrt{39}}{2} = \frac{52-6\sqrt{39}}{8} = \frac{26-3\sqrt{39}}{4}.$$

$$b = \frac{1-\sqrt{39}}{2} \Rightarrow a = \frac{1}{4}(29 - 6\cdot\frac{1-\sqrt{39}}{2}) = \frac{1}{4}\cdot\frac{58-6+6\sqrt{39}}{2} = \frac{52+6\sqrt{39}}{8} = \frac{26+3\sqrt{39}}{4}.$$

We know that the radius is $|a|$, and that the center is (a, b).

Therefore, the circle C is $(x - a)^2 + (y - b)^2 = a^2$, where $a = \frac{26\mp3\sqrt{39}}{4}$ and $b = \frac{1\pm\sqrt{39}}{2}$.